Distributed Coordination and Partial Synchronization in Complex Networks

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To my family

献给我的家人

妻子葛杉杉、
母亲董红、父亲秦有清

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1 Introduction

Coordinating behaviors in large groups of interacting units are pervasive in nature. Remarkable examples include fish schooling [1], avian flocking [2], land animals herding [3], rhythmic firefly flashing [4], and synchronized neuronal spiking [5]. Extensive efforts have been made to uncover the mechanisms behind these astonishing coordinating behaviors. There have been major progresses, and many of them have also been applied to solving various problems in engineering practice. For example, distributed weighted averaging has found applications in distributed computation in robotic networks. On the other hand, the mechanisms of many coordinating behaviors remain unknown. For example, what gives rise to a variety of synchronization patterns in the human brain is still an intriguing question. In this thesis, we first study distributed coordination algorithms in stochastic settings. We then investigate partial instead of global synchronization in complex networks, trying to reveal some possible mechanisms that could render correlations across only a part of brain regions as indicated by empirical data. In this chapter, we introduce some background knowledge of distributed coordination algorithms as well as synchronization, provide a sketch of the main contributions, and explain how this thesis is structured. Some notations used throughout the thesis are also presented.

1.1 Background

In the next two subsections, we introduce some background information of distributed coordination algorithms and synchronization, respectively.

1.1.1 Distributed Coordination Algorithms

A huge number of models have been proposed to describe coordinating behaviors in a network of autonomous agents. The DeGroot model and the Vicsek model are two of the most popular models. Introduced in 1975, the DeGroot model describes how a group of people might reach an agreement by pooling their individual opinions [6]. Proposed in 1995, the Vicsek model is used to investigate the emergence of self-organized motion in systems of particles [7]. These two models have fascinated a lot of researchers in different fields because they are very simple yet revealing, and they are capable of explaining rich collective behaviors in nature. They have also inspired the development of distributed coordination algorithms in multi-agent systems. There are two key features of distributed coordination algorithms that are inherited from the Vicsek model and the DeGroot model: 1) each agent simply needs to compute the weighted average of the states of itself and its neighbors; and 2) only local information is required for computation of the weighted averages, and thus the distributed coordination algorithms are also known as the distributed weighted averaging algorithms.

Distributed coordination algorithms in complex networks have attracted much interest in the recent two decades. Just like the Vicsek's model, each agent's nearest neighbors in distributed coordination algorithms can change with time. To study this, early works have considered dynamically changing networks and provided some connectivity conditions for convergence [8–12]. Moreover, agents may not have a common clock to synchronize their update actions in practice. Thus, asynchronous events have also been taken into account, and conditions have been obtained such that the convergence can be preserved [10, 13, 14]. Distributed coordination algorithms actually serve as a foundation for a considerable number of network algorithms for various purposes such as load balancing [15,16], information fusion [17,18], rendezvous of robots [19, 20], placement of mobile sensors [21, 22], formation control [23, 24]. More recently, distributed coordination algorithms have also been used for many other research topics including distributed optimization [25, 26], distributed observer design [27,28], solving linear equations distributively [29,30], and modeling of opinion dynamics in social networks [31–33].

Most of the aforementioned studies on distributed coordination algorithms and their applications are in deterministic settings. However, in many circumstances, the implementation of distributed coordination algorithms is often under the influence of uncertainty in the environment. Some further works have shown that the convergence can still be guaranteed even with the presence of randomly changing network topologies [34–36], random network weights [37], random communication delays [38–40], and random asynchronous events [41, 42]. Much less attention has been paid to the

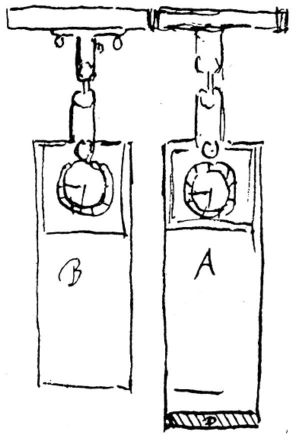


Figure 1.1: Original drawing of Christiaan Huygens: two pendulum clocks hanging side by side on a beam (source: [46])

investigation of how the presence of some randomness can be helpful for coordination in a network. Surprisingly, random noise, usually believed to be troublesome, sometimes brings benefits to a system in terms of achieving better system-level performance. For example, the survivability of a group of fish can be boosted by random schooling [43]; random deviation can enhance cooperation in social dilemmas [44]; and behavioral randomness can improve the global performance of human in a coordination game [45]. There is a great need in systematically studying stochastic distributed algorithms, which enables the analysis of coordination in networks under the influence of both detrimental and beneficial randomness.

1.1.2 Synchronization and Brain Communication

In February 1665, staring aimlessly at two pendulum clocks hanging side by side on a wooden structure (shown in Fig. 1.1), Christiaan Huygens suddenly noticed they began to swing perfectly in step. More unexpectedly, he found that they seemed to never break step. This renowned Dutch physicist, mathematician and astronomer described this surprising discovery by "*an odd sympathy*". After more than 350 years, the interesting phenomenon is nowadays termed *synchronization*.

As another form of coordinating behaviors, synchronization has attracted attention from scientists in various disciplines due to its ubiquitous occurrence in many natural, engineering, and social systems. The snowy tree crickets are found to be able to synchronize their chirping [47]; rhythmic hand clapping often appears after theater

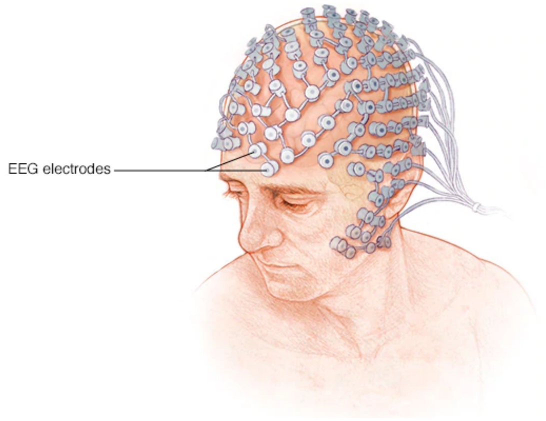
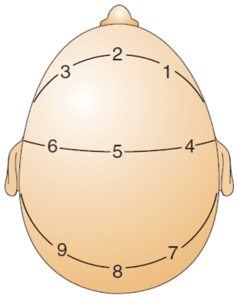


Figure 1.2: In illustration of how EEG records brain waves (source: https://www.mayoclinic.org/tests-procedures/eeg/about/pac-20393875)

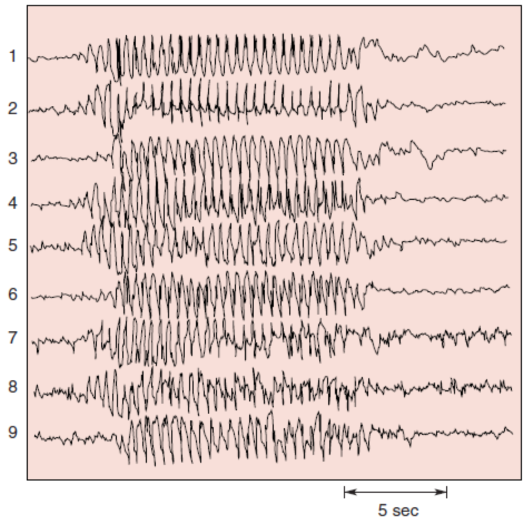
and opera performances [48]; power generators operate synchronously to function properly [49]; and circadian rhythms of almost all land animals are often in accordance with the environment [50] (e.g., sleep and wakefulness are closely related to daily cycles of daylight and darkness).

Synchronization has also been detected pervasively in neuronal systems [51–53]. It plays a central role in information processing within a brain region and neuronal communication between different regions. Investigation on synchronization of neuronal ensembles in the brain, especially in cortical regions, becomes one of the most important problems in neuroscience. The *electroencephalogram* (EEG) is a typical method to measure brain activities, and is essential to experimentally study synchronization of the cerebral cortex. Measuring brain waves using EEG is quite simple since it is noninvasive and painless. Fig. 1.2 provides an illustration of how EEG is used to record brain signals. Several early experiments indicate that synchronization of neuron spikes in the visual cortex of animals accounts for different visual stimulus features [5,53]. Interregional spike synchronization is shown to have a functional role in the coordination of attentional signals across brain areas [54, 55]. Recently, it has been shown that phase synchronization contributes mechanistically to attention [56], cognitive tasks [57], working memory [58], and particularly interregional communication [52, 59].

In fact, synchronization across brain regions is believed to facilitate interregional communication. Only cohesively oscillating neuronal groups can exchange information effectively because their communication windows are open at the same time [52]. However, abnormal synchronization in the human brain is always a sign of pathology [60, 61]. As an example, Fig. 1.3 presents the EEG recording of brain waves during



(a)



(b)

Figure 1.3: A EEG recording of an epileptic seizure (source: [50, Fig. 19.14]): (a) positions on the scalp where EEG electrodes are placed; (b) the EEG signals recorded by the electrodes.

an epileptic seizure, where synchronization across the entire brain is observed. Such strikingly abnormal behavior is never detected in a healthy brain. This suggests that there are some robust and powerful regulation mechanisms in a non-pathological brain that are able to not only facilitate but also preclude neuronal communication. Partial synchronization is believed to be such a mechanism [52]. Only necessary parts of regions are synchronized for some specific brain function. Communication between incoherent brain regions is prevented. In this case, information exchange between two neuronal groups is not possible because their communication windows are not coordinated. Synchronizing a selective set of brain regions can render and also prevent neuronal communication in a selective way.

When it comes to the study of synchronization, the Kuramoto model serves as a powerful tool. After it was first proposed in 1975 [62], the Kuramoto model rapidly became one of the most widely-accepted models in understanding synchronization phenomena in a large population of oscillators. It is simple enough for mathematical analysis, yet still capable of capturing rich sets of behaviors. Thus, it has been extended to many variations [63]. The Kuramoto model and its generalizations are also widely used to model the dynamics of coupled neuronal ensembles in the human brain. It is of great interest to analytically study partial synchronization with the help of the Kuramoto model and its variations, trying to reveal the possible underlying mechanisms that can give rise to different synchrony patterns in the human brain.

1.2 Contributions

In the first part of this thesis, we restrict our attention to distributed coordination algorithms in stochastic settings since the implementation of them is often under random influences and the introduction of some randomness sometimes can be beneficial.

Study of stochastic distributed coordinate algorithms is often associated with stability analysis of stochastic discrete-time systems. There are some noticeable Lyapunov theories on stability of stochastic systems including Khasminskii's book [64], and Kushner's works [65–67]. Particularly in [66, 67], the expectation of a Lyapunov function is required to decrease after every time step, in order to show the stability of a stochastic discrete-time system. However, it is not always easy to construct such a Lyapunov function. Therefore, we propose some new Lyapunov criteria for asymptotic and exponential stability analysis of stochastic discrete-time systems. We allow the expectation of Lyapunov function candidates to decrease after some finite steps instead of every step. This relaxation enlarges the range of applicable Lyapunov functions and also provides us with the possibility of working on systems with non-Markovian states.

Using these new Lyapunov criteria, we then study the convergence of products of random stochastic matrices. While implementing distributed coordinate algorithms, one always encounters the need to prove the convergence of products of stochastic matrices, or equivalently the convergence of inhomogeneous Markov chains. The study of products of stochastic matrices dates back to more than 50 years ago in Wolfowitz's paper [68]. Since then, a lot of progress has been made [69–73], and many applications have been implemented [8–11, 74]. Recent years have witnessed an increasing interest in studying products of random sequences of stochastic matrices [35, 75, 76]. Nevertheless, most of the existing results rely on the assumption that each matrix in a sequence has strictly positive diagonal entries. Without this assumption, many existing results do not hold anymore. Moreover, the underlying random processes driving the random sequences are usually confined to some special types, such as independent and identically distributed (i.i.d) sequences [35], stationary ergodic sequences [36], or independent sequences [75, 76]. The new Lyapunov criteria we obtained enable us to work on more general classes of random sequences of stochastic matrices without the assumption of nonzero diagonal entries. We obtain some quite mild conditions compared to the existing results on random sequences of stochastic matrices such that convergence of the products can be guaranteed. The convergence speed, which is believed to be quite challenging, is also estimated. We also consider some special random sequences including stationary processes and stationary ergodic processes.

As another application, we study agreement of multi-agent systems in periodic networks. Periodic networks often lead to oscillating behavior, but we show that agreement can surprisingly be reached if the agents activate and update their states asynchronously. We relax the requirement that networks need to be aperiodic, and obtain a necessary and sufficient condition for the network topology such that agreement can take place almost surely. We further apply our Lyapunov criteria to solving linear equations distributively. We relax the existing conditions in [77] on the changing network topology such that equations can be solved almost surely.

In the second part of this thesis, we study partial synchronization in complex networks. As we have discussed in the previous section, partial synchronization is perhaps more common than global synchronization in nature. Particularly, global synchronization in the human brain is often a symptom of serious diseases [60]. Unlike global synchronization, partial synchronization is a phenomenon that only a specific portion of units in a network are synchronized, while the rest remains incoherent. Unlike global synchronization, on which a lot of results have been obtained (we refer the readers to a survey paper [78]), the study on partial synchronization is much less. However, it has attracted growing interests recently. Cluster synchronization is a type of partial synchronization, which describes the situation where more than one synchronized groups of oscillators coexist in a network. It has been shown that network topology and the presence of time delays are quite important to render cluster synchronization [79–85]. The Chimera state is another interesting type of partial synchronization, which is characterized by the coexistence of both coherent and incoherent groups within the same network. Chimera states were initially discovered by Kuramoto *et al.* in 2002. Since then several investigations have been made [86–88]. We refer the readers to a survey for more details [89].

With the help of the Kuramoto model and its variations, we identify two mechanisms that can account for the emergence and stability of partial synchronization: 1) strong local or regional connections, and 2) network symmetries. Inspired by some empirical works [90, 91], we show that a part of oscillators in a network can be quite coherent if they are directly connected and the connections between them are strong, while the rest that are weakly connected remain incoherent. In addition, we also show that oscillators that are not directly connected can also be synchronized, with the ones connecting them having different dynamics, if they are located at symmetric positions in a network. Such a phenomenon is called *remote synchronization*, which has also been widely detected in the human brain, where distant cortical regions without direct neural links also experience functional correlations [92].

In the first case, we utilize the incremental 2-norm and the incremental ∞-norm based Lyapunov functions to study partial synchronization. Sufficient conditions on the network parameters (i.e., algebraic connectivity and nodal degrees) are obtained such that partial synchronization can take place. We calculate the regions of attraction and estimate the ultimate level of synchrony. The results using incremental ∞-norm are the first known ones that are used to study synchronization in non-complete networks.

In the second case, we study remote synchronization in star networks by using the Kuramoto-Sakaguchi model. The phase shift in the Kuramoto-Sakaguchi model is usually used to model synaptic connection delays [93]. A star network is simple in structure, but has basic morphologically symmetric properties. The peripheral nodes have no direct connection, but obviously play similar roles in the whole network. The node at the center acts as a relay or mediator. As an example, the thalamus is such a relay in neural networks. It is connected to all the cortical regions, and is believed to enable separated regions to be completely synchronized [94, 95]. We show that network symmetries indeed play a central role in giving rise to remote synchronization as is predicted in some works such as [80, 96]. We reveal that the symmetry of outgoing connections from the central oscillator is crucial to shaping remote synchronization, and is possible to render several clusters for the peripheral oscillators. Note that the coupling strengths of incoming links to the central oscillator are not required to be symmetric.

Motivated by some experimental works [97, 98], we then study how detuning the natural frequency of the central oscillator in a star network with two peripheral nodes can enhance remote synchronization. To analyze this interesting problem, we obtained some new Lyapunov criteria for partial stability of nonlinear systems. Partial stability describes the behavior of a dynamical system in which only a given part of its state variables, instead of all, are stable. To show partial asymptotic or exponential stability, the time derivative of a Lyapunov function candidate is required to be negative definite according to the existing results [99–101]. We relax this condition by allowing the time derivative of the Lyapunov function to be positive, as long as the Lyapunov function *per se* decreases after a finite time. We then establish some further criteria for partial exponential stability of slow-fast systems using periodic averaging methods. We prove that partial exponential stability of the averaged system implies that of the original one. As some intermediate results, a new converse Lyapunov theorem and some perturbation theorems are also obtained for partial exponential stability systems. Finally, we use the obtained Lyapunov criteria to prove that natural frequency detuning of the central oscillator actually strengthens the remote synchronization, making it robust against to the phase shift. The proof reduces to the demonstration of the partial exponential stability of a slow-fast system.

1.3 Thesis Outline

The remainder of this thesis is organized as follows. Chapter 2 provides some preliminary concepts and theories that will be used throughout the thesis, including probability theory, graph theory, and some concepts related to stochastic matrices.

The main body of the thesis is divided into two parts. The first part consists of two chapters, i.e., Chapters 3 and 4, in which we focus on stochastic distributed coordination algorithms. In Chapter 3, we propose some new Lyapunov criteria for stability and convergence of stochastic discrete-time systems. The results in Chapter 3 provide some tests for stability analysis of asymptotic convergence, exponential convergence, asymptotic stability in probability, exponential stability in probability, almost sure asymptotic stability, or almost sure exponential stability of a stochastic discrete-time system. These criteria are then used in Chapter 4, where the convergence of products of random stochastic matrices, agreement problems induced by asynchronous events, and solving linear equations by distributed algorithms are studied. The content of Chapter 3 is based on [102], and that of Chapter 4 on [102] and [103].

The second part of the thesis consists of three chapters, i.e., Chapters 5, 6, and 7. In this part, we aim at identifying some possible underlying mechanisms that could lead to partial synchronization in complex networks. We first investigate in Chapter 5 how partial synchronization can take place among directly connected regions. We find that strong local or regional coupling is a possible mechanism. Tightly connected oscillators can have coherent behaviors, while other oscillators that are weakly connected to them can evolve quite differently. In addition, we also study how partial synchronization is possible to occur among oscillators that have no direct connections, namely remote synchronization phenomena. In order to study remote synchronization, we develop some new criteria for partial stability of nonlinear systems in Chapter 6. In Chapter 7, we analytically study remote synchronization in star networks. We employ the Kuramoto model and the Kuramoto-Sakaguchi model to describe the dynamics of the oscillators. Some sufficient conditions are obtained such that remote synchronization can emerge and remain stable. The content of Chapter 5 is based on [104] and [105], Chapter 6 on [106] and [107], and Chapter 7 on [107] and[108].

1.4 List of Publications

Journal articles

[1] **Y. Qin**, M. Cao, and B. D. O. Anderson, "Lyapunov criterion for stochastic systems and its applications in distributed computation." *IEEE Transactions on Automatic Control*, doi: 10.1109/TAC.2019.2910948, to appear as a *full paper*.

[2] **Y. Qin**, Y. Kawano, O. Portoles and M. Cao. "Partial phase cohesiveness in networks of Kuramoto oscillator networks." *IEEE Transactions on Automatic Control*, under review as a *technical note*.

[3] **Y. Qin**, Y. Kawano, B. D. O. Anderson, and M. Cao. "Partial Exponential Stability Analysis of Slow-fast Systems via Periodic Averaging." *IEEE Transactions on Automatic Control*, under review as a *full paper*.

[4] M. Ye, **Y. Qin**, A. Govaert, B. D. O. Anderson, and M. Cao. "An influence network model to study discrepancies in expressed and private Opinions," *Automatica*, 107: 371-381, 2019, *full paper*.

Conference papers

[1] **Y. Qin**, Y. Kawano and M. Cao, "Stability of remote synchronization in star networks of Kuramoto oscillators," in *Proceedings of the 57th IEEE Conference on Decision and Control*, Miami Beach, FL, USA, 2018, pp. 5209-5214.

[2] **Y. Qin**, Y. Kawano, and M. Cao, "Partial phase cohesiveness in networks of communitinized Kuramoto oscillators," in *Proceedings of IEEE European Control Conference*, Limassol, Cyprus, 2018, pp. 2028-2033.

