On Perold’s ‘Fundamentally Flawed Indexing’.
Comments are invited.

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Arnott et al (2005) argue that the use of weights in proportion to market capitalization in indexes and portfolios creates possibly a performance drag, because overvalued companies will be overweighted, as undervalued companies will be underweighted. The performance drag occurs when there is a tendency in the market to correct for mispricing. Perold (2007) takes issue with Arnott’s proposition. He contends that the market price will not tell whether the stock is over- or undervalued, and therefore cap-weighting entails no performance drag. Both authors take the (market) price $P$ to be the product of (fair) value $V$ and (pricing) error $E$, so $P = V \cdot E$, and both take fair value and pricing error to be independent. ($P$, $V$ and $E$ are positive random variables). Perold argues that if nothing is known a priori about fair value, which he models by a log-uniform distribution for $V$, then pricing error and market price are also independent, hence the absence of the performance drag. The purpose of this note is to show that Perold’s result is carried entirely by the log-uniform distribution. In other words, he has identified the singular situation, the unique distribution (and an improper one at that), that allows independence between value and error and between price and error$^1$. Any other distribution does not allow of that and thereby supports Arnott’s proposition. The fundamental issue in fundamental indexation is therefore not the one identified by Perold, but the existence of fair value and mean reversion of errors. Direct observations cannot, by definition, settle this. Empirical ‘success’ (sustained relative outperformance of valuation indifferent portfolios and indexes in various circumstances), and theoretical usefulness (in explaining e.g. well-established anomalies) will be decisive in assessing the value of ‘noisy prices’. Some relevant references are Arnott & Hsu (2008) and Arnott et al (2005).

$^1$This curious property of the log-uniform distribution has been misinterpreted in some critical responses to Perold (2007): Treynor (2008) and Arnott and Markowitz (2008) thought that Perold had replaced the independence between value and error by independence between price and error. But in fact he could maintain, in a way, both. Arguments against Perold’s model in terms of covariances presuppose the existence of second order moments.
The remainder of this note aims at proving

\[ V \perp E \& P \perp E \implies f_V(v) \propto 1/v \& f_E(e) \propto 1/e \& f_P(p) \propto 1/p. \]  

(1)

Here \( V \perp E \) is short for ‘\( V \) and \( E \) are independent’, and \( f_V(v) \propto 1/v \) says that the density of \( V \) evaluated at any positive number \( v \) is proportional to one over \( v \), in other words that \( \log(V) \) has a uniform (maximum entropy) distribution on the real line\(^2\). So if we maintain that \( V \perp E \) and any one of \( V \), \( E \) or \( P \) is not log-uniform, then \( P \) cannot be independent of \( E \). A particular counter-example would be the usual workhorse of mathematical finance, the log-normal distribution for \( P \), the maximum entropy distribution for \( \log(P) \) with finite variance. In this situation \( V \) and \( E \) are log-normal too\(^3\), and price and error are positively correlated.

We will prove (1) by establishing two other simple statements which may be of some independent interest. The first one is:

\[ V \perp E \& f_V(v) \propto 1/v \iff P \perp E \& f_P(p) \propto 1/p. \]  

(2)

Proof: ( \( \implies \) ) \( \text{Prob}\{P \leq p | E = e\} = \text{Prob}\{V \leq \frac{p}{e} | E = e\} = F_V\left(\frac{p}{e}\right) \). And so \( f_{P|E=e}(p) = f_V\left(\frac{p}{e}\right) \frac{1}{e} \propto \frac{1}{p} \). The conditional density of \( P \) given \( E = e \) does not depend on \( e \), so \( P \perp E \) and \( P \) is log-uniform.

Conversely, \( \text{Prob}\{V \leq v | E = e\} = \text{Prob}\{P \leq v \cdot e | E = e\} = F_P(v \cdot e) \) and so \( f_{V|E=e}(v) = f_P(v \cdot e) e \propto \frac{1}{v} e = 1/v \). QED.

The next statement is

\[ V \perp E \& P \perp E \implies f_V(v) \propto 1/v. \]  

(3)

Proof: As before we have \( \text{Prob}\{P \leq p | E = e\} = \text{Prob}\{V \leq \frac{p}{e} | E = e\} = F_V\left(\frac{p}{e}\right) \) where we used \( V \perp E \). And therefore again \( f_{P|E=e}(p) = f_V\left(\frac{p}{e}\right) \frac{1}{e} \). Now \( f_{P|E=e}(p) \) must not depend on \( e \), since \( P \perp E \), so the product \( f_V\left(\frac{p}{e}\right) \frac{1}{e} \) cannot depend on \( e \). This requires \( f_V(v) \propto 1/v \). QED.

Now if \( V \perp E \& P \perp E \) then statements (2) and (3) together imply that \( f_V(v) \propto 1/v \& f_P(p) \propto 1/p \). It remains to show that also \( f_E(e) \propto 1/e \). This is done as follows. The joint density of \( V \) and \( E \), \( f_{V,E}(v,e) \) must be proportional to \( \frac{1}{e} f_E(e) \) because of the independence and the log-uniformity

\(^2\)\(F_V(.)\) will be the distribution function, most of the notation will be self-explanatory. We will integrate and differentiate formally, ignoring the fact that the log-uniform distribution is improper. A proper treatment would use a limiting process starting from a bounded interval, see Zellner (1971), p.45.

\(^3\)This follows from a well-known characterization of normality: if the sum of two independent random variables is normal, then the random variables must be normal as well, see Cramér (1962)
of $V$. So, integrating formally,

$$
\text{Prob} \{ P \leq p \} = \text{Prob} \{ V \cdot E \leq p \} \propto \int_0^\infty \frac{1}{v} \left( \int_0^{p/v} f_E(e) \, de \right) \, dv.
$$

(4)

Therefore $f_P(p) \propto \int_0^\infty \frac{1}{v} \left[ \frac{1}{v} f_E \left( \frac{p}{v} \right) \right] \, dv$ and this must be proportional to $1/p$. Consequently, we have that

$$
\int_0^\infty \frac{1}{v} \left[ \frac{p}{v} f_E \left( \frac{p}{v} \right) \right] \, dv
$$

is a constant, the same for all values of $p$. This requires $f_E(e) \propto 1/e$. QED.

As a final observation we have:

$$
V \perp E \& P \perp E \implies P \perp V.
$$

(6)

This is true because

$$
\text{Prob} \{ P \leq p | V = v \} = \text{Prob} \left\{ E \leq \frac{p}{v} | V = v \right\} = F_E \left( \frac{p}{v} \right)
$$

(7)

where we used $V \perp E$, and since the conjunction with $P \perp E$ implies $f_E(e) \propto 1/e$ we get $f_{P|V=v}(p) = f_E \left( \frac{p}{v} \right) \frac{1}{v} \propto 1$ and $f_{P}(p) \propto 1/p$.

References


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*The statement concerning the density of $E$ is most easily seen by postulating a power series for $\frac{p}{v} f_E \left( \frac{p}{v} \right)$ in $\frac{p}{v}$ and integrating with respect to the proper density $\frac{1}{v} \log(v)$ over intervals of the form $(1/B, B)$ where $B$ tends to $\infty$. Since the outcome of the integration on a bounded interval is a power series in $p$ that cannot vary with $p$, all coefficients in the expansion, except the constant, are zero.*