[3] **Y. Qin**, M. Cao, and B. D. O. Anderson, "Asynchronous agreement through distributed coordination algorithms associated with periodic matrices," in *Proceedings of the 20th IFAC World Congress*, Toulouse, France, 2017, 50(1): 1742-1747.

[4] A Govaert, **Y. Qin**, and M. Cao. "Necessary and sufficient conditions for the existence of cycles in evolutionary dynamics of two-strategy games on networks," in *Proceedings of IEEE European Control Conference*, Limassol, Cyprus, 2018, pp. 2182-2187.

1.5 Notation

Sets

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}_0$ the set of non-negative integers, and $\mathbb{N}$ the collection of positive integers. Let $\mathbb{R}^q$ denote the real q-dimensional vector space, $\mathbf{1}_q$ the q-dimensional vector consisting of all ones, and for any $n \in \mathbb{N}$ let $\mathbf{N} = \{1, 2, \dots, n\}$. For any $\delta > 0, x \in \mathbb{R}^n$, define $\mathcal{B}_\delta(x) := \{y \in \mathbb{R}^n : \|y - x\| < \delta\}$ and $\bar{\mathcal{B}}_\delta(x) := \{y \in \mathbb{R}^n : \|y - x\| \leq \delta\}$. Particularly, let $\mathcal{B}_\delta = \{y \in \mathbb{R}^n : \|y\| < \delta\}$ and $\bar{\mathcal{B}}_\delta = \{y \in \mathbb{R}^n : \|y\| \leq \delta\}$.

Norms

Let $\|\cdot\|_p$, $p \geq 1$, be any p-norm for both vectors and matrices.

Comparison functions

A continuous function $h(x) : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $h(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ function if $a = \infty$ and $h(r) \to \infty$ as $r \to \infty$.

Other Notation

Given two sets $\mathcal{A}$ and $\mathcal{B}$, the union of them is denoted by $\mathcal{A} \cup \mathcal{B}$, the intersection is denoted by $\mathcal{A} \cap \mathcal{B}$, and $\mathcal{A} \backslash \mathcal{B}$ presents the difference between $\mathcal{A}$ and $\mathcal{B}$, i.e., $\mathcal{A} \backslash \mathcal{B} = \{x : x \in \mathcal{A}, x \notin \mathcal{B}\}$. Given $x \in \mathbb{R}^n, y \in \mathbb{R}^m$, denote $\mathrm{col}(x, y) = (x^\top, y^\top)^\top$. With a bit abuse of notation, we denote $\mathrm{col}(f_1, f_2) = (f_1(x)^\top, f_2(x)^\top)^\top$ for two given functions $f_1 : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $f_2 : \mathbb{R}^{n+m} \to \mathbb{R}^m$.

In Part I of this thesis, we let x^i denote the ith element of a given vector $x \in \mathbb{R}^n$ for the purpose of notational clarity; in Part II, we denote the ith element of x in the conventional way, i.e., x_i. Given a vector $x \in \mathbb{R}^n$, let

$$\mathrm{diag}(x) = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix}.$$

For any $x \in \mathbb{R}$, Let $\lfloor x \rfloor$ denote the largest integer that is less than or equal to x, and $\lceil x \rceil$ the smallest integer that is greater than or equal to x.

2 Preliminaries

In this chapter, we introduce some theories and concepts that will be used in the remainder of this thesis.

2.1 Probability Theory

Probability Space and Random Variables

The *sample space* Ω of an experiment is the set of all possible outcomes. A collection $\mathcal{F}$ of subsets of Ω is called a σ-field if it satisfies: 1) $\emptyset \in \mathcal{F}$; 2) if $A_1, A_2, \dots \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$; and 3) $A \in \mathcal{F}$, then its complement $A^c \in \mathcal{F}$. A probability space is defined by a triple $(\Omega, \mathcal{F}, \Pr)$, where $\Pr : \mathcal{F} \to [0, 1]$ is a function (called a *probability measure*) that assigns probabilities to events [109].

A random variable X is a measurable function from a sample space to the set of real numbers $\mathbb{R}$, i.e., $X : \Omega \to \mathbb{R}$. We are only concerned with discrete random variables in this thesis. Thus, the subsequent concepts are all associated with discrete random variables. A vector-valued random variable Y is defined by $Y : \Omega \to \mathbb{R}^n$.

Conditional Probability and Conditional Expectation

In probability, a conditional probability measures the probability of an event A occurring given that another event B has occurred. It is usually denoted by $\Pr[A|B]$, and can be calculated by

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]},$$

assuming that $P(B) > 0$.

A conditional expectation of a random variable X is its expected value given an event has already occurred. It can be calculated in the following way

$$\mathbb{E}[X|B] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega|B].$$

Stochastic Processes

A stochastic process is an infinite collection of (vector-valued) random variables, indexed by an integer often interpreted as time, usually denoted by $\{X(k) : k \in \mathbb{N}_0\}$.

Joint Probability Distribution

Given n random variables $X_1, X_2, \ldots, X_n$, the joint probability distribution of them is

$$p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \Pr[X_1 = x_1, \ldots, X_n = x_n].$$

2.2 Graph Theory

Graphs are used to describe network topologies. An n-node graph is defined by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of nodes, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. A *directed graph* is a graph where all the edges are directed from one node to another. We use (i, j) to denote a directed edge from i to j; i is said to be the *source*, and j is said to be the *target*. Given $\mathcal{E}_p \in \mathcal{E}$, we let $s(\mathcal{E}_p)$ denote the source of $\mathcal{E}_p$, and $t(\mathcal{E}_p)$ the target of $\mathcal{E}_p$. A directed *path* is a sequence of edges of the form $(p_1, p_2), (p_2, p_3), \ldots, (p_{m-1}, p_m)$, where p_i are distinct nodes in $\mathcal{V}$, and $(p_j, p_{j+1}) \in \mathcal{E}$.

On the other hand, a graph, in which all the edges are undirected, is called an *undirected graph*. An undirected path is defined in the same way as the directed one, but the edges are undirected.

Directed Graph

A directed graph is said to be *strongly connected* if there is a path from every node to every other node [110]. A directed graph is said to be a *directed spanning tree* if there is exactly one node, called root, such that any other node can be reached from it via exactly one directed path. A directed graph is said to be *rooted* if it contains a directed spanning tree that contains all the nodes.

Given two directed graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with the same node set $\mathcal{V}$, the *composition* of them, denoted by $\mathcal{G}_2 \circ \mathcal{G}_1$, is a directed graph with the node set $\mathcal{V}$ and edge set defined in such a way that (i, j) is an edge of the composition if there is a node i_1 such that (i, i_1) is an edge in $\mathcal{G}_1$ and meanwhile (i_1, j) is an edge in $\mathcal{G}_2$. Given a sequence of graphs $\{\mathcal{G}(1), \mathcal{G}(2), \ldots, \mathcal{G}(k)\}$, a *route* over it is a sequence of vertices $i_0, i_1, \ldots, i_k$ such that (i_{j-1}, i_j) is an edge in $\mathcal{G}(j)$ for all $1 \leq j \leq k$.

Undirected Graph

An undirected graph is said to be *connected* if there is an undirected path between any pair of nodes. A *complete graph* is a graph in which each node is directly connected to all the other nodes.

Laplacian Matrices and Incidence Matrices

Let $w_{ij} > 0$, $i, j \in \mathcal{V}$, be the weight of the direct edge from i to j in the directed graph $\mathcal{G}$ (if there is no edge between them, $w_{ij} = 0$). The weighted adjacency matrix is defined by $W = [w_{ij}]_{n \times n}$. The degree matrix of this graph is given by $D = \operatorname{diag}(W\mathbf{1}_n)$.The Laplacian matrix of this direct graph is then defined by

$$L = D - W = \operatorname{diag}(W\mathbf{1}_n) - W.$$

If $\mathcal{G}$ is an undirected graph, the Laplacian matrix L is symmetric, i.e., $L^\top = L$. For an undirected graph, the second smallest eigenvalue of L, denoted by $\lambda_2(L)$, is referred to as the *algebraic connectivity* [110].

For a directed graph with edge set $\mathcal{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$, its incidence matrix is an $n \times m$ matrix, denoted by $B = [b_{ij}]_{n \times m}$, whose elements are defined by

$$b_{ip} = \begin{cases} 1, & \text{if } s(\mathcal{E}_p) = i; \\ -1, & \text{if } t(\mathcal{E}_p) = i; \\ 0, & \text{otherwise.} \end{cases}$$

For an undirected graph, its incidence matrix and Laplacian matrix satisfy the equality $L = B\mathcal{W}B^\top$, where $\mathcal{W} \in \mathbb{R}^{m \times m}$ is a diagonal matrix whose elements represent the weights of the edges. We let B_c denote the incidence matrix of a complete graph.

2.3 Stochastic Matrices

A matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is said to be (row) *stochastic* if $a_{ij} \geq 0$ for any i, j, and it satisfies

$$\sum_{j=1}^{n} a_{ij} = 1, \qquad\qquad \forall i = 1, 2, \ldots, n.$$

A stochatic matrix A is said to be *irreducible* if for any pair (i, j), there exists an $m \in \mathbb{N}$ such that $A_{ij}^m > 0$. On the other hand, it is said to be *reducible* if it is not irreducible [71]. A stochastic matrix A is *indecomposable* and *aperiodic* (SIA) if

$$Q = \lim_{k\to\infty} A^k$$

exists and all the rows of Q are identical [68].

A stochastic matrix $A \in \mathbb{R}^{n\times n}$ is said to be: 1) *scrambling* if no two rows are orthogonal; 2) *Markov* if it has a column with all positive elements [71]. If two stochastic matrices A_1 and A_2 have zero elements in the same positions, we say these two matrices are of the same type, denoted by $A_1 \sim A_2$.

Given a stochastic matrix $A \in \mathbb{R}^{n\times n}$, we can associate it with a directed, and weighted graph $\mathcal{G}_A = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} := \{1, \ldots, n\}$ is the set of vertices, and $\mathcal{E}$ is the set of edges. A directed edge $\mathcal{E}_{ij} = (i, j)$ is in the set of $\mathcal{E}$ if $a_{ji} > 0$, and then its weight is a_{ji}.

Part I

Stochastic Distributed Coordination Algorithms:

Stochastic Lyapunov Methods

Overview of Part I

The past few decades have witnessed the fast development of network computational algorithms, in which computational processes are carried out in coupled computational units. The distributed coordination algorithms [111] are a typical type of network algorithms. Units in a network compute individually, but communicate and coordinate locally. They repeatedly update their states (computed results) to the weighted average of their neighbors', seeking for coordination. This type of algorithms are widely applied to many research topics, including distributed optimization [25,26], distributed control of networked robots [112], distributed linear equation solving [29, 30, 113, 114], and opinion dynamics modeling [6, 32, 115, 116].

When applying distributed coordination algorithms, one cannot ignore the fact that the computational processes are usually under inevitable random influences, resulting from random changes of network structures [36, 37, 117, 118], stochastic communication delays [38–40], and random asynchronous updating events [41, 42]. Moreover, some randomness may also be introduced deliberately to improve the global performance in a network [44, 45]. Traditional methods for stability analysis of deterministic systems cannot be directly applied due to the presence of random uncertainty in the system dynamics. Instead, the stochastic Lyapunov theory serves as a powerful tool for the analysis of such stochastic systems. Different from deterministic Lyapunov theory, one needs to evaluate the expectation of a constructed Lyapunov function. For example, if the expectation of a Lyapunov candidate decreases at every time step along the solution to a stochastic discrete-time system, the stability of this system can be shown [65, 66]. However, it is sometimes quite difficult to construct a Lyapunov function using the existing stochastic Lyapunov theory, especially when the systems are influenced by non-Markovian random processes.

The purpose of this part of the thesis is to further develop Lyapunov criteria for stochastic discrete-time systems, and use them to study stochastic distributed coordination algorithms. In Chapter 3, we establish some finite-step stochastic Lyapunov criteria, which enlarge the range of choices of applicable Lyapunov functions for stochastic stability analysis. In Chapter 4 , we show how these new criteria can be applied to the analysis of some stochastic distributed coordination algorithms.

3 New Lyapunov Criteria for Discrete-Time Stochastic Systems

More recently, with the fast development of network algorithms, more and more distributed computational processes are carried out in networks of computational units. Such dynamical processes are usually modeled by stochastic discrete-time dynamical systems since they are usually under inevitable random influences or deliberately randomized to improve performance. So there is a great need to further develop the Lyapunov theory for stochastic dynamical systems, in particular in the setting of network algorithms for distributed computation. And this is exactly the aim of this chapter.

3.1 Introduction

Stability analysis for stochastic dynamical systems has always been an active research field. Early works have shown that stochastic Lyapunov functions play an important role, and to use them for discrete-time systems, a standard procedure is to show that they decrease in *expectation* at every time step [65–67, 119]. Properties of supermartingales and LaSalle's arguments are critical to establishing the related proofs. However, most of the stochastic stability results are built upon a crucial assumption, which requires that the state of a stochastic dynamical system under study is Markovian (see e.g., [64–67]), and very few of them have reported bounds for the convergence speed.

In this chapter, we aim at further developing the Lyapunov criterion for stochastic discrete-time systems in order to solve the problems we encounter in studying distributed coordination algorithms in the next chapter. Inspired by the concept of *finite-step Lyapunov functions* for deterministic systems [120–122], we propose to define a *finite-step stochastic Lyapunov function*, which decreases in expectation, not necessarily at every step, but after a finite number of steps. The associated new Lyapunov criterion not only enlarges the range of choices of candidate Lyapunov functions but also implies that the systems that can be analyzed do not need to have Markovian states. An additional advantage of using this new criterion is that we are enabled to construct conditions to guarantee exponential convergence and estimate convergence rates [102].

Outline

The remainder of this chapter is structured as follows. First, we introduce the system dynamics and formulate the problem in Section 3.2. Main results on finite-step Lyapunov functions are provided in Section 3.3. Finally, some concluding remarks appear in Section 3.4.

3.2 Problem Formulation

Consider a stochastic discrete-time system described by

$$x_{k+1} = f(x_k, y_{k+1}), \qquad k \in \mathbb{N}_0, \tag{3.1}$$

where $x_k \in \mathbb{R}^n$, and $\{y_k : k \in \mathbb{N}\}$ is a $\mathbb{R}^d$-valued stochastic process on a probability space $(\Omega, \mathcal{F}, \Pr)$. Here $\Omega = \{\omega\}$ is the sample space; $\mathcal{F}$ is a set of events which is a σ-field; y_k is a measurable function mapping Ω into the state space $\Omega_0 \subseteq \mathbb{R}^d$, and for any $\omega \in \Omega$, $\{y_k(\omega) : k \in \mathbb{N}\}$ is a realization of the stochastic process $\{y_k\}$ at ω. Let $\mathcal{F}_k = \sigma(y_1, \ldots, y_k)$ for $k \geq 1$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, so that evidently $\{\mathcal{F}_k\}, k = 1, 2, \ldots,$ is an increasing sequence of σ-fields. Following [123], we consider a constant initial condition $x_0 \in \mathbb{R}^n$ with probability one. It then can be observed that the solution to (3.1), $\{x_k\}$, is a $\mathbb{R}^n$-valued stochastic process adapted to $\mathcal{F}_k$. The randomness of y_k can be due to various reasons, e.g., stochastic disturbances or noise. Note that (3.1) becomes a stochastic switching system if $f(x, y) = g_y(x)$, where y maps Ω into the set $\Omega_0 := \{1, \ldots, p\}$, and $\{g_p(x) : \mathbb{R}^n \to \mathbb{R}^n, p \in \Omega_0\}$ is a given family of functions.

A point x^* is said to be an *equilibrium* of system (3.1) if $f(x^*, y) = x^*$ for any $y \in \Omega_0$. Without loss of generality, we assume that the origin $x = 0$ is an equilibrium. Researchers have been interested in studying the limiting behavior of the solution $\{x_k\}$, i.e., when and to where x_k converges as $k \to \infty$. Most noticeably, Kushner developed classic results on stochastic stability by employing stochastic Lyapunov functions [65–67]. We introduce some related definitions before recalling some of Kushner's results. Following [124, Sec. 1.5.6] and [125], we first define convergence and exponential convergence of a sequence of random variables.

Definition 3.1 (Convergence)**.** *A random sequence* $\{x_k \in \mathbb{R}^n\}$ *in a sample space* Ω *converges to a random variable* x almost surely *if* $\Pr\left[\omega \in \Omega : \lim_{k\to\infty} \|x_k(\omega) - x\| = 0\right] = 1$. *The convergence is said to be exponentially fast with a rate no slower than* γ^{-1} *for some* $\gamma > 1$ *independent of* ω *if* $\gamma^k \|x_k - x\|$ *almost surely converges to* y *for some finite* $y \geq 0$. *Furthermore, let* $\mathcal{D} \subset \mathbb{R}^n$ *be a set; a random sequence* $\{x_k\}$ *is said to converge to* $\mathcal{D}$ *almost surely if* $\Pr\left[\omega \in \Omega : \lim_{k\to\infty} \operatorname{dist}(x_k(\omega), \mathcal{D}) = 0\right] = 1$, *where* $\operatorname{dist}(x, \mathcal{D}) := \inf_{y \in \mathcal{D}} \|x - y\|$.

Here "almost surely" is exchangeable with "with probability one", and we sometimes use the shorthand notation "a.s.". We now introduce some stability concepts for stochastic discrete-time systems analogous to those in [64] and [126] for continuous-time systems[1].

Definition 3.2. *The origin of (1) is said to be:*

1) stable in probability *if* $\lim_{x_0 \to 0} \Pr\left[\sup_{k \in \mathbb{N}} \|x_k\| > \varepsilon\right] = 0$ *for any* $\varepsilon > 0$*;*

2) asymptotically stable in probability *if it is stable in probability and moreover* $\lim_{x_0 \to 0} \Pr\left[\lim_{k\to\infty} \|x_k\| = 0\right] = 1$*;*

3) exponentially stable in probability *if for some* $\gamma > 1$ *independent of* ω*, it holds that* $\lim_{x_0 \to 0} \Pr\left[\lim_{k\to\infty} \|\gamma^k x_k\| = 0\right] = 1$*;*

Definition 3.3. *For a set* $\mathcal{Q} \subseteq \mathbb{R}^n$ *containing the origin, the origin of (1) is said to be:*

1) locally a.s. asymptotically stable in $\mathcal{Q}$ (globally a.s. asymptotically stable, respectively) *if a) it is stable in probability, and b) starting from* $x_0 \in \mathcal{Q}$ *(*$x_0 \in \mathbb{R}^n$*, respectively) all the sample paths* x_k *stay in* $\mathcal{Q}$ *(*$\mathbb{R}^n$*, respectively) for all* $k \geq 0$ *and converge to the origin almost surely;*

2) locally a.s. exponentially stable in $\mathcal{Q}$ (globally a.s. exponentially stable, respectively) *if it is locally (globally, respectively) a.s. asymptotically stable and the convergence is exponentially fast.*

Now let us recall some Kushner's results on convergence and stability, where stochastic Lyapunov functions have been used.

Lemma 3.1 (Asymptotic Convergence and Stability [67, 127])**.** *For the stochastic discrete-time system* (3.1)*, let* $\{x_k\}$ *be a Markov process. Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuous positive definite and radially unbounded function. Define the set* $\mathcal{Q}_\lambda := \{x : 0 \leq V(x) < \lambda\}$ *for some* $\lambda > 0$*, and assume that*

$$\mathbb{E}\left[V(x_{k+1}) \,|x_k\right] - V(x_k) \leq -\varphi(x_k), \forall k, \tag{3.2}$$

[1] Note that 1) and 2) of Definition 3.2 follow from the definitions in [64, Chap. 5], in which an arbitrary initial time s rather than just 0 is actually considered; we define 3) following the same lines as 1) and 2). In Definition 3.3, 1) follows from the definitions in [126], and we define 2) following the same lines as 1).

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $\varphi(x) \geq 0$ for any $x \in \mathcal{Q}_\lambda$. Then the following statements apply:

(i) for any initial condition $x_0 \in \mathcal{Q}_\lambda$, x_k converges to $\mathcal{D}_1 := \{x \in \mathcal{Q}_\lambda : \varphi(x) = 0\}$ with probability greater than or equal to $1 - V(x_0)/\lambda$ [67];

(ii) if moreover $\varphi(x)$ is positive definite on $\mathcal{Q}_\lambda$, and $h_1(\|s\|) \leq V(s) \leq h_2(\|s\|)$ for two class $\mathcal{K}$ functions h_1 and h_2, then $x = 0$ is asymptotically stable in probability [67], [127, Theorem 7.3].

Lemma 3.2 (Exponential Convergence and Stability [66, 127])**.** *For the stochastic discrete-time system* (3.1)*, let $\{x_k\}$ be a Markov process. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonnegative function. Assume that*

$$\mathbb{E}\left[V\left(x_{k+1}\right)|x_k\right] - V\left(x_k\right) \leq -\alpha V(x_k), \qquad 0 < \alpha < 1. \tag{3.3}$$

Then the following statements apply:

(i) for any given x_0, $V(x_k)$ almost surely converges to 0 *exponentially fast with a rate no slower than $1 - \alpha$ [66, Th. 2, Chap. 8], [127];*

(ii) if moreover V satisfies $c_1\|x\|^a \leq V(x) \leq c_2\|x\|^a$ for some $c_1, c_2, a > 0$, then $x = 0$ is globally a.s. exponentially stable [127, Theorem 7.4].

To use these two lemmas to prove asymptotic (or exponential) stability for a stochastic system, the critical step is to find a stochastic Lyapunov function such that (3.2) (respectively, (3.3)) holds. However, it is not always obvious how to construct such a stochastic Lyapunov function. We use the following simple but suggestive example to illustrate this point.

Example 3.1 Consider a randomly switching system described by $x_k = A_{y_k} x_{k-1}$, where y_k is the switching signal taking values in a finite set $\mathcal{P} := \{1, 2, 3\}$, and

$$A_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix}.$$

The stochastic process $\{y_k\}$ is described by a Markov chain with initial distribution $v = \{v_1, v_2, v_3\}$. The transition probabilities are described by a transition matrix

$$\pi = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

whose ijth element is defined by $\pi_{ij} = \Pr[y_{k+1} = j | y_k = i]$. Since $\{y_k\}$ is not independent and identically distributed, the process $\{x_k\}$ is not Markovian. Nevertheless, we might conjecture that the origin is globally a.s. exponentially stable. In order to try to prove this, we might choose a stochastic Lyapunov function candidate $V(x) = \|x\|_\infty$, but the existing results introduced in Lemma 3.2 cannot be used since $\{x_k\}$ is not Markovian. Moreover, by calculation we can only observe that $\mathbb{E}\left[V\left(x_{k+1}\right)\middle|\, x_k, y_k\right] \leq V\left(x_k\right)$ for any y_k, which implies that (3.3) is not necessarily satisfied. Thus $V(x)$ is not an appropriate stochastic Lyapunov function for which Lemma 3.2 can be applied. As it turns out however, the same $V(x)$ can be used as a Lyapunov function to establish exponentially stability via the alternative criterion set out subsequently. $\triangle$

It is difficult, if not impossible, to construct a stochastic Lyapunov function, especially when the state of the system is not Markovian. So it is of great interest to generalize the results in Lemmas 3.1 and 3.2 such that the range of choices of candidate Lyapunov functions can be enlarged. For deterministic systems, Aeyels et al. have introduced a new Lyapunov criterion to study asymptotic stability of continuous-time systems [120]; a similar criterion has also been obtained for discrete-time systems, and the Lyapunov functions satisfying this criterion are called *finite-step Lyapunov functions* [121, 122]. A common feature of these works is that the Lyapunov function is required to decrease along the system's solutions after a finite number of steps, but not necessarily at every step. We now use this idea to construct stochastic finite-step Lyapunov functions, a task which is much more challenging compared to the deterministic case due to the uncertainty present in stochastic systems. The tools for analysis are totally different from what are used for deterministic systems. We will exploit supermartingales [109] and their convergence property, as well as another lemma found in [66, P.192]; these concepts are introduced in the two following lemmas.

Lemma 3.3 ([109, Sec. 5.2.9])**.** *Let the sequence $\{X_k\}$ be a nonnegative* supermartingale *with respect to $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$, i.e., suppose: (i) $\mathbb{E}X_n < \infty$; (ii) $X_k \in \mathcal{F}_k$ for all k; (iii) $\mathbb{E}\left(X_{k+1}\middle|\,\mathcal{F}_k\right) \leq X_k$. Then there exists some random X such that $X_k \xrightarrow{a.s.} X, k \to \infty$, and $\mathbb{E}X \leq \mathbb{E}X_0$.*

Lemma 3.4 ([66, P.192])**.** *Let $\{X_k\}$ be a nonnegative random sequence. If $\sum_{k=0}^{\infty} \mathbb{E}X_k < \infty$, then $X_k \xrightarrow{a.s.} 0$.*

Lemma 3.4 is also called Borel-Cantelli Lemma by Kushner in his book [66]. However, it is a bit different from the standard Borel-Cantelli Lemma (see [109, Chap. 2]). We provide a proof of Lemma 3.4 following the ideas in [66], which can be found in Section 3.5.

3.3 Finite-Step Stochastic Lyapunov Criteria

In this subsection, we present some finite-step stochastic Lyapunov criteria for stability analysis of stochastic discrete-time systems, which are the main results in the chapter. In these criteria, the expectation of a Lyapunov function is not required to decrease at every time step, but is allowed to decrease after some finite steps. The relaxation enlarges the range of choices of candidate Lyapunov functions. In addition, these criteria can be used to analyze non-Markovian systems.

Theorem 3.1. *For the stochastic discrete-time system* (3.1)*, let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuous nonnegative and radially unbounded function. Define the set* $\mathcal{Q}_\lambda := \{x : V(x) < \lambda\}$ *for some* $\lambda > 0$*, and assume that*

(a) $\mathbb{E}\left[V\left(x_{k+1}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq 0$ *for any* k *such that* $x_k \in \mathcal{Q}_\lambda$*;*

(b) there is an integer $T \geq 1$*, independent of* ω*, such that for any* k*,*

$$\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\varphi(x_k),$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is continuous and satisfies* $\varphi(x) \geq 0$ *for any* $x \in \mathcal{Q}_\lambda$*.*

Then the following statements apply:

(i) for any initial condition $x_0 \in \mathcal{Q}_\lambda$*,* x_k *converges to* $\mathcal{D}_1 := \{x \in \mathcal{Q}_\lambda : \varphi(x) = 0\}$ *with probability greater than or equal to* $1 - V(x_0)/\lambda$*;*

(ii) if moreover $\varphi(x)$ *is positive definite on* $\mathcal{Q}_\lambda$*, and* $h_1\left(\|s\|\right) \leq V(s) \leq h_2\left(\|s\|\right)$ *for two class* $\mathcal{K}$ *functions* h_1 *and* h_2*, then* $x = 0$ *is asymptotically stable in probability.*

Proof. Before proving (i) and (ii), we first show that starting from $x_0 \in \mathcal{Q}_\lambda$ the sample paths $x_k(\omega)$ stay in $\mathcal{Q}_\lambda$ with probability greater than or equal to $1 - V(x_0)/\lambda$ if Assumption a) is satisfied. This has been proven in [66, p. 196] by showing that

$$\Pr\left[\sup_{k\in\mathbb{N}} V\left(x_k\right) \geq \lambda\right] \leq V\left(x_0\right)/\lambda. \tag{3.4}$$

Let $\bar{\Omega}$ be a subset of the sample space Ω such that for any $\omega \in \bar{\Omega}$, $x_k(\omega) \in \mathcal{Q}_\lambda$ for all k. Let J be the smallest $k \in \mathbb{N}$ (if it exists) such that $V(x_k) \geq \lambda$. Note that, this integer J does not exist when $x_k(\omega)$ stays in $\mathcal{Q}_\lambda$ for all k, i.e., when $\omega \in \bar{\Omega}$.

We first prove (i) by showing that the sample paths staying the $\mathcal{Q}_\lambda$ converge to $\mathcal{D}_1$ with probability one, i.e., $\Pr[x_k \to \mathcal{D}_1|\bar{\Omega}] = 1$. Towards this end, define a new function $\tilde{\varphi}(x)$ such that $\tilde{\varphi}(x) = \varphi(x)$ for $x \in \mathcal{Q}_\lambda$, and $\tilde{\varphi}(x) = 0$ for $x \notin \mathcal{Q}_\lambda$. Define another random process $\{\tilde{z}_k\}$. If J exists, when $J > T$ let

$$\begin{aligned} \tilde{z}_k &= x_k, \qquad && k < J - T, \\ \tilde{z}_k &= \epsilon, && k \geq J - T, \end{aligned}$$

where ϵ satisfies $V(\epsilon) = \tilde{\lambda} > \lambda$; when $J \leq T$, let $\tilde{z}_k = \epsilon$ for any $k \in \mathbb{N}_0$. If J does not exist, we let $\tilde{z}_k = x_k$ for all $k \in \mathbb{N}_0$. Then it is immediately clear that $\mathbb{E}\left[V\left(\tilde{z}_{k+T}\right)|\mathcal{F}_k\right] - V\left(\tilde{z}_k\right) \leq -\tilde{\varphi}(\tilde{z}_k) \leq 0$. By taking the expectation on both sides of this inequality, we obtain

$$\mathbb{E}\left[V\big(\tilde{z}_{k+T}\big)\right] - \mathbb{E}V\big(\tilde{z}_k\big) \leq -\mathbb{E}\tilde{\varphi}\big(\tilde{z}_k\big), k \in \mathbb{N}_0. \tag{3.5}$$

For any $k \in \mathbb{N}$, there is a pair $p, q \in \mathbb{N}_0$ such that $k = pT + q$. From (3.5) one obtains that

$$\mathbb{E}\left[V\big(\tilde{z}_{pT+j}\big)\right] - \mathbb{E}V\big(\tilde{z}_{(p-1)T+j}\big) \leq -\mathbb{E}\tilde{\varphi}\big(\tilde{z}_{(p-1)T+j}\big)$$

holds for all $j = 0, \ldots, q$, and

$$\mathbb{E}\left[V\big(\tilde{z}_{iT+m}\big)\right] - \mathbb{E}V\big(\tilde{z}_{(i-1)T+m}\big) \leq -\mathbb{E}\tilde{\varphi}\big(\tilde{z}_{(i-1)T+m}\big)$$

holds for all $i = 1, \ldots, p-1$ and $m = 0, \ldots, T-1$. By summing up all the left and right sides of these inequalities respectively for all the i, j and m, we have

$$\begin{aligned} &\sum_{m=0}^{T-1} \Big(\mathbb{E}\big[V(\tilde{z}_{(p-1)T+m} - \mathbb{E}V(\tilde{z}_m)\big]\Big) \\ &\qquad + \sum_{j=1}^{q} \Big(\mathbb{E}\big[V(\tilde{z}_{pT+j} - \mathbb{E}V(\tilde{z}_{(p-1)T+j})\big]\Big) \leq -\sum_{i=1}^{k-T} \mathbb{E}\tilde{\varphi}\big(\tilde{z}_i\big). \end{aligned} \tag{3.6}$$

As $V(x)$ is nonnegative for all x, from (3.5) it is easy to observe that the left side of (3.6) is greater than $-\infty$ even when $k \to \infty$ since T and q are finite numbers, which implies that $\sum_{i=0}^{\infty} \mathbb{E}\tilde{\varphi}\big(\tilde{z}_k\big) < \infty$. By Lemma 3.4, ones knows that $\tilde{\varphi}\big(\tilde{z}_k\big) \xrightarrow{a.s.} 0$ as $k \to \infty$. For $\omega \in \bar{\Omega}$, one can observe that $\tilde{\varphi}(x_k(\omega)) = \varphi(x_k(\omega))$ and $\tilde{z}_k\left(\omega\right) = x_k(\omega)$ according to the definitions of $\tilde{\varphi}$ and $\{\tilde{z}_k\}$, respectively. Therefore, $\tilde{\varphi}(\tilde{z}_k(\omega)) = \varphi(x_k(\omega))$ for all $\omega \in \bar{\Omega}$, and subsequently

$$\Pr[\varphi\left(x_k\right) \to 0|\bar{\Omega}] = \Pr[\tilde{\varphi}\left(\tilde{z}_k\right) \to 0|\bar{\Omega}] = 1.$$

From the continuity of $\varphi(x)$ it can be seen that $\Pr[x_k \to \mathcal{D}_1|\bar{\Omega}] = 1$. The proof of (i) is complete since (3.4) means that the sample paths stay in $\mathcal{Q}_\lambda$ with probability greater than or equal to $1 - V(x_0)/\lambda$.

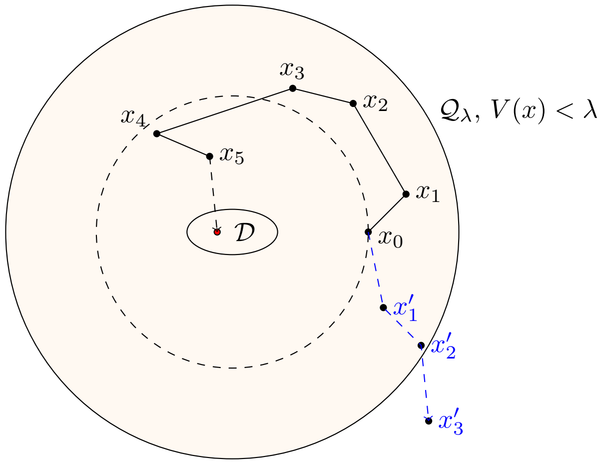


Figure 3.1: An illustration of the asymptotic behavior in $\mathcal{Q}_\lambda$.

Next, we prove (ii) in two steps. We first prove that the origin $x = 0$ is stable in probability. The inequalities $h_1(\|s\|) \leq V(s) \leq h_2(\|s\|)$ imply that $V(x) = 0$ if and only if $x = 0$. Moreover, it follows from $h_1(\|s\|) \leq V(s)$ and the inequality (3.4) that for any initial condition $x_0 \in \mathcal{Q}_\lambda$,

$$\Pr\left[\sup_{k\in\mathbb{N}} h_1(\|x_k\|) \geq \lambda_1\right] \leq \Pr\left[\sup_{k\in\mathbb{N}} V(x_k) \geq \lambda_1\right] \leq \frac{V(x_0)}{\lambda_1}$$

for any $\lambda_1 > 0$. Since h_1 is a class $\mathcal{K}$ function and thus invertible, it can be observed that

$$\Pr\left[\sup_{k\in\mathbb{N}} \|x_k\| \geq h_1^{-1}(\lambda)\right] \leq V(x_0)/\lambda \leq h_2(\|x_0\|)/\lambda.$$

Then for any $\varepsilon > 0$, it holds that

$$\lim_{x_0\to 0} \Pr\left[\sup_{k\in\mathbb{N}} \|x_k\| > \varepsilon\right] \leq \Pr\left[\sup_{k\in\mathbb{N}} \|x_k\| \geq \varepsilon\right] = 0,$$

which means that the origin is stable in probability.

Second, we show the probability that $x_k \to 0$ tends to 1 as $x_0 \to 0$. One knows that $\mathcal{D}_1 = \{0\}$ since φ is positive definite in $\mathcal{Q}_\lambda$. From (i) one knows that x_k converges to $x = 0$ with probability greater than or equal to $1 - V(x_0)/\lambda$. Since $V(x) \to 0$ as $x_0 \to 0$, it holds that $\lim_{x_0\to 0} \Pr[\lim_{k\to\infty} \|x_k\| = 0] \to 1$. The proof is complete. □

With the help of Fig. 3.1, let us provide some explanations on what have been mainly stated in Theorem 3.1. The sample paths x_k always have a possibility to leave the set $\mathcal{Q}_\lambda$, but with probability less than $V(x_0)/\lambda$ (see the blue trajectory $\{x'_k\}$). In other words, they stay in $\mathcal{Q}_\lambda$ with probability no less than $1 - V(x_0)/\lambda$. If $\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\varphi(x_k)$ for a finite positive integer T, all the sample paths remaining in $\mathcal{Q}_\lambda$ will converge to the set $\mathcal{D}_1$ (see the black trajectory $\{x_k\}$). If moreover, $\mathcal{D}_1$ is a singleton $\{0\}$, and $h_1\left(\|s\|\right) \leq V(s) \leq h_2\left(\|s\|\right)$ for two class $\mathcal{K}$ functions h_1 and h_2, then $x = 0$ is asymptotically stable *in probability*.

Particularly, if $\mathcal{Q}_\lambda$ is positively invariant, i.e., starting from $x_0 \in \mathcal{Q}_\lambda$ all sample paths x_k will stay in $\mathcal{Q}_\lambda$ for all $k \geq 0$, this corollary follows from Theorem 3.1 straightforwardly.

Corollary 3.1. *Assume that $\mathcal{Q}_\lambda$ is positively invariant along the system* (3.1)*, and there hold that*

(a) $\mathbb{E}\left[V\left(x_{k+1}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq 0$ *for any k such that $x_k \in \mathcal{Q}_\lambda$;*

(b) there is an integer $T \geq 1$, independent of ω, such that for any k,

$$\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\varphi(x_k),$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $\varphi(x) \geq 0$ for any $x \in \mathcal{Q}_\lambda$.

Then the following statements apply:

(i) for any initial condition $x_0 \in \mathcal{Q}_\lambda$, x_k converges to $\mathcal{D}_1$ with probability one;

(ii) if moreover $\varphi(x)$ is positive definite on $\mathcal{Q}_\lambda$, and $h_1\left(\|s\|\right) \leq V(s) \leq h_2\left(\|s\|\right)$ for two class $\mathcal{K}$ functions h_1 and h_2, then $x = 0$ is locally a.s. asymptotically stable in $\mathcal{Q}_\lambda$. Furthermore, if $\mathcal{Q}_\lambda = \mathbb{R}^n$, then $x = 0$ is globally a.s. asymptotically stable.

Theorem 3.1 and Corollary 3.1 provide some Lyapunov criteria for asymptotic stability and convergence of stochastic discrete-time systems. The next theorem provides a new criterion for exponential convergence and stability of stochastic systems, relaxing the conditions required by Lemma 3.2.

Theorem 3.2. *Suppose that the following conditions are satisfied*

(a) $\mathbb{E}\left[V\left(x_{k+1}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq 0$ *for any k such that $x_k \in \mathcal{Q}_\lambda$;*

(b) there is an integer $T \geq 1$, independent of ω, such that for any k,

$$\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\alpha V(x_k), \qquad 0 < \alpha < 1, \tag{3.7}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $\varphi(x) \geq 0$ for any $x \in \mathcal{Q}_\lambda$.

Then, the following statements apply:

(i) for any given $x_0 \in \mathcal{Q}_\lambda$, $V(x_k)$ converges to 0 exponentially at a rate no slower than $(1-\alpha)^{1/T}$, and x_k converges to $\mathcal{D}_2 := \{x \in \mathcal{Q}_\lambda : V(x) = 0\}$, with probability greater than or equal to $1 - V(x_0)/\lambda$;

(ii) if moreover V satisfies that $c_1\|x\|^a \le V(x) \le c_2\|x\|^a$ for some $c_1, c_2, a > 0$, then $x = 0$ is exponentially stable in probability.

Proof. We first prove (i). From the proof of Theorem 3.1, we know that the sample paths x_k stay in $\mathcal{Q}_\lambda$ with probability greater than or equal to $1 - V(x_0)/\lambda$ for any initial condition $x_0 \in \mathcal{Q}_\lambda$ if the assumption a) is satisfied. We next show that for any sample path that always stays in $\mathcal{Q}_\lambda$, $V(x_k)$ converges to 0 exponentially fast. Towards this end, we define a random process $\{\hat{z}_k\}$. Let J be as defined in the proof of Theorem 3.1. If J exists, when $J > T$, let

$$\begin{aligned} \hat{z}_k &= x_k, && k < J - T, \\ \hat{z}_k &= \varepsilon, && k \ge J - T, \end{aligned}$$

where ε satisfies $V(\varepsilon) = 0$, when $J \le T$, let $\hat{z}_k = \varepsilon$ for any $k \in \mathbb{N}_0$; if J does not exist, we let $\hat{z}_k = x_k$ for all $k \in \mathbb{N}_0$.

If the inequality (3.7) is satisfied, one has $\mathbb{E}\left[V\left(\hat{z}_{k+T}\right)|\mathcal{F}_k\right] - V\left(\hat{z}_k\right) \le -\alpha V(\hat{z}_k)$. Using this inequality, we next show that $V\left(\hat{z}_{k+T}\right)$ converges to 0 exponentially. To this end, define a subsequence

$$Y_m^{(r)} := V(\hat{z}_{mT+r}), \qquad m \in \mathbb{N}_0,$$

for each $0 \le r \le T-1$. Let $\mathcal{G}_m^{(r)} := \sigma(Y_0^{(r)}, Y_1^{(r)}, \dots, Y_m^{(r)})$, and one knows that $\mathcal{G}_m^{(r)}$ is determined if we know $\mathcal{F}_{mT+r}$. It then follows from the inequality (3.7) that for any r, $\mathbb{E}[Y_{m+1}^{(r)}|\mathcal{G}_m^{(r)}] - Y_m^{(r)} \le -\alpha Y_m^{(r)}$. We observe from this inequality that

$$\mathbb{E}\left[(1-\alpha)^{-(m+1)}Y_{m+1}^{(r)}|\mathcal{G}_m^{(r)}\right] - (1-\alpha)^{-m}Y_m^{(r)} \le 0.$$

This means that $(1-\alpha)^{-m}Y_m$ is a supermartingale, and thus there is a finite random number $\bar{Y}^{(r)}$ such that $(1-\alpha)^{-m}Y_m^{(r)} \xrightarrow{a.s.} \bar{Y}^{(r)}$ for any r. Let $\gamma = \sqrt[T]{1/(1-\alpha)}$, and then by the definition of $Y_m^{(r)}$ we have

$$\gamma^{mT}V(\hat{z}_{mT+r}) \xrightarrow{a.s.} \bar{Y}^{(r)}.$$

Straightforwardly, it follows that $\gamma^{mT+r}V(\hat{z}_{mT+r}) \xrightarrow{a.s.} \gamma^r\bar{Y}^{(r)}$. Let $k = mT + r, \bar{Y} = \max_r\{\gamma^r\bar{Y}^{(r)}\}$, then it almost surely holds that $\lim_{k\to\infty}\gamma^k V(\hat{z}_k) \le \bar{Y}$. From Definition 3.1, one concludes that $V(\hat{z}_k)$ almost surely converges to 0 exponentially no slower than $\gamma^{-1} = (1-\alpha)^{1/T}$. From the definition of $\hat{z}_k$, we know that $V(\hat{z}_k(\omega)) = V(x_k(\omega))$ for all $\omega \in \bar{\Omega}$, with $\bar{\Omega}$ defined in the proof of Theorem 3.1. Consequently, it holds that

$$\Pr\left[\lim_{k\to\infty} \gamma^k V(x_k) \leq \bar{Y} | \bar{\Omega}\right] = \Pr\left[\lim_{k\to\infty} \gamma^k V(\hat{z}_k) \leq \bar{Y} | \bar{\Omega}\right] = 1. \tag{3.8}$$

The proof of (i) is complete since the sample paths stay in $\mathcal{Q}_\lambda$ with probability greater than or equal to $1 - V(x_0)/\lambda$.

Next, we prove (ii). If the inequalities $c_1\|x\|^a \leq V(x) \leq c_2\|x\|^a$ are satisfied, and then we know that $V(x) = 0$ if and only if $x = 0$. Moreover, it follows from (3.8) that for all the sample paths that stay in $\mathcal{Q}_\lambda$ it holds that $c_1\gamma^k\|x\|^a \leq \gamma^k V(x_k) \leq \bar{Y}$ since $c_1\|x_k\|^a \leq V(x)$. Hence,

$$\|x_k(\omega)\| \leq \left(\bar{V}/c_1\right)^{1/a} \gamma^{-k/a}, \qquad \forall \omega \in \bar{\Omega},$$

and one can check that this inequality holds with probability greater than or equal to $1 - V(x_0)/\lambda$. If $x_0 \to 0$, we know that $1 - V(x_0)/\lambda \to 1$, which completes the proof. $\square$

If $\mathcal{Q}_\lambda$ is positively invariant, the following corollary follows straightforwardly.

Corollary 3.2. *Suppose that $\mathcal{Q}_\lambda$ is positively invariant along the system* (3.1)*, and the following conditions are satisfied*

a) $\mathbb{E}\left[V\left(x_{k+1}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq 0$ *for any k such that $x_k \in \mathcal{Q}_\lambda$;*

b) *there is an integer $T \geq 1$, independent of ω, such that for any k,*

$$\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\alpha V(x_k), \qquad 0 < \alpha < 1, \tag{3.9}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $\varphi(x) \geq 0$ for any $x \in \mathcal{Q}_\lambda$.

Then, the following statements apply:

(i) for any given $x_0 \in \mathcal{Q}_\lambda$, $V(x_k)$ converges to 0 exponentially no slower than $(1-\alpha)^{1/T}$ with probability one;

(ii) if moreover V satisfies that $c_1\|x\|^a \leq V(x) \leq c_2\|x\|^a$ for some $c_1, c_2, a > 0$, then $x = 0$ is locally a.s. exponentially stable in $\mathcal{Q}_\lambda$. Furthermore, if $\mathcal{Q}_\lambda = \mathbb{R}^n$, then $x = 0$ is globally a.s. exponentially stable.

The following corollary, which can be proven following the same lines as Theorems 3.1 and 3.2, shares some similarities to LaSalle's theorem for deterministic systems. It is worth mentioning that the function V here does not have to be radially unbounded.

Corollary 3.3. *Let $\mathbb{D} \subset \mathbb{R}^n$ be a compact set that is positively invariant along the system* (3.1)*. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonnegative function, and $\bar{\mathcal{Q}}_\lambda := \{x \in \mathbb{D} : V(x) < \lambda\}$ for some $\lambda > 0$. Assume that $\mathbb{E}\left[V\left(x_{k+1}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq 0$ for all k such that $x_k \in \bar{\mathcal{Q}}_\lambda$, then*

(i) if there is an integer $T \geq 1$, independent of ω, such that for any $k \in \mathbb{N}_0$, $\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\varphi(x_k)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $\varphi(x) \geq 0$ for any $x \in \bar{\mathcal{Q}}_\lambda$, then for any initial condition $x_0 \in \bar{\mathcal{Q}}_\lambda$, x_k converges to $\bar{\mathcal{D}}_1 := \{x \in \bar{\mathcal{Q}}_\lambda : \varphi(x) = 0\}$ with probability greater than or equal to $1 - V(x_0)/\lambda$;

(ii) if the inequality in a) is strengthened to $\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\alpha V(x_k)$ for some $0 < \alpha < 1$, then for any given $x_0 \in \bar{\mathcal{Q}}_\lambda$, $V(x_k)$ converges to 0 exponentially at a rate no slower than $(1-\alpha)^{1/T}$, and x_k converges to $\bar{\mathcal{D}}_2 := \{x \in \bar{\mathcal{Q}}_\lambda : V(x) = 0\}$, with probability greater than or equal to $1 - V(x_0)/\lambda$;

(iii) if $\bar{\mathcal{Q}}_\lambda$ is positively invariant w.r.t the system (3.1)*, then all the convergence in both (i) and (ii) takes place* almost surely*.*

Continuation of Example 3.1 Now let us look back at Example 1 and still choose $V(x) = \|x\|_\infty$ as a stochastic Lyapunov function candidate. It is easy to see that $V(x)$ is a nonnegative supermartingale. To show the stochastic convergence, let $T = 2$ and one can calculate the conditional expectations

$$\begin{aligned}
&\mathbb{E}\left[V\left(x_{k+T}\right)|\, x_k, y_k = 1\right] - V\left(x_k\right) \\
&= 0.5\left\| \begin{array}{c} 0.2x_k^1 \\ 0.8x_k^2 \end{array} \right\|_\infty + 0.5\left\| \begin{array}{c} 0.2x_k^1 \\ 0.6x_k^2 \end{array} \right\|_\infty - \left\| \begin{array}{c} x_k^1 \\ x_k^2 \end{array} \right\|_\infty \\
&\leq -0.3V\left(x_k\right), \forall x_k \in \mathbb{R}^2.
\end{aligned}$$

When $y_k = 2, 3$, it analogously holds that

$$\mathbb{E}[V\left(x_{k+T}\right)|\, x_k, y_k] - V\left(x_k\right) \leq -0.3V(x_k), \forall x_k \in \mathbb{R}^2.$$

From these three inequalities one can observe that starting from any initial condition x_0, $\mathbb{E}V(x)$ decreases at an exponential speed after every two steps before it reaches 0. By Corollary 3.2, one knows that origin is globally a.s. exponentially stable, consistent with our conjecture. △

Kushner and other researchers have used more restricted conditions to construct Lyapunov functions than those appearing in our results to analyze asymptotic or exponential stability of random processes [66, 67, 119]. It is required that $\mathbb{E}[V(x_k)]$ decreases strictly at every step, until $V(x_k)$ reaches a limit value. However, in our result, this requirement is relaxed. In addition, Kushner's results rely on the assumption that the underlying random process is Markovian, but we work with more general random processes.

3.4 Concluding Remarks

Many distributed coordination algorithms are stochastic since they are often under inevitable random influences, or randomness is deliberately introduced into them to improve global performance. Stochastic Lyapunov theory is often needed to study them. However, it is not always easy to construct a stochastic Lyapunov function using the existing criteria. In this chapter, we have further developed a tool, termed finite-step stochastic Lyapunov criteria, using which one can study the convergence and stability of a stochastic discrete-time system together with its convergence rate. Unlike what is required in the existing Lyapunov criteria [65–67, 119], the constructed Lyapunov function does not have to decrease after every time step. Instead, decreasing after some finite time steps is sufficient to guarantee the asymptotic or exponential convergence and stability of a system, which makes the construction of a Lyapunov function easier. In addition, the states of a system under study do not have to be Markovian. The tool we developed in this chapter plays a very important role in studying some stochastic coordination algorithms, which we will discuss in more detail in the next chapter.

3.5 Appendix: Proof of Lemma 3.4

We first provide two lemmas and a definition that will be used in the proof.

Lemma 3.5 (Borel-Cantelli lemma [109, Chap. 2]). *Let $\{A_k\}$ be a sequence of events in some probability space. If the sum of the probabilities of the A_k is finite*

$$\sum_{k=1}^{\infty} \Pr[A_k] < \infty,$$

then the probability that infinitely many of them occur is 0, that is,

$$\Pr\left(\limsup_{k\to\infty} A_k\right) = 0.$$

Lemma 3.6 (Markov's inequality [109, Chap. 1]). *If X is a nonnegative random variable and $a > 0$, then $\Pr[X \geq a] \leq \mathbb{E}X/a$.*

Definition 3.4. *We say the nonnegative sequence X_k converges to* 0 almost surely *if*

$$\Pr\left[\liminf_{k\to\infty} X_k < \epsilon\right] = 1, \forall\epsilon > 0.$$

Note that this definition is equivalent to the almost sure convergence defined in Definition 3.1 in Section 3.2. We are now ready to provide the proof of Lemma 3.4.

Proof of Lemma 3.4. We complete the proof in two steps. First, we show

$$\sum_{k=1}^{\infty} \Pr\left[X_k \geq \epsilon\right] < \infty, \forall\epsilon \qquad \Rightarrow \qquad X_k \xrightarrow{a.s.} 0.$$

Second, we prove

$$\sum_{k=1}^{\infty} \mathbb{E}[X_k] < \infty \qquad \Rightarrow \qquad \sum_{k=1}^{\infty} \Pr\left[X_k \geq \epsilon\right] < \infty, \forall\epsilon.$$

Let us start with the first step. From the Borel-Cantelli lemma, one knows that if $\sum_{k=1}^{\infty} \Pr\left[X_k \geq \epsilon\right] < \infty$ for all $\epsilon > 0$, then

$$\Pr\left[\limsup_{k\to\infty}(X_k \geq \epsilon)\right] = 0, \forall\epsilon.$$

Let A^c denote the complementary of the event A. Using the property that

$$\left(\limsup_{k\to\infty}(X_k \geq \epsilon)\right)^c = \liminf_{k\to\infty}(X_k < \epsilon),$$

we have

$$\Pr\left[\liminf_{k\to\infty}(X_k < \epsilon)\right] = 1 - \Pr\left[\limsup_{k\to\infty}(X_k \geq \epsilon)\right] = 1, \forall\epsilon > 0.$$

Then one can say that $X_k \xrightarrow{a.s.} 0$.

We finally use the Markov's inequality to show the second step. Using the lemma, we know that $\mathbb{E}X_k \geq \epsilon \Pr[X_k \geq \epsilon]$ for any $\epsilon > 0$. Then there holds that

$$\epsilon \sum_{n=1}^{\infty} \Pr[X_k \geq \epsilon] \leq \sum_{k=1}^{\infty} \mathbb{E}[X_k] < \infty,$$

which implies that $\sum_{n=1}^{\infty} \Pr[X_k \geq \epsilon] < \infty$ for any $\epsilon > 0$. The proof is complete. $\square$

4 Stochastic Distributed Coordination Algorithms

In this chapter, we deal with several stochastic distributed coordination algorithms, which is the central aim of Part I. The new stochastic Lyapunov criteria developed in Chapter 3 will be used to prove the convergence of these stochastic algorithms.

4.1 Introduction

Distributed coordination algorithms, known as distributed weighted averaging algorithms, have been playing crucial roles in various distributed systems and algorithms, including distributed optimization [25, 26], distributed control of networked robots [112], opinion dynamics [6, 32, 115, 116], and many other distributed algorithms [8, 9, 9–11, 35, 36]. In order to analyze such systems and algorithms, one frequently encounters the need to prove convergence of inhomogeneous Markov chains, or equivalently the convergence of backward products of random sequences of stochastic matrices $\{W(k)\}$. Most of the existing results assume exclusively that all the $W(k)$ in the sequence have all positive diagonal entries, see e.g., [73, 128, 129]. This assumption simplifies the analysis of convergence significantly; moreover, without this assumption, the existing results do not always hold. For example, from [35, 36] one knows that the product of $W(k)$ converges to a rank-one matrix almost surely if exactly one of the eigenvalues of the expectation of $W(k)$ has the modulus of one, which can be violated if $W(k)$ has zero diagonal elements. Note also that most of the existing results are confined to special random sequences, e.g., independently distributed sequences [35], stationary ergodic sequences [36], or independent sequences [75, 76]. In the first part of this chapter, we work on more general classes of random sequences of stochastic matrices without the assumption of non-zero diagonal entries. Using the novel Lyapunov criteria we developed in Chapter 3, we show that if there exists a fixed length such that the product of any successive subsequence of matrices of this length has the *scrambling* property (see the definition in Section 2.3) with positive probability, the convergence to a rank-one matrix for the infinite product can be guaranteed almost surely. We also prove that the convergence can be exponentially fast if this probability is lower bounded by some positive number, and the greater the lower bound is, the faster the convergence becomes. For some particular random sequences, we further relax this "scrambling" condition. If the random sequence is driven by a *stationary* process, the almost sure convergence can be ensured as long as the product of any successive subsequence of finite length has a positive probability to be indecomposable and aperiodic (SIA). The exponential convergence rate follows without other assumptions if the random process that governs the evolution of the sequence is a *stationary ergodic* process.

Using these results on products of random stochastic matrices, we then investigate a classic agreement problem, in which agents coupled by a network repeatedly update their states to the weighted average of their neighbors' states and their own. This problem is usually modeled by a linear recursion equation $x(k) = Wx(k-1)$ with W a stochastic matrix describing the interaction structure. The agreement problem is equivalent to studying whether W^k converge to a rank-one matrix. Usually, W is required to be indecomposable and aperiodic matrix (SIA) [68, 71]. However, the case when W is not an SIA matrix has not been studied before. For example, a periodic W leads to oscillating behaviors. We address the agreement problem when W is periodic in Section 4.3. We show that, instead of oscillation, agreement takes place if the agents update asynchronously. Specifically, we assume that each agent has access to its own state while executing averaging actions at every time instant. In other words, at each time step, a random number of agents are activated and then update. In sharp contrast to the existing works, e.g. [130, 131] and [129], agents do not need to use their own states to update. The obtained results reveal that asynchrony can play a very important role in giving rise to an agreement.

We then investigate another distributed coordination algorithm for solving linear algebraic equations of the form $Ax = b$, as another application of the finite-step stochastic Lyapunov criteria in Chapter 3. The problem is to design a distributed algorithm such that the equations are solved in parallel by n agents, each of whom just knows a subset of the rows of the matrix $[A, b]$. Each agent recursively updates its estimate of the solution using the current estimates from its neighbors. Recently several solutions under different sufficient conditions have been proposed [29, 30, 77], and particularly in [77], the sequence of the neighbor relationship graphs $\mathcal{G}(k)$ is required to be repeated jointly strongly connected. We show that a much weaker condition is sufficient to solve the problem almost surely, namely the algorithm in [77] works if there exists a fixed length such that any subsequence of $\{\mathcal{G}(k)\}$ at this length is jointly strongly connected with positive probability. The proof also relies on the new Lyapunov criteria we developed in the previous section.

Outline

The remainder of this chapter is structured as follows. Products of random sequences of stochastic matrices are studied in Section 4.2. We investigate asynchronous updating induced agreement problem in Section 4.3. A distributed algorithm to solve linear equation is studied in Section 4.4. Concluding remarks appear in Section 4.5.

4.2 Products of Random Sequences of Stochastic Matrices

In this section, we study the convergence of products of stochastic matrices, where the obtained results on finite-step Lyapunov functions are used for analysis. Let $\Omega_0 := \{1, 2, \dots, m\}$ be the state space and $\mathcal{M} := \{F_1, F_2, \dots, F_m\}$ be the set of m stochastic matrices $F_i \in \mathbb{R}^{n\times n}$. Consider a random sequence $\{W_\omega(k) : k \in \mathbb{N}\}$ on the probability space $(\Omega, \mathcal{F}, \Pr)$, where Ω is the collection of all infinite sequences $\omega = (\omega_1, \omega_2, \dots)$ with $\omega_k \in \Omega_0$, and we define $W_\omega(k) := F_{\omega_k}$. For notational simplicity, we denote $W_\omega(k)$ by $W(k)$. For the backward product of stochastic matrices

$$W(t + k, t) = W(t + k) \cdots W(t + 1), \tag{4.1}$$

where $k \in \mathbb{N}, t \in \mathbb{N}_0$, we are interested in establishing conditions on $\{W(k)\}$, under which it holds that $\lim_{k\to\infty} W(k, 0) = L$ for a random matrix $L = \mathbf{1}\xi^\top$ where $\xi \in \mathbb{R}^n$ satisfies $\xi^\top\mathbf{1} = 1$.

Before proceeding, let us introduce some concepts in probability. Let $\mathcal{F}_k = \sigma(W(1), \dots, W(k))$, so that evidently $\{\mathcal{F}_k\}$, $k = 1, 2, \dots$, is an increasing sequence of σ-fields. Let $\chi : \Omega \to \Omega$ be the shift operator, i.e., $\chi(\omega_1, \omega_2, \dots) = (\omega_2, \omega_3, \dots)$. A random sequence of stochastic matrices $\{W(1), W(2), \dots, W(k), \dots\}$ is said to be *stationary* if the shift operator is measure-preserving. In other words, for any $k_1, k_2, \dots, k_r$ and $\tau \in \mathbb{N}$, the sequence

$$\{W(k_1 + \tau), W(k_2 + \tau), \dots, W(k_r + \tau)\}$$

has the same joint distribution as $\{W(k_1), W(k_2), \dots, W(k_r)\}$. Moreover, a sequence is said to be *stationary ergodic* if it is stationary, and every invariant set $\mathcal{B}$ is trivial, i.e., for every $A \in \mathcal{B}$, $\Pr[A] \in \{0, 1\}$. Here by an invariant set $\mathcal{B}$, we mean $\chi^{-1}\mathcal{B} = \mathcal{B}$.

4.2.1 Convergence Results

In this subsection, we provide some sufficient conditions such that the backward product of the sequence $\{W(k)\}$ converges to a rank one matrix.

We first recall three classes of stochastic matrices defined in Section 2.3, denoted by $\mathcal{M}_1, \mathcal{M}_2$, and $\mathcal{M}_3$, respectively. Give a stochastic matric $A \in \mathbb{R}^{n\times n}$, we say $A \in \mathcal{M}_1$ if A is SIA (stochastic, indecomposable, and aperiodic); $A \in \mathcal{M}_2$ if A is scrambling; and $A \in \mathcal{M}_3$ if A is Markov.

Coefficients of ergodicity serve as a fundamental tool in analyzing the convergence of products of stochastic matrices. In this chapter, we employ a standard one. For a stochastic matrix $A \in \mathbb{R}^{n\times n}$, the coefficient of ergodicity $\tau(A)$ is defined by

$$\tau\left(A\right) = 1 - \min_{i,j} \sum\nolimits_{s=1}^{n} \min(a_{is}, a_{js}). \tag{4.2}$$

It is known that this coefficient of ergodicity satisfies $0 \le \tau(A) \le 1$, and $\tau(A)$ is proper since $\tau(A) = 0$ if and only if all the rows of A are identical. Importantly, it holds that

$$\tau(A) < 1 \tag{4.3}$$

if and only if $A \in \mathcal{M}_2$ (see [71, p.82]). For any two stochastic matrices A, B, there is an important property for this coefficient of ergodicity

$$\tau(AB) \le \tau(A)\tau(B). \tag{4.4}$$

This property will be used also in the proof in Section 4.6. Before providing our first results in this subsection, we make the following assumption for the random sequence $\{W(k)\}$.

Assumption 4.1. *Suppose the sequence of stochastic matrices $\{W(k) : k \in \mathbb{N}\}$ is driven by a random process satisfying the following conditions:*

a) There exists an integer $h > 0$ such that for any $k \in \mathbb{N}_0$, it holds that

$$\Pr\left[W(k+h,k) \in \mathcal{M}_2\right] > 0, \tag{4.5}$$

$$\sum_{i=1}^{\infty} \Pr\left[W\big(k+ih, k+(i-1)\,h\big) \in \mathcal{M}_2\right] = \infty; \tag{4.6}$$

b) There is a positive number α such that $W_{ij}(k) \ge \alpha$ for any $i, j \in \mathbf{N}, k \in \mathbb{N}_0$ satisfying $W_{ij}(k) > 0$.

In other words, Assumption 4.1 requires that any corresponding matrix product of length h becomes a scrambling matrix with positive probability, and the positive elements for any matrix in $\mathcal{M}$ are uniformly lower bounded away from some positive value. Now we are ready to provide our main result on the convergence of stochastic matrices' products.

Theorem 4.1. *Under Assumption 4.1, the product of the random sequence of stochastic matrices $W(k,0)$ converges to a random matrix $L = \mathbf{1}\xi^\top$ almost surely.*

To prove Theorem 4.1, consider the stochastic discrete-time dynamical system described by

$$x_{k+1} = F_{y(k+1)}x_k := W(k+1)x_k, \qquad k \in \mathbb{N}_0, \tag{4.7}$$

where $x_k \in \mathbb{R}^n$; the initial state x_0 is a constant with probability one; $y(k) \in \{1, \dots, m\}$ is regarded as the randomly switching signal; and $\{W(1), W(2), \dots\}$ is the random process of stochastic matrices we are interested in. One knows that x_k is adapted to $\mathcal{F}_k$. Thus, to investigate the limiting behavior of the product (4.1), it is sufficient to study the limiting behavior of system dynamics (4.7). We say the state of system (4.7) reaches an *agreement* state if $\lim_{k\to\infty} x_k = \mathbf{1}\xi$ for some $\xi \in \mathbb{R}$. Then, from [75] one can say that the agreement of system (4.7) for any initial state x_0 implies that $W(k,0)$ converges to a rank-one matrix as $k \to \infty$.

To investigate the agreement problem, we define

$$\lceil x_k \rceil := \max_{i\in\mathbf{N}} x_k^i, \qquad \lfloor x_k \rfloor := \min_{i\in\mathbf{N}} x_k^i$$

where $k \in \mathbb{N}_0$, and

$$v_k = \lceil x_k \rceil - \lfloor x_k \rfloor . \tag{4.8}$$

For any $k \in \mathbb{N}$, v_k is adapted to $\mathcal{F}_k$ since x_k is. The agreement is said to be reached asymptotically almost surely if $v_k \xrightarrow{a.s.} 0$ as $k \to \infty$; and it is said to be reached exponentially almost surely with convergence rate no slower than γ^{-1} for some $\gamma > 1$ if $\gamma^k v_k \xrightarrow{a.s.} y$ for some finite $y \geq 0$. The random variable v_k has some important properties given by the following proposition.

Proposition 4.1. *Consider a system $x_{k+1} = Ax_k, k \in \mathbb{N}_0$, where A is a stochastic matrix. For v_k defined in* (4.8)*, it follows that $v_{k+1} \leq v_k$, and the strict inequality holds for any $x_k \notin \operatorname{span}(\mathbf{1})$* if and only if *$A$ is scrambling.*

Proof. It is shown in [71] that $v_{k+1} \leq \tau(A)v_k$ with $\tau(\cdot)$ defined in (4.2). Therefore, the sufficiency follows from (4.3) straightforwardly. We then prove the necessity by contradiction. Suppose A is not scrambling, and then there must exist at least two rows, denoted by i, j, that are orthogonal. Define the two sets $\mathbf{i} := \{l : a_{il} > 0, l \in \mathbf{N}\}$ and $\mathbf{j} := \{m : a_{jm} > 0, m \in \mathbf{N}\}$, respectively. It follows then from the scrambling property that $\mathbf{i} \cap \mathbf{j} = \emptyset$. Let $x_k^q = 1$ for all $q \in \mathbf{i}$, $x_k^q = 0$ for all $q \in \mathbf{j}$, and let x_k^m be any arbitrary positive number less than 1 for all $m \in \mathbf{N}\backslash(\mathbf{i} \cup \mathbf{j})$ if $\mathbf{N}\backslash(\mathbf{i} \cup \mathbf{j})$ is not empty. Then the states of i and j at time $k+1$ become

$$x_{k+1}^{i} = \sum\nolimits_{l=1}^{n} a_{il} x_k^l = \sum\nolimits_{l \in \mathbf{i}} a_{il} x_k^l = 1,$$
$$x_{k+1}^{j} = \sum\nolimits_{l=1}^{n} a_{jl} x_k^l = \sum\nolimits_{l \in \mathbf{j}} a_{jl} x_k^l = 0,$$

and $0 \leq x_{k+1}^m \leq 1$ for all $m \in \mathbf{N}\backslash(i \cup j)$. This results in $v_{k+1} = v_k = 1$. By contradiction one knows that a scrambling A is necessary for $v_{k+1} < v_k$, which completes the proof. □

In order to prove Theorem 4.1, we obtain the following intermediate result.

Proposition 4.2. *For any scrambling matrix $A \in \mathbb{R}^{n \times n}$, the coefficient of ergodicity $\tau(A)$ defined in* (4.2) *satisfies*

$$\tau(A) \leq 1 - \gamma$$

if all the positive elements of A are lower bounded by $\gamma > 0$.

Proof. Consider any two rows of A, denoted by i, j. Define two sets, $\mathbf{i} := \{s : a_{is} > 0\}$ and $\mathbf{j} := \{s : a_{js} > 0\}$. From the scrambling hypothesis, one knows that $\mathbf{i} \cap \mathbf{j} \neq \emptyset$. Thus it holds that

$$\sum_{s=1}^{n} \min\left(a_{is}, a_{js}\right) = \sum_{s \in \mathbf{i} \cap \mathbf{j}} \min\left(a_{is}, a_{js}\right) \geq \gamma.$$

Then from the definition of $\tau(A)$, it is easy to see

$$\tau\left(A\right) = 1 - \min_{i,j} \sum_{s=1}^{n} \min\left(a_{is}, a_{js}\right) \leq 1 - \gamma,$$

which completes the proof. □

We are in the position to prove Theorem 4.1 by showing that $v_k \xrightarrow{a.s.} 0$ as $k \to \infty$, where the results obtained in Corollary 3.3 will be used.

Proof of Theorem 4.1. Let $V(x_k) = v_k$ be a finite-step stochastic Lyapunov function candidate for the system dynamics (4.7). It is easy to see $V(x) = 0$ if and only if $x \in \text{span}(\mathbf{1})$. Since all $W(k)$ are stochastic matrices, we observe that

$$\mathbb{E}[V(x_{k+1})|\mathcal{F}_k] - V(x_k) \leq 0$$

from Proposition 4.1, which implies that $V(x_k)$ is exactly a supermartingale with respect to $\mathcal{F}_k$. From Lemma 3.3, we know $V(x_k) \xrightarrow{a.s.} \bar{V}$ for some $\bar{V}$ because $V(x_k) \geq 0$

and $\mathbb{E}V(x_k) < \infty$. From Assumption 4.1, we know that there is an h such that the product $W(k+h,k)$ is scrambling with positive probability for any k. Let $\mathcal{W}_k$ be the set of all possible $W(k+h,k)$ at time k, and n_k the cardinality of $\mathcal{W}_k$. Let n_k^s be the number of scrambling matrices in $\mathcal{W}_k$. We denote each of these scrambling matrices and each of non-scrambling matrices by $S_k^i, i=1,\dots,n_k^s$ and $\bar{S}_k^j, j=1,\dots,n_k-n_k^s$, respectively. The probabilities of all the possible $W(k+h,k)$ sum to 1, i.e.,

$$\sum_{i=1}^{n_k^s} \Pr\left[S_k^i\right] + \sum_{j=1}^{n_k-n_k^s} \Pr\left[\bar{S}_k^j\right] = 1. \tag{4.9}$$

Then the conditional expectation of $V(x)$ after finite steps for any k becomes

$$\begin{aligned}
&\mathbb{E}\left[V\left(x_{k+h}\right)\middle|\mathcal{F}_k\right] - V\left(x_k\right)\\
&\qquad = \mathbb{E}\left[V\big(W\left(k+h,k\right)x_k\big)\right] - V\left(x_k\right)\\
&\qquad \le \mathbb{E}\left[\tau\big(W\left(k+h,k\right)\big)\right] V\big(x_k\big) - V\left(x_k\right),
\end{aligned}$$

where $\tau(\cdot)$ is given by (4.2). One can calculate that

$$\begin{aligned}
&\mathbb{E}\left[\tau\Big(W\left(k+h,k\right)\Big)\right] - 1\\
&= \sum\nolimits_{i=1}^{n_k^s} \Pr\left[S_k^i\right]\tau\left(S_k^i\right) + \sum\nolimits_{j=1}^{n_k-n_k^s} \Pr\left[\bar{S}_k^j\right]\tau\left(\bar{S}_k^j\right) - 1\\
&\le \sum\nolimits_{i=1}^{n_k^s} \Pr\left[S_k^i\right]\Big(\tau\left(S_k^i\right) - 1\Big),
\end{aligned}$$

where Proposition 4.1 and equation (4.9) have been used. From Assumption 4.1.b), we know that the positive elements of $W(k)$ are lower bounded by α, and thus the positive elements of S_k^i in (4.10) are lower bounded by α^h. Thus $\tau(S_k^i) \le 1-\alpha^h$ according to Proposition 4.2, and it follows that

$$\begin{aligned}
&\mathbb{E}[V\left(x_{k+h}\right)|\,\mathcal{F}_k] - V\left(x_k\right)\\
&\qquad \le -\sum\nolimits_{i=1}^{n_k^s} \Pr\left[S_k^i\right]\alpha^h \mathbb{E}V(x_k) := \varphi_k\left(x_k\right).
\end{aligned} \tag{4.10}$$

By iterating, one can easily show that

$$\begin{aligned}
\mathbb{E}\left[V\left(x_{nh}\right)\right] - V\left(x_0\right) &\le -\sum\nolimits_{k=0}^{n-1} \varphi_k\left(x_k\right)\\
&= -\sum\nolimits_{k=0}^{n-1}\sum\nolimits_{i=1}^{n_k^s} \Pr\left[S_k^i\right]\alpha^h \mathbb{E}V(x_k).
\end{aligned} \tag{4.11}$$

It then follows that $V\left(x_0\right) - \mathbb{E}\left[V\left(x_{nh}\right)\right] < \infty$ even when $n \to \infty$, since $V(x) \ge 0$. According to the condition (4.6), we know $\sum_{k=0}^{n-1}\sum_{i=1}^{n_k^s} \Pr\left[S_k^i\right] = \infty$. By contradiction, it is easy to infer that $\mathbb{E}V(x_k) \xrightarrow{a.s.} 0$. Since we have already shown that $V(x_k) \xrightarrow{a.s.} \bar{V}$

for some random $\bar{V} \geq 0$, one can conclude that $V(x_k) \xrightarrow{a.s.} 0$. For any given $x_0 \in \mathbb{R}^n$, define the compact set $\mathcal{Q} := \{x : \lceil x \rceil \leq \lceil x_0 \rceil, \lfloor x \rfloor \geq \lfloor x_0 \rfloor$. For any random sequence $\{W(k)\}$, it follows from the system dynamics (4.7) that

$$\lceil x_k \rceil \leq \lceil x_{k-1} \rceil \leq \cdots \leq \lceil x_1 \rceil \leq \lceil x_0 \rceil,$$
$$\lfloor x_k \rfloor \geq \lfloor x_{k-1} \rfloor \geq \cdots \geq \lfloor x_1 \rfloor \geq \lfloor x_0 \rfloor,$$

and thus x_k will remain within $\mathcal{Q}$. From Corollary 3.3, we know that x_k asymptotically converges to $\{x \in \mathcal{Q} : \varphi_k(x) = 0\}$, or equivalently, $\{x \in \mathcal{Q} : V(x) = 0\}$ almost surely as $k \to \infty$ since $V(x)$ is continuous. In other words, for any $x_0 \in \mathbb{R}^n$, $x_k \xrightarrow{a.s.} \zeta\mathbf{1}$ for some $\zeta \in \mathbb{R}$, which proves Theorem 4.1. $\square$

Compared to the existing results, Theorem 4.1 has provided a quite relaxed condition for the convergence of the backward product (4.1) determined by the random sequence $\{W(k)\}$ to a rank-one matrix: over any time interval of length h, i.e., $[h+k, k]$ for any $k \in \mathbb{N}_0$, the product $W(k+h) \cdots W(k+1)$ has positive probability to be scrambling. The following corollary follows straightforwardly since a Markov matrix is certainly scrambling.

Corollary 4.1. *For a random sequence $\{W(k) : k \in \mathbb{N}\}$, the product* (4.1) *converges to a random matrix $L = \mathbf{1}\xi^\top$ almost surely if there exists an integer h such that for any k the product $W(k+h, k)$ is a Markov matrix with positive probability and*

$$\sum_{i=1}^{\infty} \Pr\left[W\left(k+ih, k+(i-1)\,h\right) \in \mathcal{M}_3\right] = \infty.$$

Next we assume that the sequence $\{W(k)\}$ is driven by an underlying *stationary* process. Then the condition in Theorem 4.1 can be further relaxed. Let us make the following assumption and provide another theorem in this subsection.

Assumption 4.2. *Suppose the random sequence of stochastic matrices $\{W(k) : k \in \mathbb{N}\}$ is driven by a stationary process satisfying the following conditions:*

a) There exists an integer $h > 0$ such that for any $k \in \mathbb{N}_0$, it holds that

$$\Pr\left[W(k+h, k) \in \mathcal{M}_1\right] > 0; \tag{4.12}$$

b) There is a positive number α such that $W_{ij}(k) \geq \alpha$ for any $i, j \in \mathbf{N}, k \in \mathbb{N}_0$ satisfying $W_{ij}(k) > 0$.

In other words, Assumption 4.2 requires that any corresponding matrix product of length h becomes an SIA matrix with positive probability, and the positive elements for any matrix in $\mathcal{M}$ are uniformly lower bounded away from some positive value.

Theorem 4.2. *Under Assumption 4.2, the product of the random sequence of stochastic matrices $W(k,0)$ converges to a random matrix $L = \mathbf{1}\xi^\top$ almost surely.*

Recall in Section 2.3 that we denote $A_1 \sim A_2$ if these two stochastic matrices are of the same type (have zero elements in the same positions). Obviously, it holds the trivial case $A_1 \sim A_1$. One knows that for any SIA matrix A, there exists an integer l such that A^l is scrambling; it is easy to extend this to the inhomogeneous case, i.e., any product of l stochastic matrices of the same type of A is scrambling if all the matrices' elements are lower bounded away by some positive number. We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Since $\{W(k)\}$ is driven by a stationary process, we know that for any $t \in \mathbb{N}_0, h \in \mathbb{N}$, $\{W\left(t+h\right), \ldots, W\left(t+1\right)\}$ has the same joint distribution as $\{W\left(t+2h\right), \ldots, W\left(t+h+1\right)\}$. For the h given in Assumption 4.2, there exists an SIA matrix A such that $\Pr[W\big(t+kh+h, t+kh+1\big) = A] > 0$. Thus it follows that $\Pr[W\big(t+kh+2h, t+kh+1\big) = A] > 0$ for any $k \in \mathbb{N}_0$. Thus

$$\Pr\left[\begin{array}{c} W\big(t+(k+2)h, t+(k+1)h\big) \\ \sim W\big(t+(k+1)h, t+kh\big) \end{array} \middle| W\left(h, t+kh\right)\right] > 0.$$

When $W(t+h,t) \in \mathcal{M}_1$, which happens with positive probability, we have

$$\begin{aligned} &\Pr\left[W(t+2h,t+h) \sim W(t+h,t), W(t+h,t) \in \mathcal{M}_1\right] \\ &= \Pr\left[\begin{array}{c} W(t+2h,t+h) \\ \qquad \sim W(t+h,t) \end{array} \middle| \Pr\left[W(t+h,t) \in \mathcal{M}_1\right]\right] \Pr\left[W(t+h,t) \in \mathcal{M}_1\right] > 0. \end{aligned}$$

By recursion one can conclude that all the m products $W(t+(k+1)h, t+kh), k \in \{0, \ldots, m-1\}$, occur as the same SIA type with positive probability. Since all the products $W(t+(k+1)h, t+kh)$ are of the same type, one can choose m such that $W(t+mh,t)$ is scrambling. This in turn implies that $\Pr\left[W(t+mh,t) \in \mathcal{M}_2\right] > 0$, and the property of stationary process makes sure that (4.6) holds. The conditions in Assumption 4.1 are therefore all satisfied, and then Theorem 4.2 follows from Theorem 4.1. $\square$

Remark 4.1. *Theorems 4.1 and 4.2 have established some sufficient conditions for the convergence of a random sequence of stochastic matrices to a rank-one matrix. A further question is how these results can be applied to controlling distributed computation processes. To answer this question, let us still consider a finite set of stochastic matrices $\mathcal{M} = \{F_1 \ldots, F_m\}$, from which each $W(k)$ in the random sequence $\{W(k)\}$ is sampled. It is defined in [132] that $\mathcal{M}$ is a consensus set if the arbitrary product $\prod_{i=1}^k W(i), W(i) \in \mathcal{M}$, converges to a rank-one matrix. However, it has also been* *shown that to decide whether $\mathcal{M}$ is a consensus set is an NP-hard problem [132, 133]. For a non-consensus set $\mathcal{M}$, it is always not obvious how to find a deterministic sequence that converges, especially when $\mathcal{M}$ has a large number of elements and F_i has zero diagonal entries. However, the convergence can be ensured almost surely by introducing some randomness in the sequence, provided that there is a convergent deterministic sequence intrinsically.*

4.2.2 Estimate of Convergence Rate

In Subsection 4.2.1, we have shown how the product $W(k,0)$ determined by a random process asymptotically converges to a rank-one matrix W a.s. as $k \to \infty$. However, the convergence rate for such a randomized product is not yet clear. It is quite challenging to investigate how fast the process converges, especially when each $W(k)$ may have zero diagonal entries. In this subsection, we address this problem by employing finite-step stochastic Lyapunov functions. Now let us present the main result on convergence rate.

Theorem 4.3. *In addition to Assumption 4.1, if there exists a number p, $0 < p < 1$, such that for any $k \in \mathbb{N}_0$*

$$\Pr\left[W(h,k) \in \mathcal{M}_2\right] \geq p > 0,$$

then the almost sure convergence of the product $W(k,0)$ to a random matrix $L = \mathbf{1}\xi^\top$ is exponential, and the rate is no slower than $\left(1 - p\alpha^h\right)^{1/h}$.

Proof. Choosing $V(x_k) = v_k$ as a finite-step stochastic Lyapunov function candidate, from (4.10) we have

$$\mathbb{E}\left[V(x_{k+h})|\,\mathcal{F}_k\right] - V(x_k) \leq -\sum\nolimits_{i=1}^{n_k^s} \Pr\left[S_k^i\right]\alpha^h V(x_k). \tag{4.13}$$

Furthermore, it is easy to see that

$$\sum\nolimits_{i=1}^{n_k^s} \Pr\left[S_k^i\right] = \Pr\left[W(h,t) \in \mathcal{M}_2\right] \geq p,$$

Substituting it into (4.13) yields

$$\mathbb{E}\left[V(x_{k+h})|\,\mathcal{F}_k\right] \leq \left(1 - p\alpha^h\right) V(x_k).$$

It follows from Corollary 3.3 that $V(x_{k+h}) \xrightarrow{a.s.} 0$, with an convergence rate no slower than $\left(1 - p\alpha^h\right)^{1/h}$. In other words, the agreement is reached exponentially almost surely, which, in turn, completes the proof. $\square$

Theorem 4.3 has established the almost sure exponential convergence rate for the product of $\{W(k)\}$. If any subsequence $\{W(k+1), \dots, W(k+2), W(k+h)\}$ can result in a scrambling product $W(k+h, k)$ with positive probability and this probability is lower bounded away by some positive number, and then the convergence rate is exponential. *Interestingly, the greater this lower bound is, the faster the convergence becomes.* If we consider a special random sequence which is driven by a stationary ergodic process, the exponential convergence rate follows without any other conditions apart from Assumption 4.2, and an alternative proof is given in Appendix 4.6.

Corollary 4.2. *Suppose the random process governing the evolution of the sequence* $\{W(k) : k \in \mathbb{N}\}$ *is* stationary ergodic*, then the product* $W(k, 0)$ *converges to a random rank-one matrix at an exponential rate almost surely under Assumption 4.2.*

4.2.3 Connections to Markov Chains

In this subsection, we show that Theorems 4.2, and 4.3 are the generalizations of some well known results for Markov chains in [68, 71]. A fundamental result on inhomogeneous Markov chains is as follows.

Lemma 4.1 ([71, Th. 4.10], [68])**.** *If the product* $W(k, t)$*, formed from a sequence* $\{W(k)\}$*, satisfies* $W(t + k, t) \in \mathcal{M}_1$ *for any* $k \geq 1, t \geq 0$*, and* $W_{ij}(k) \geq \alpha$ *whenever* $W_{ij}(k) > 0$*, then* $W(k, 0)$ *converges to a rank-one matrix as* $k \to \infty$*.*

Let h be the number of distinct types of scrambling matrices of order n. It is known that the product $W(k + h, k)$ is scrambling for any k. In this case, we may take the probability of each product $W(k+h, k)$ being scrambling as $p = 1$, and as an immediate consequence of Theorem 4.3, we know that $W(k, 0)$ converges to a rank-one matrix at a exponential rate that is no slower than $(1 - \alpha^h)^{1/h}$. This convergence rate is consistent with what is estimated in [71, Th. 4.10]. This also applies to the homogeneous case where $W(k) = W$ for any k with W being scrambling. Moreover, it is known that the condition can be relaxed by just requiring W to be SIA to ensure the convergence, which is an immediate consequence of Theorem 4.2.

In next section, we discuss how the results in this section can be further applied to the context of asynchronous computations.

4.3 Agreement Induced by Stochastic Asynchronous Events

In this section, we study the agreement problem of multi-agent systems in networks that are allowed to be periodic (which will be defined later in this section). Periodic networks often lead to oscillating behaviors, but we show that asynchronous updating can induce agreement even the network is periodic. The results on products of random sequences of stochastic matrices obtained in Section 4.2 will be used to construct the proofs.

We take each component x^j in x from (4.7) as the state of agent i in an n-agent system. Define the distributed coordination algorithm

$$x^i(t_{k+1}) = \sum\nolimits_{j=1}^{n} w_{ij} x^j(t_k), \qquad k \in \mathbb{N}_0, i \in \mathbf{N}, \tag{4.14}$$

where the averaging weights $w_{ij} \geq 0$, $\sum_{j=1}^{n} w_{ij} = 1$, and t_k denote the time instants when updating actions happen. Here we assume the initial state $x(t_0)$ is given. It is always assumed that $T_1 \leq t_{k+1} - t_k \leq T_2$, where $t_0 = 0$ and T_1, T_2 are positive numbers. We say the states of system (4.14) reach *agreement* if $\lim_{k\to\infty} x(t_k) = \mathbf{1}\zeta$, mentioned in Section 4.2. Let $W = [w_{ij}] \in \mathbb{R}^{n\times n}$, and obviously W is a stochastic matrix. The algorithm (4.14) can be rewritten as

$$x(t_{k+1}) = Wx(t_k). \tag{4.15}$$

In fact, the matrix W can be associated with a directed, weighted graph $\mathcal{G}_W = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, 2, \cdots, n\}$ is the vertex set and $\mathcal{E}$ is the edge set for which $(i, j) \in \mathcal{E}$ if $w_{ji} > 0$. The graph $\mathcal{G}_W$ is called a *rooted* one if there exists at least one vertex, called a *root*, from which any other vertex can be reached. It is known that agents are able to reach agreement for all $x(0)$ if W is SIA ([68, 71]). However, the situations when W is not SIA have not been studied before, although they appear often in real systems, such as social networks.

In the context of distributed computation, it is always assumed that each computational unit in the network has access to its own latest state while implementing the iterative update rules [10, 25]. A class of situations that has received considerably less attention in the literature arise when some individuals are not able to obtain their own states, a case which can result from memory loss. Similar phenomena have also been observed in social networks while studying the evolution of opinions. Self-contemptuous people change their opinions solely in response to the opinions of others. The existence of computational units or individuals who are not able to access their own states sometimes might result in the computational failure or opinions' disagreement. As such an example, a periodic matrix W, which must has all zero diagonal entries (no access to their own states for all individuals), always leads the system (4.14) to oscillation. This is because for a periodic W, W^k never converges to a matrix with identical rows as $k \to \infty$. Instead, the positions of W^k that have positive values are periodically changing with k, resulting in a periodically changing value of $W^k x(0)$. We illustrate this point by the following example.

Example 4.1. *For system (4.15), the initial state is given by $x(0) = [1, 2, 3, 4]^T$, and the matrix P is*

$$W = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

By simple computation, one can check that $x(t_1) = [4, 1, 2, 3]^T, x(t_2) = [3, 4, 1, 2]^T, x(t_3) = [2, 3, 4, 1]^T, x(t_4) = [1, 2, 3, 4]^T = x(0)$. It is easy to see that the state equals the initial state after updating for four times. Then the same process will repeat again, which obviously implies a oscillating behavior instead of agreement. △

This motivates us to investigate the particular case where W is periodic. In the following two definitions, we provide the formal definitions of periodic stochastic matrices. We first introduce the definition of periodic irreducible matrices found in [71, Def. 1.6], and then extend this definition to the case when the matrices do not have to be irreducible.

Definition 4.1 ([71, Def. 1.6])**.** *Consider an irreducible stochastic matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. An index $i \in \{1, 2, \cdots, n\}$ is said to have period $d(i)$ if $d(i)$ is the common divisor of those $m \in \mathbb{N}^+$ for which $a_{ii}^{(m)} > 0$. The matrix A is said to be periodic with period d if $d(i) = d > 1$ for all i.*

Definition 4.2. *Consider a stochastic matrix $A \in \mathbb{R}^{n\times n}$, and let $\mathcal{P} := \{i : \exists m \in \mathbb{N}^+ : a_{ii}^{(m)} > 0\}$. An index $i \in \mathcal{P}$ is said to have period $d(i)$ if $d(i)$ is the common divisor of those m for which $a_{ii}^{(m)} > 0$. The matrix A is said to be periodic if $d(i) > 1$ for any $i \in \mathcal{P}$, and the period d is the common divisor of those m such that $a_{ii}^{(m)} > 0$ for all $i \in \mathcal{P}$.*

Definition 4.2 is a generalization of Definition 4.1. In this definition, a periodic stochastic matrix is not necessarily irreducible. The following example provides some intuition on these two definitions.

Example 4.2. *Consider the following two matrices:*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

One can see that A is irreducible, and B, C are reducible. According to Definition 4.1, it can be calculated that the indices $1, 2$ and 3 of A all have period 3, which means A is periodic with period 3. According to Definition 4.2, $\mathcal{P} = 1, 2$ for B, and the indices $1, 2$ have period 2. Then it is clear that the period of B is 2. Likewise, one can check that the period of C is 6. △

With a slight abuse of terminology, we say the graph $\mathcal{G}_W$ is *periodic* if the associated matrix W is. In this section, we show that agreement can be reached even when W is periodic, *just by introducing asynchronous updating events to the coupled agents.* In fact, perfect synchrony is hard to realize in practice as it is difficult for all agents to have access to a common clock according to which they coordinate their updating actions, while asynchrony is more likely. Researchers have studied how an agreement can be preserved with the existence of asynchrony, see e.g., [13, 14]. Unlike these works, we approach the same problem from a different aspect, where agreement occurs just because of asynchrony.

To proceed, we define a framework of randomly asynchronous updating events. It is usually legitimate to postulate that on occasions more than one, but not all, agents may update. Assume that each agent is equipped with a clock, which need not be synchronized with other clocks. The state of each agent remains unchanged except when an activation event is triggered by its own clock. Denote the set of event times of the ith agent by $\mathcal{T}^i = \{0, t_1^i, \cdots, t_k^i, \cdots\}, k \in \mathbb{N}$. At the event times, agent i updates its state obeying the asynchronous updating rule

$$x_i\left(t_{k+1}^i\right) = \sum\nolimits_{j=1}^{n} w_{ij} x_j\left(t_k^i\right), \tag{4.16}$$

where $i \in \mathbf{N}$. We assume that the clocks which determine the updating events for the agents are driven by an underlying random process. The following assumption is important for the analysis.

Assumption 4.3. *For any agent i, the intervals between two event times, denoted by $h_k^i = t_k^i - t_{k-1}^i$, are such that*

(i) h_k^i are upper bounded with probability 1 for all k and all i;

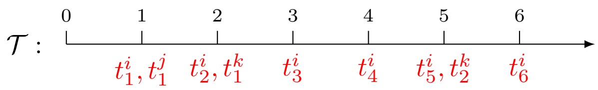


Figure 4.1: Event times of all agents: one (or more) agents can be activated simultaneously.

(ii) $\{h_k^i : k \in \mathbb{N}_0\}$ *is a random sequence, with* $\{h_k^1\}$, $\{h_k^2\}$, …, $\{h_k^n\}$ *being mutually independent.*

Assumption 4.3 ensures that an agent can be activated again within finite time after it is activated at t_{k-1}^i for all $k \in \mathbb{N}$, which implies that all agents will update their states for infinitely many times in the long run. In fact, Assumption 4.3 can be satisfied if the agents are activated by mutually independent Poisson clocks or at rates determined by mutually independent Bernoulli processes ([134, Ch. 6], [124, Ch. 2]).

Let $\mathcal{T} = \{t_0, t_1, t_2, \cdots, t_k, \cdots\}$ denote all event times of all the n agents, in which the event times have been relabeled in a way such that $t_0 = 0$ and $t_\tau < t_{\tau+1}, \tau = \{0, 1, 2, \cdots\}$. This idea has been used in [135] and [10] to study asynchronous iterative algorithms. One situation may occur in which there exists some k such that $t_k \in \mathcal{T}^i$ and $t_k \in \mathcal{T}^j$ for some i, j, which implies more than one agent is activated at some event times. Although this is not likely to happen when the underlying process is some special random ones like Poisson, our analysis and results will not be affected. The arrangement of $\mathcal{T}$ is illustrated clearly by Figure 4.1. For simplicity, we rewrite the set of event times as $\mathcal{T} = \{0, 1, 2, \cdots, k, \cdots\}$. Then the system with asynchronous updating can be treated as one with discrete-time dynamics in which the agents are permitted to update only at certain event times $k, k \in \mathbb{N}$, according to the updating rule (4.16) at each time k. Since each $k \in \mathcal{T}$ can be the event time of any subset of agents, we can associate any set of event times $\{k+1, k+2, \dots, k+h\}$ with the updating sequence of agents $\{\lambda(k+1), \lambda(k+2), \dots, \lambda(k+h)\}$ with $\lambda(i) \in \mathcal{V}$. Under Assumption 4.3, one knows that this updating sequence can be *arbitrarily ordered*, and each possible sequence can occur with positive probability, though the particular value is not of concern.

Assume at time k, $m \geq 1$ agents are activated, labeled by $k_1, k_2, \dots, k_m$, then we define the following matrices

$$W(k) = \left[u_1, \cdots, w_{k_1}^\top, u_{k+1}, \cdots, w_{k_m}^\top, \cdots, u_n\right]^\top, \tag{4.17}$$

where $u_i \in \mathbb{R}^n$ is the ith column of the identity matrix I_n and $w_k \in \mathbb{R}^n$ denotes the kth row of W. We call $W(k)$ the *asynchronous updating matrix* at time k. Then the asynchronous updating rule (4.16) becomes

$$x_k = W(k)x_{k-1}, \qquad k \in \mathbb{N}, \tag{4.18}$$

where $\{W(k)\}$ is a random sequence of asynchronous updating matrices which are stochastic, and $x_0 \in \mathbb{R}^n$ is a given initial state. We say the *asynchronous agreement* is reached if x_k converges to a scaled all-one vector when the agents update asynchronously. It suffices to study the convergence of the product $W(k)\dots W(2)W(1)$ to a rank-one matrix.

In Subsection 4.3.1, we consider the agents are coupled by a strongly connected and periodic network, and show that agreement is reached almost surely if the agents update their states asynchronously under Assumption 4.3. In Subsection 4.3.2, we identify a necessary and sufficient condition on the graph structure for asynchronous agreement, where aperiodicity is not required anymore.

4.3.1 Asynchronous Agreement over Strongly Connected Periodic Networks

In this subsection, we assume that the agents are coupled by a strongly connected and periodic network $\mathcal{G}_W$. Equivalently, the associated stochastic matrix W in the system (4.15) is irreducible and periodic (see Definition 4.1). We show in the following theorem that agreement can be reached if the agents update their states asynchronously.

Theorem 4.4. *Suppose that the agents are coupled by a strongly connected and periodic graph $\mathcal{G}_W$. Then, they can reach agreement almost surely if they update asynchronously under Assumption 4.3.*

We use the results in Corollary 4.1 to construct the proof. Then, it suffices to prove that there is a class of updating sequence of finite length such that the product of the corresponding asynchronous updating matrices, i.e., $W(k)$ in (4.18), is a Markov matrix, and this class of updating sequence appears with positive probability. This is formally stated in the following proposition.

Proposition 4.3. *There exists $T \in \mathbb{N}$ such that the product of the asynchronous updating matrices $W(k+T)W(k+T-1)\cdots W(k+1)$ have a positive probability to be a Markov matrix for any $k \in \mathbb{N}_0$.*

To prove this proposition, we define an operator $\mathcal{N}(\cdot,\cdot)$ for any stochastic matrix and any subset $\mathcal{S} \in \mathcal{V}$

$$\mathcal{N}(A, \mathcal{S}) := \{j : A_{ij} > 0, i \in \mathcal{S}\}, \tag{4.19}$$

and we write $\mathcal{N}(A,\{i\})$ as $\mathcal{N}(A,i)$ for brevity. It is easy to check then for any two stochastic matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ and for any subset $\mathcal{S} \in \mathcal{V}$, it holds that

$$\mathcal{N}\left(A_2A_1,\mathcal{S}\right)=\mathcal{N}\left(A_1,\mathcal{N}\left(A_2,\mathcal{S}\right)\right). \tag{4.20}$$

Proof of Proposition 4.3. This proposition can be proved by considering a special class of updating sequences, which appears with probability greater than 0. Since the directed graph $\mathcal{G}_W=(\mathcal{V},\mathcal{E})$ considered in this chapter is strongly connected (W is irreducible), for any fixed node $\lambda(1)\in\mathcal{V}$ one can always find some directed paths starting from $\lambda(1)$ and passing through all other nodes with finite lengths. Choose the path with the minimal length $T-1$, denoted by

$$\lambda\left(1\right)\to\lambda\left(2\right)\to\cdots\lambda\left(T-1\right)\to\lambda\left(T\right).$$

Obviously, it satisfies $\bigcup_{i=1}^{T}\lambda\left(i\right)=\mathcal{V}$. Now we assume that the updating sequence of the agents is $\{\lambda\left(1\right),\lambda\left(2\right),\cdots,\lambda(T)\}$, where only one agent updates at the corresponding time. Let $\{W_{\lambda(1)},W_{\lambda(2)},\cdots,W_{\lambda(T)}\}$ denote the sequence of the updating matrices. Let Φ be the backward product of this sequence, and it is given by

$$\Phi=W_{\lambda(T)}W_{\lambda(T-1)}\cdots W_{\lambda(2)}W_{\lambda(1)} \tag{4.21}$$

We next show Φ in (4.21) has at least one positive column. One knows Φ has a positive column if only if all the nodes in the associated graph $\mathcal{G}_\Phi$ share a common neighbor. Then we will prove all the nodes in $\mathcal{G}_\Phi$ share a common neighbor, i.e.,

$$\bigcap_{i=1}^{n}\mathcal{N}\left(\Phi,i\right)\neq\emptyset. \tag{4.22}$$

We first define the following iteration

$$\begin{aligned}&s_m=\{\lambda\left(k_{m-1}\right)\}\cup s_{m-1},\\&k_m=\max\left\{k:\lambda\left(k\right)\notin s_m,1\le k\le T\right\}\end{aligned}$$

where $m=2,\cdots,n$. Let $s_1=\emptyset, k_1=T$. Since $\bigcup_{i=1}^{T}\lambda\left(i\right)=\mathcal{V}$, it holds that $\bigcup_{i=1}^{T}\lambda\left(k_i\right)=\mathcal{V}$. For any k_i, it is obvious to see

$$\begin{aligned}&\mathcal{N}\left(W_{\lambda(T)}\cdots W_{\lambda(k_i+1)}W_{\lambda(k_i)}\cdots W_{\lambda(2)}W_{\lambda(1)},\lambda\left(k_i\right)\right)\\&=\mathcal{N}\left(W_{\lambda(k_i)}\cdots W_{\lambda(2)}W_{\lambda(1)},\lambda\left(k_i\right)\right).\end{aligned}$$

As $\lambda(k_i-1)$ is one of the neighbors of $\lambda(k_i)$, i.e.,

$$\lambda\left(k_i-1\right)\in\mathcal{N}\left(W_{\lambda(k_i)},\lambda\left(k_i\right)\right),$$

it follows that

$$\begin{aligned}&\mathcal{N}\left(W_{\lambda(k_i)}W_{\lambda(k_i-1)}\cdots W_{\lambda(2)}W_{\lambda(1)},\lambda\left(k_i\right)\right)\\&\supseteq\mathcal{N}\left(W_{\lambda(k_i-1)}\cdots W_{\lambda(2)}W_{\lambda(1)},\lambda\left(k_i-1\right)\right)\end{aligned} \tag{4.23}$$

where the inequality (4.20) has been used. Also, $\lambda(k_i-2)$ is an neighbor of $\lambda(k_i-1)$, then

$$\begin{aligned} &\mathcal{N}\left(W_{\lambda(k_i-1)}W_{\lambda(k_i-2)}\cdots W_{\lambda(2)}W_{\lambda(1)},\lambda\left(k_i\right)\right)\\ &\supseteq \mathcal{N}\left(W_{\lambda(k_i-2)}\cdots W_{\lambda(2)}W_{\lambda(1)},\lambda\left(k_i-2\right)\right) \end{aligned} \tag{4.24}$$

By recurrence one can conclude that

$$\begin{aligned} &\mathcal{N}\left(W_{\lambda(k_i-m)}\cdots W_{\lambda(2)}W_{\lambda(1)},\lambda\left(k_i-m\right)\right)\\ &\supseteq \mathcal{N}\left(W_{\lambda(k_i-m-1)}\cdots W_{\lambda(2)}W_{\lambda(1)},\lambda\left(k_i-m-1\right)\right), \end{aligned}$$

where $0\leq m\leq k_i-2$. It is straightforward to see

$$\mathcal{N}\left(\Phi,\lambda(k_i)\right)\supseteq \mathcal{N}\left(W_{\lambda(1)},\lambda\left(1\right)\right)=\mathcal{N}\left(W,\lambda\left(1\right)\right) \tag{4.25}$$

It is worth mentioning that (4.25) holds for any $i=1,2,\cdots,n$, which implies (4.22). Till here we know that all the nodes in the associated graph $\mathcal{G}_\Phi$ have at least one common neighbor which is the neighbor of $\lambda(1)$ in $\mathcal{G}_W$. It is easy to see that Φ has at least one positive column, which implies that it is a Markov matrix.

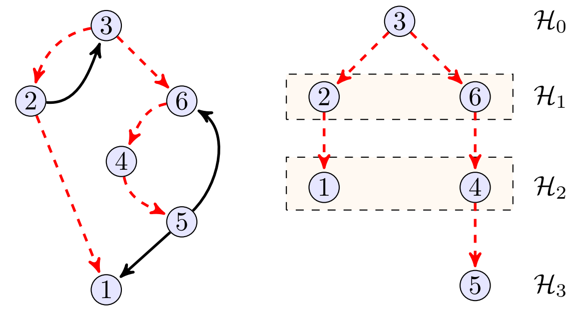
The updating sequence $\{\lambda\left(1\right),\lambda\left(2\right),\cdots,\lambda(T)\}$ can appear with positive probability at every interval of T time steps. This means that the product of the asynchronous updating matrices $W(k+T)W(k+T-1)\cdots W(k+1)$ have a positive probability to be a Markov matrix for any k, which completes the proof. □

4.3.2 A Necessary and Sufficient Condition for Asynchronous Agreement

In the previous subsection, we prove that the agents coupled by a strongly connected and periodic graph can reach an agreement if the agents update asynchronously. It is surprising since it has been believed that agreement through weighted averaging algorithms like (4.16) requires the graph to be aperiodic. In this subsection, we generalize the result in the previous subsection, and obtain a necessary and sufficient condition on the graph structure of $\mathcal{G}_W$ such that asynchronous agreement is ensured. The main result is presented in the following theorem.

Theorem 4.5. *Suppose the agents coupled by a network update asynchronously under Assumption 4.3, then they reach agreement almost surely if and only if the network is rooted, i.e., the matrix W is indecomposable.*

To prove this theorem, we need to introduce some additional concepts and results. It is equivalent to say the associated graph $\mathcal{G}_W$ is rooted if W is indecomposable. Denote the set of all the roots of $\mathcal{G}_W$ by $\mathbf{r}\subseteq\mathcal{V}$. We can partition the vertices of $\mathcal{G}_W$ into some hierarchical subsets as follows. For any $\kappa\in\mathbf{r}$, there must exist at least one directed spanning tree rooted at κ, see e.g., Fig. 4.2 (a). We select any of these



(a) The original graph. (b) Partition of the vertices.

Figure 4.2: An illustration of the graph partition; the hierarchical subsets: $\mathcal{H}_0 = \{3\}$,$\mathcal{H}_1 = \{2,6\}$,$\mathcal{H}_2 = \{1,4\}$,$\mathcal{H}_3 = \{5\}$; for example, {3,2,6,1,4,5} is a hierarchical updating vertex sequence.

directed spanning trees, denoted by $\mathcal{G}_W^s$. There exists a directed path from κ to any other vertex $i \in \mathcal{V}\backslash\kappa$, see e.g., Fig. 4.2 (b). Let l_i be the length of the directed path from κ to i, and there exists an integer $L \le n$ such that $l_i < L$ for all i. Define

$$\mathcal{H}_r := \{i : l_i = r\}, r = 1, \cdots, L-1,$$

and $\mathcal{H}_0 = \{\kappa\}$. From this definition, one can partition the vertices of $\mathcal{G}_W^s$ into L hierarchical subsets, i.e., $\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_{L-1}$, according to the vertices' distances to the root κ. Let n_r be the number of vertices in the subset $\mathcal{H}_r$, $0 \le r \le L-1$ (see the example in Fig. 4.2 (b)). Note that given a spanning tree, its corresponding hierarchical subsets H_r's are uniquely determined.

Definition 4.3. *An updating vertex sequence of length n is said to be hierarchical if it can be partitioned into some successive subsequences, denoted by $\{\mathcal{A}_0, \ldots, \mathcal{A}_{L-1}\}$ with $\mathcal{A}_r = \{\lambda_r(1), \lambda_r(2), \cdots, \lambda_r(n_r)\}$, such that $\bigcup_{k=1}^{n_r} \lambda_r(k) = \mathcal{H}_r$ for all $r = 0, \cdots, L-1$, where $\mathcal{H}_r$'s are the hierarchical subsets of some spanning tree $\mathcal{G}_W^s$ in $\mathcal{G}_W$.*

Proposition 4.4. *If agents coupled by $\mathcal{G}_W$ update in a hierarchical sequence $\{a_1, \cdots, a_n\}, a_i \in \mathcal{V}$ for all i, the product of the corresponding asynchronous updating matrices,*

$$\Phi := W_{a_n} \cdots W_{a_2} W_{a_1}$$

is a Markov matrix.

Proof of Proposition 4.4. It suffices to show that all $i \in \mathcal{V}$ share at least one common neighbor in the graph $\mathcal{G}_\Phi$, i.e.,

$$\bigcap_{i=1}^{n} \mathcal{N}(\Phi, i) \neq \emptyset. \tag{4.26}$$

We rewrite the product of asynchronous updating matrices into

$$\Phi = \left\{ W_{\lambda_{L-1}(1)} \cdots W_{\lambda_{L-1}(n_{L-1})} \cdots W_{\lambda_{L-2}(1)} \cdots W_{\lambda_0(1)} \right\}.$$

For any distinct $i, j \in \mathcal{V}$, we know that $\mathcal{N}(W_j, i) = \{i\}$ from the definition of asynchronous updating matrices. Then for any $\lambda_r(t) \in \mathcal{H}_r, t \in \{1, \cdots, n_r\}, r \in \{1, \cdots, L-1\}$, it holds that

$$\begin{aligned}
&\mathcal{N}(\Phi, \lambda_r(t)) \\
&= \mathcal{N}\left(W_{\lambda_r(t)} W_{\lambda_r(t+1)} \cdots W_{\lambda_r(n_r)} \cdots W_{\lambda_0(1)}, \lambda_r(t)\right) \\
&= \mathcal{N}\left(W_{\lambda_r(t+1)} \cdots W_{\lambda_r(n_r)} \cdots W_{\lambda_0(1)}, \mathcal{N}\left(W_{\lambda_r(t)}, \lambda_r(t)\right)\right),
\end{aligned}$$

where the property (4.20) has been used. From Definition 4.3, one knows that there exists at least one vertex $\lambda_{r-1}(t_1) \in \mathcal{H}_{r-1}$ that can reach $\lambda_r(t)$ in $\mathcal{G}_W$ and subsequently in $\mathcal{G}_{W_{\lambda_r(t)}}$, which implies

$$\lambda_{r-1}(t_1) \in \mathcal{N}\left(W_{\lambda_r(t)}, \lambda_r(t)\right).$$

It then follows

$$\begin{aligned}
&\mathcal{N}\left(W_{\lambda_r(t+1)} \cdots W_{\lambda_r(n_r)} \cdots W_{\lambda_0(1)}, \lambda_{r-1}(t_1)\right) \\
&\subseteq \mathcal{N}(\Phi, \lambda_r(t)).
\end{aligned}$$

Similarly, one obtains

$$\begin{aligned}
&\mathcal{N}\left(W_{\lambda_r(t+1)} \cdots W_{\lambda_r(n_r)} \cdots W_{\lambda_0(1)}, \lambda_{r-1}(t_1)\right) \\
&= \mathcal{N}\left(W_{\lambda_{r-1}(t_1)} \cdots W_{\lambda_r(n_r)} \cdots W_{\lambda_0(1)}, \lambda_{r-1}(t_1)\right) \\
&= \mathcal{N}\left(W_{\lambda_{r-1}(t_1+1)} \cdots W_{\lambda_0(1)}, \mathcal{N}\left(W_{\lambda_{r-1}(t_1)}, \lambda_{r-1}(t_1)\right)\right) \\
&\supseteq \mathcal{N}\left(W_{\lambda_{r-1}(t_1+1)} \cdots W_{\lambda_0(1)}, \lambda_{r-2}(t_2)\right).
\end{aligned}$$

As a recursion, it must be true that

$$\mathcal{N}\left(W_{\lambda_0(1)}, \kappa\right) \subseteq \mathcal{N}(\Phi, \lambda_r(t)), \tag{4.27}$$

where κ is a root of $\mathcal{G}_W^s$. In fact, it holds that $\lambda_0(1) = \kappa$, and then we know

$$\mathcal{N}\left(W_{\lambda_0(1)}, \kappa\right) = \mathcal{N}(W_\kappa, \kappa) = \mathcal{N}(W, \kappa). \tag{4.28}$$

Substituting (4.28) into (4.27) leads to

$$\mathcal{N}(W, \kappa) \subseteq \mathcal{N}(\Phi, \lambda_r(t))$$

for all $\lambda_r(t)$. Since $\bigcup_{r,t} \{\lambda_r(t)\} = \mathcal{V}$, we know

$$\mathcal{N}(W, \kappa) \subseteq \bigcap_{r,t} \mathcal{N}(\Phi, \lambda_r(t)).$$

Straightforwardly, (4.26) follows, which completes the proof. $\square$

Since the hierarchical sequences will appear with positive probability in any sequence of length n, one can easily prove the following proposition by letting $l = n$.

Proposition 4.5. *There exists an integer l such that the product $W(k+l)\cdots W(k+1)$, where $W(k)$ is given in* (4.18)*, is a Markov matrix with positive probability for any $k \in \mathbb{N}$.*

Proof of Theorem 4.5. We prove the necessity by contradiction. Suppose the matrix W is decomposable. Then there are at least two sets of vertices that are isolated from each other. Then agreement will never happen between these two isolated groups if they have different initial states. Let $l = n$, in view of Corollary 4.1, the sufficiency follows directly from Proposition 4.5, which completes the proof. □

Note that the hierarchical sequence is a particular type of updating orders that results in a Markov matrix as the product of the corresponding updating matrices. We have identified another type of updating orders in our earlier work when W is irreducible and periodic in the previous subsection. It is of great interest for future work to look for other updating mechanisms to enable the appearance of Markov matrices or scrambling matrices, which plays a crucial role in giving rise to an asynchronous agreement.

In the next subsection, we demonstrate the obtained results in the two subsections by simulation.

4.3.3 Numerical Examples

In this section, we demonstrate the obtained results by a numerical example. Consider the system (4.15) with the following periodic matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding graph is given by Fig. 4.3, which is strongly connected and periodic. Let the initial state be $x(0) = [1.1, 4.2, 7.3, 3.4, 4.5, 5.6]^T$. If the agents in the network have a common clock to synchronize the updating actions, the states of the agents cannot reach an agreement, instead, a oscillating behavior takes place, as shown in Fig. 4.4.

However, if individuals update according to their own clocks under Assumption 4.3, the agreement can be reached. To illustrate this, we assume the clocks are driven

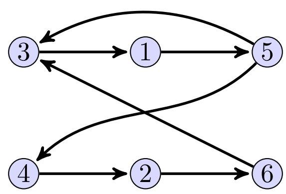


Figure 4.3: Associated graph of P.

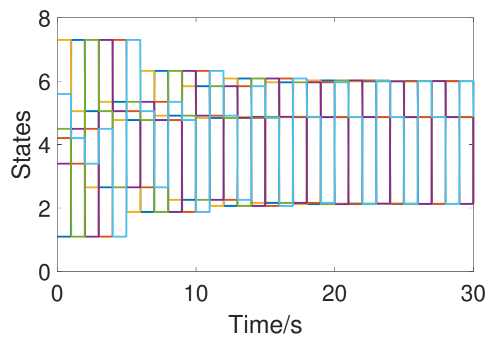


Figure 4.4: Update synchronously: oscillation.

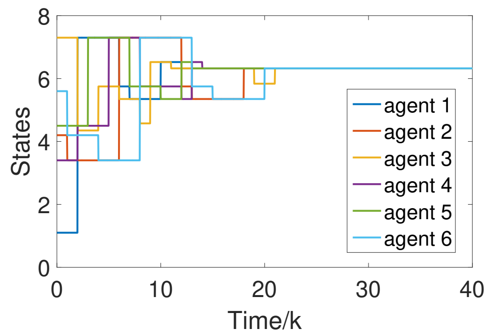


Figure 4.5: Update asynchronously: agreement.

by mutually independent Poisson processes in which the interarrival intervals have the density functions

$$f_i(x) = \lambda_i e^{-\lambda_i x}, \qquad\qquad \text{for } x \geq 0,$$

where $i = 1, 2, \cdots, n$. Let $\lambda_i = 2$ for all i. The evolution of the agents' states is shown in Fig. 4.3.3, which shows that the states converge to a common value instead of an oscillation although the network is periodic. Thus one observes that asynchronous updating events have played a fundamental role in giving rise to agreement.

4.4 A Linear Algebraic Equation Solving Algorithm

In this section, we apply the finite step Lyapunov criteria obtained in Chapter 3 to solving linear algebraic equations distributively.

Researchers have been quite interested in solving a system of linear algebraic equations in the form of $Ax = b$ in a distributed way [29, 30, 113, 114]. In this section we deal with the problem under the assumption that this system of equations has at least one solution. The set of equations is decomposed into smaller sets and distributed to a network of n processors, referred to as agents, to be solved in parallel. Agents can receive information from their neighbors and the neighbor relationships are described by a time-varying n-vertex directed graph $\mathcal{G}(t)$ with self-arcs. When each agent knows only the pair of real-valued matrices $(A_i^{n_i \times m}, b_i^{n_i \times 1})$, the problem of interest is to devise local algorithms such that all n agents can iteratively compute the same solution to the linear equation $Ax = b$, where $A = [A_1^\top, A_2^\top, \ldots, A_n^\top]^\top, b = [b_1^\top, b_2^\top, \ldots, b_n^\top]^\top$ and $\sum_{i=1}^n n_i = m$.

A distributed algorithm to solve the problem is introduced in [77], where the iterative updating rule for each agent i is described by

$$x_{k+1}^i = x_k^i - \frac{1}{d_k^i} P_i \Big(d_k^i x_k^i - \sum_{j \in \mathcal{N}_i(k)} x_k^j \Big), k \in \mathbb{N}, \tag{4.29}$$

where $x_k^i \in \mathbb{R}^m$, d_k^i is the number of neighbors of agent i at time k, $\mathcal{N}_i(k)$ is the collection of i's neighbors, P_i is the orthogonal projection on the kernel of A_i, and the initial value x_1^i is any solution to the equations of $A_i x = b_i$.

The results in [77] have shown that all x_k^i converge to the same solution exponentially fast if the sequence of graphs $\mathcal{G}(t)$ is repeatedly jointly strongly connected. This condition requires that for some integer l, the composition of the sequence of graphs, $\{\mathcal{G}(k), \ldots, \mathcal{G}(k + l - 1)\}$, must be strongly connected for any t. It is not so easy to satisfy this condition if the network is changing randomly. Now assume that the evolution of the sequence of graphs $\{\mathcal{G}(1), \ldots, \mathcal{G}(k), \ldots\}$ is driven by a random process. In this case, results in Theorem 3.1 and Corollary 3.1 can be applied to relaxing the condition in [77] to achieve the following more general result.

Theorem 4.6. *Suppose that each agent updates its state x_k^i according to the rule (4.29). All states x_k^i converge to the same solution to $Ax = b$ almost surely if the following two conditions are satisfied:*

a) there exists an integer l such that for any $k \in \mathbb{N}$ the composition of the sequence of randomly changing graphs $\{\mathcal{G}(k), \mathcal{G}(k+1), \dots, \mathcal{G}(k+l-1)\}$ is strongly connected with positive probability $p(k) > 0$;

b) for any $k \in \mathbb{N}$, it holds that $\sum_{i=0}^{\infty} p\,(k + il) = \infty$.

To prove the theorem, we define an error system. Let x^* be any solution to $Ax = b$, so $A_i x^* = b_i$ for any i. Then, we define

$$e_k^i = x_k^i - x^*, i \in \mathcal{V}, k \in \mathbb{N},$$

which, as is done in [77], can be simplified into

$$e_{k+1}^i = \frac{1}{d_k^i} P_i \sum\nolimits_{j \in \mathcal{N}_i(k)} P_j e_k^j. \tag{4.30}$$

Let $e_k = [{e_k^1}^\top, \dots, {e_k^n}^\top]^\top$, $A(k)$ be the adjacency matrix of the graph $\mathcal{G}(k)$, $D(k)$ be the diagonal matrix whose ith diagonal entry is d_k^i, and $W(k) = D^{-1}(k)A^\top(k)$. It is clear that $W(k)$ is a stochastic matrix, and $\{W(k)\}$ is a stochastic process. Now we write equation (4.30) into a compact form

$$e_{k+1} = P(W(k) \otimes I)Pe_k, k \in \mathbb{N}, \tag{4.31}$$

where $\otimes$ denotes the Kronecker product, $P := \text{diag}\{P_1, P_2, \dots, P_n\}$, and $\{W(k)\}$ is a random process. We will show this error system is globally a.s. asymptotically stable. Define the transition matrix of this error system by

$$\Phi(k + T, k) = P(W(k + T - 1) \otimes I)P \cdots P(W(k) \otimes I)P.$$

In order to study the stability of the error system (4.31), we define a mixed-matrix norm for an $n \times n$ block matrix $Q = [Q_{ij}]$ whose ijth entry is a matrix $Q_{ij} \in \mathbb{R}^{m \times m}$, and

$$[\![Q]\!] = |\langle Q \rangle|_\infty,$$

where $\langle Q \rangle$ is the matrix in $\mathbb{R}^{n \times n}$ whose ijth entry is $|Q_{ij}|_2$. Here $\| \cdot \|_2$ and $\| \cdot \|_\infty$ denote the induced 2 norm and infinity norm, respectively. It is easy to show that $[\![\cdot]\!]$ is a norm. Since $\|Ax\|_2 \leq \|A\|_2 \|x\|_2$ for $x \in \mathbb{R}^{nm \times nm}$, it follows straightforwardly that $[\![Ax]\!] \leq [\![A]\!] \, [\![x]\!]$. It has been proven in [77] that $\Phi(k + T, k)$ is non-expansive for any $k > 0, T \geq 0$. In other words, it holds that

$$[\![\Phi(k + T, k)]\!] \leq 1.$$

Moreover, the transition matrix is a contraction, i.e., $\llbracket \Phi(k+T,k) \rrbracket < 1$, if there exists a route $j = i_0, i_1, \ldots, i_T = i$ over the sequence $\{\mathcal{G}(k), \ldots, \mathcal{G}(k+T-1)\}$ for any $i, j \in \mathcal{V}$ that satisfies $\bigcup_{k=0}^{T} \{i_k\} = \mathcal{V}$. Now we are ready to prove Theorem 4.6.

Proof of Theorem 4.6. Let $V(e_k) = \llbracket e_k \rrbracket$ be a finite-step stochastic Lyapunov function candidate. Let $\{\mathcal{F}_k\}$, where $\mathcal{F}_k = \sigma(\mathcal{G}(1), \cdots, \mathcal{G}(k), \cdots)$, be an increasing sequence of σ-fields. We first show that $V(e_k)$ is a supermartingale with respect to $\mathcal{F}_k$ by observing

$$\mathbb{E}\big[V\big(e_{k+1}\big)\big|\, \mathcal{F}_k\big] = \mathbb{E}\,\llbracket \Phi_k e_k \rrbracket \leq \mathbb{E}\,\llbracket \Phi_k \rrbracket \, \llbracket e_k \rrbracket \leq \llbracket e_k \rrbracket \ ,$$

where $\Phi_k = \Phi(k,k) = P(W(k) \otimes I)Pe_k$. The last inequality follows from the fact that $\mathbb{E}\,\llbracket \Phi_k \rrbracket \leq 1$ since all the possible Φ_k are non-expansive. Consider the sequence of randomly changing graphs $\{\mathcal{G}(1), \mathcal{G}(2), \cdots, \mathcal{G}(q)\}$, where $q = (n-1)^2 l$. Let $r = n-1$, and partition this sequence into r successive subsequences $\mathcal{G}_1 = \{\mathcal{G}(1), \ldots, \mathcal{G}(rl)\}$, $\mathcal{G}_2 = \{\mathcal{G}(rl+1), \ldots, \mathcal{G}(2rl)\}, \cdots$, $\mathcal{G}_r = \{\mathcal{G}((r-1)l+1), \ldots, \mathcal{G}(r^2 l)\}$. Let $\mathbb{C}_z$ denote the composition of the graphs in the zth subsequence, i.e., $\mathbb{C}_z = \mathcal{G}(zl) \circ \cdots \circ \mathcal{G}((z-1)l+2) \circ \mathcal{G}((z-1)l+1)$, $z = 1, 2, \ldots, r$. Since all the subsequences have the length rl, each can be further partitioned into r successive sub-subsequences of length l. From the condition of Theorem 4.6, one knows that the composition of the graphs in any sub-subsequence has positive probability to be strongly connected. The event that the composition of the graphs in each of the r sub-subsequences in $\mathcal{G}_z$ is strongly connected also has positive probability. This holds for all z. We know that the composition of any r or more strongly connected graphs, within which each vertex has a self-arc, results in a complete graph [9]. It follows straightforwardly that the graphs $\mathbb{C}_1, \cdots, \mathbb{C}_r$ have positive probability to be all complete. Therefore, for any pair $i, j \in \mathcal{V}$, there exists a route from j to i over the graph $\mathbb{C}_z$ for any z. It is easy to check that there exists a route $i_1, i_2, \ldots, i_n$ over the graphs $\mathbb{C}_1, \cdots, \mathbb{C}_r$, where $i_1, i_2, \ldots, i_n$ can be any reordered sequence of $\{1, 2, \ldots, n\}$. Similarly, for any x there must exist a route of length rl, $i_z = i_z^1, i_z^2, \ldots, i_z^{rl} = i_{z+1}$, over $\mathcal{G}_z$. Thus there is a route $i_1^1, i_1^2, \ldots, i_1^{rl}, i_2^2, \ldots, i_2^{rl} \ldots, i_r^{rl}$ over the graph sequence $\{\mathcal{G}(1), \mathcal{G}(2), \cdots, \mathcal{G}(q)\}$ so that $\bigcup_{\delta=1}^{r} \bigcup_{\theta=1}^{rl} \{i_\delta^\theta\} = \mathcal{V}$. This implies that the probability that $\Phi(q,1)$ being a contraction is positive. Since all $\Phi(q,1)$ are non-expansive, there is a number $\rho(1) < 1$ such that $\mathbb{E}\,\llbracket \Phi(q,1) \rrbracket = \rho(1)$. Straightforwardly, it also holds $\mathbb{E}\,\llbracket \Phi(k+q,k) \rrbracket = \rho(k) < 1$ for all $k < \infty$. Thus there a.s. holds that

$$\begin{aligned} &\mathbb{E}\big[V\left(e_{k+q}\right)\big|\, \mathcal{F}_k\big] - V(e_k) = \mathbb{E}\,\llbracket \Phi\left(k+q,k\right) e_k \rrbracket - V(e_k) \\ &\leq \mathbb{E}\,\llbracket \Phi\left(k+q,k\right) \rrbracket \cdot \llbracket e_k \rrbracket - V(e_k) = (\rho(k)-1)V(e_k). \end{aligned}$$

Similarly as in the proof of Theorem 4.1, the condition b) in Theorem 4.6 ensures that $\sum_{i=1}^{\infty}(1-\rho(k)) = \infty$. It follows that $V\left(e_k\right) \xrightarrow{a.s.} 0$ as $t \to \infty$ since $V(e_0) -$

$\mathbb{E}\big[V(e_{nq})|\mathcal{F}_k\big] < \infty$ for any N. Define the set $\mathcal{Q} := \{e : V(e) \leq V(e_1)\}$ for any initial e_1 corresponding to x_1. For any random sequence $\{\mathcal{G}(k)\}$, it follows from the system dynamics (4.31) that

$$V(e_k) \leq V(e_{k-1}) \cdots \leq V(e_2) \leq V(e_1),$$

and thus e_k will stay within the set $\mathcal{Q}$ with probability 1. From Theorem 3.1 and Corollary 3.1, it follows that e_k asymptotically converges to $\{e : V(e) = 0\}$ almost surely. Moreover, since $V(e)$ is a norm of e, it can be concluded from Corollary 3.1 that the error system (4.31) is globally a.s. asymptotically stable. The proof is complete. □

It is worth mentioning that the error system is globally a.s. exponentially stable under the assumption that the probability of the composition of any sequence of randomly-changing graphs, $\{\mathcal{G}(k), \ldots, \mathcal{G}(k+1), \mathcal{G}(k+l-1)\}$, for any $k \in \mathbb{N}$, being strongly connected is lower bounded by some positive number. This can be proven with the help of Theorem 3.2 and Corollary 3.2.

4.5 Concluding Remarks

In this chapter, we have shown how the finite-step Lyapunov criteria established in the Chapter 3 can be applied to studying several distributed coordination algorithms. As the first application, we look at the product of random sequences of stochastic matrices, including those with zero diagonal entries, and obtain sufficient conditions to ensure that the product almost surely converges to a matrix with identical rows; we also show that the rate of convergence can be exponential under additional conditions. Using these results, we have further investigated how asynchronous updating events can induce agreement among agents coupled by periodic networks. As another application, we have studied a distributed network algorithm for solving linear algebraic equations. We relax the existing conditions on the network structures, while still guaranteeing the equations are solved asymptotically.

4.6 Appendix: An Alternative Proof of Corollary 4.2

For ergodic stationary sequences, the following important property is the key to construct the convergence rate.

Lemma 4.2 (Birkhoff's Ergodic Theorem, see [109, Th. 7.2.1])**.** *For an ergodic sequence* $\{X_k\}, k \in \mathbb{N}_{\geq 0}$*, of random variables, it holds that*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} X_k \xrightarrow{a.s.} E(X_0) \tag{4.32}$$

For the product given in (4.1), we say $W(k,0)$ converges to a rank-one matrix $W = 1\xi^\top$ a.s. as $k \to \infty$ if $\tau(W(k,0)) \to 0$ as $k \to \infty$, where $\tau(\cdot)$ is defined in (4.2). According to Definition 3.1, if there exists $\beta > 1$ such that

$$\beta^k \tau\big(W(k,0)\big) \xrightarrow{a.s.} 0, k \to \infty, \tag{4.33}$$

then the convergence rate is said to be exponential at the rate no slower than β^{-1}. We are now ready to present the proof of Corollary 4.2.

Proof of Corollary 4.2. Let h be the same as that in Assumption 4.2. There is an integer $\theta \in \mathbb{N}$ such that $W(t+\theta h, t)$ is scrambling with positive probability. Let $T = \theta h$. Consider a sufficiently large r, and then $W(r,0)$ can be written as

$$W(r,0)) = \bar{W} \cdot W\big(mT, (m-1)\,T\big) \cdots W\,(T,0)\,,$$

where m is the largest integer such that $mT \leq r$, $W\,(kT+T, kT)\,, k = 0, \cdots, m-1$, are the matrix products defined by (4.1), and $\bar{W} = W(r, mT)$ is the remaining part, which is obviously a stochastic matrix. To study the limiting behavior of $W(r,0)$, we compute its coefficients of ergodicity

$$\begin{aligned}\tau\big(W\,(r,0)\big) &\leq \tau\,(\bar{W}) \prod_{k=0}^{m-1} \tau\big(W\,(kT+T, kT)\,\big)\\ &\leq \prod_{k=0}^{m-1} \tau\big(W\,(kT+T, kT)\,\big),\end{aligned}$$

where the property (4.4) has been used. The last inequality follows from the property of coefficients of ergodicity, i.e., $\tau(A) \leq 1$ for a stochastic matrix A. Taking logarithms yields that

$$\log \tau\big(W\,(r,0)\big) \leq \sum_{k=0}^{m-1} \log \tau\big(W\,(KT+T, kT)\,\big). \tag{4.34}$$

Since the sequence $\{W(k)\}$ is ergodic, it is easy to see that the sequence of products $\{W\,(kT+T, kT)\}$, $k = 0, \cdots, m-1$, over non-overlapping intervals of length T, is also ergodic. It follows in turn that $\{\log \tau\big(W\left(kT+T,kT\right)\big)\}$ is ergodic. From Lemma 4.2, one can further obtain

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}\log\tau\big(W\left(kT+T,kT\right)\big)$$
$$\xrightarrow{a.s.}\mathbb{E}\Big[\log\tau\big(W\left(T,0\right)\big)\Big]\le\log\mathbb{E}\big[\tau\big(W\left(T,0\right)\big)\big].$$

The last inequality follows from Jensen's inequality (see [109, Th. 1.5.1]) since $\log(\cdot)$ is concave. According to Assumption 4.1, one knows that $W(t+h,t)$ is scrambling with positive probability, and thus it follows that $0<\mathbb{E}\big[\tau\left(W\left(T,0\right)\right)\big]<1$. Taking a positive number λ satisfying $\lambda<-\log\mathbb{E}\big[\tau\big(W\left(T,0\right)\big)\big]$, one obtains

$$m\lambda+\sum\nolimits_{k=0}^{m-1}\log\tau\big(W\left(KT+T,kT\right)\big)\xrightarrow{a.s.}-\infty.$$

Adding $m\lambda$ to both sides of (4.34) yields that

$$m\lambda+\log\tau\big(W\left(r,0\right)\big)$$
$$\le m\lambda+\sum\nolimits_{k=0}^{m-1}\log\tau\big(W\left(kT+T,kT\right)\big)\xrightarrow{a.s.}-\infty.$$

It follows straightforwardly that

$$\big(e^{\lambda}\big)^{m}\tau\big(W\left(r,0\right)\big)\xrightarrow{a.s.}0.$$

Let $\beta=e^{\lambda}$, which apparently satisfies $\beta>1$. From Definition 3.1, one can conclude that the product $W(k,0)$ almost surely converges to a rank-one stochastic matrix exponentially at a rate no slower than β^{-1}, which completes the proof. $\square$

Part II

Partial Synchronization of Kuramoto Oscillators:

Partial Stability Methods

Overview of Part II

Synchronization is a ubiquitous phenomenon that has been observed pervasively in many natural, social and man-made systems [46, 136–138]. Remarkable examples include synchronized flashing of fireflies [4], animal flocking [7], pedestrian footwalk synchrony on London's Millennium Bridge [139], phase synchronization of coupled Josephson junction circuits [140], and synchronous operation of power generators [49].

Global synchronization describes the situation where all units in a network evolve in unison. Strong network coupling plays a fundamental role in the emergence and stability of global synchronization [78]. Recently, another form of synchronization, termed partial synchronization, has attracted a lot of attention [82, 141, 142]. In contrast to global synchronization, partial synchronization characterizes a circumstance in which only some parts of, instead of all, units in a network have similar dynamics. It is believed to be more common [82] in nature, for example in the human brain.

Neuronal synchronization across cortical regions of the human brain, which has been widely detected through recording and analyzing brain waves, is believed to facilitate communication among neuronal ensembles [55]. Only closely correlated oscillating neuronal ensembles can exchange information effectively, because their input and output windows are open at the same time [52]. In healthy human brain, it is frequently observed that only a part of its cortical regions are synchronized [59], and such a phenomenon is commonly referred to as partial phase cohesiveness or partial synchronization of brain neural networks. In contrast, in the pathological brain of an epileptic patient, global synchronization of neural activities are detected to take place across the entire brain [60]. These observations suggest that healthy brain has powerful regulation mechanisms that are not only able to render synchronization, but also capable of preventing unnecessary synchronization among neuronal ensembles. Partly motivated by these experimental studies, researchers are interested in theoretically studying cluster synchronization [82, 85, 142, 143] and chimera states [88], even though analytical results are much more difficult to obtain, while analytical results for global synchronization are ample, e.g., [78, 144, 145].

In this part of the thesis, our objective is to identify some possible underlying mechanisms that could give rise to partial synchronization in complex networks, particularly in human brain networks. The Kuramoto model and its variations [62] will be used to describe the dynamics of oscillators. We first investigate in Chapter 5 how partial synchronization can take place among directly connection regions. We find that strong local or regional coupling is a possible mechanism. Oscillators that are tightly connected can exhibit coordinating behavior, while the rest that are weakly connected to them remain different. In addition, we also study how remote synchronization, a phenomenon also detected in the human brain [92], can take place in star networks. In order to study remote synchronization, we develop some new criteria for partial stability of nonlinear systems in Chapter 6. These new criteria are then used to analytically study remote synchronization in Chapter 7.

5 Partial Phase Cohesiveness in Networks of Kuramoto Oscillator Networks

In this chapter, we aim at identifying a mechanism that could account for the emergence of partial synchronization among adjacent brain regions. We use the Kuramoto model to describe the dynamics of neural ensembles. The oscillators are assumed to have heterogeneous natural frequencies, and thus phase synchronization is not possible to take place among them. We employ another terminology, *partial phase cohesiveness*, to describe the situation where the oscillators have different phases, but the phase differences are bounded to be small. Motivated by the organization of cortical neurons, we assume that each region consists of a number of Kuramoto oscillators that are fully connected, and different regions are interconnected with each other. We try to identify some sufficient conditions such that partial phase cohesiveness of Kuramoto oscillators coupled by this type of network-of-networks structure can occur.

5.1 Introduction

As a powerful tool for understanding synchronization patterns emerged in the human brain, the Kuramoto model has fascinated researchers in neuroscience. It has been widely used to describe the dynamics of coupled neural ensembles [146, 147]. In this chapter, we employ the Kuramoto model and analytically study how partial phase cohesiveness can occur in a network motivated by the organization structure of neurons. In the human brain, the organization of cortical neurons exhibits a "network-of-networks" structure in the sense that a cortical region is typically composed of strongly connected ensembles of neurons that interact not only locally but also with ensembles in other regions [148]. As neural ensembles in a cortical region are adjacent in space, it is thus reasonable to assume that oscillators within a brain region are coupled through an all-to-all network, forming local communities at the lower level; at the higher level, the communities are interconnected by a sparse network facilitated through bundles of neural fibers connecting regions of the brain. Motivated by these facts, we consider in this chapter the networks describing the interaction between Kuramoto oscillators have this two-level structure.

The main contributions of this chapter are some new sufficient conditions for partial phase cohesiveness by using Lyapunov functions utilizing the incremental 2-norm and ∞-norm. The incremental 2-norm was first proposed in [145, 149], in which some conditions for locally exponentially stable synchronization was obtained. Later on, it was also employed in the study of non-complete networks [150, 151]. Inspired by these works, we first employ the incremental 2-norm and obtain a sufficient condition for the algebraic connectivity $\lambda_2(L)$ of the considered subnetwork, and then estimate the region of attraction and the ultimate boundedness of phase cohesiveness. This critical value for $\lambda_2(L)$ depends on the natural frequency heterogeneity of the oscillators within the subnetwork and the strength of the connections from its outside to this subnetwork. Since the incremental 2-norm depends greatly on the scale, the obtained critical value and the estimated region of attraction are both conservative, especially when there are large numbers of oscillators in the considered subnetwork.

On the other hand, the incremental ∞-norm is scale-independent. It has been utilized to prove the existence of phase-locking manifolds and their local stability. Existing conditions are usually expressed implicitly by a combined measure [152, 153], and the regions of attraction are not estimated [85, 154]. To the authors' best knowledge, the best result on explicit conditions utilizing the incremental ∞-norm is given in [144], which has only studied unweighted complete networks. It is challenging to extend it to the non-complete or even weighted complete networks. To meet the challenges, we introduce a concept of the *generalized complement graph* in this chapter, which, in turn, enables us to make use of the incremental ∞-norm and obtain an explicit condition. Compared to the results obtained by the incremental 2-norm: 1) the established sufficient condition is less conservative if the dissimilarity of natural frequencies and the strengths of external connections are noticeable; 2) more importantly, the region of attraction we identified is much larger. After simplifying the network structure, our results on partial phase cohesiveness can reduce to some results on complete phase cohesiveness. The reduced results are sharper than the best-known result obtained by using incremental 2-norm for the case of weighted complete and non-complete networks [151, Theorem 4.6] (especially in terms of the region of attraction), and are identical to the sharpest one found in [144] for the case of unweighted complete networks. The only drawback of our condition is that each oscillator is required to be connected to a minimum number of other oscillators. Finally, we perform some simulations using the anatomical brain network data obtained in

[155]; the simulation results show how our theoretical findings may reveal a possible mechanism that gives rise to various patterns of synchrony detected in empirical data of the human brain [156].

Outline

The remainder of this chapter is structured as follows. We introduce the model on the two-level networks and formulate the problem of partial phase cohesiveness in Section 5.2. The first result is obtained by using the incremental 2-norm in Section 5.3. Section 5.4 introduces the notion of generalized complement graphs and derives the main result utilizing the incremental ∞-norm. Some simulations are performed in Section 5.5. Concluding remarks appear in Section 5.6.

5.2 Problem Formulation

We consider a network of $M > 1$ communities, each of which consists of $N \geq 1$ fully connected heterogeneous Kuramoto oscillators. The graph of the network, which describes which community is interconnected to which other communities, is in general not a complete graph. The dynamics of the oscillators are described by

$$\dot{\theta}_i^p = \omega_i^p + K^p \sum\nolimits_{n=1}^{N} \sin(\theta_n^p - \theta_i^p) + \sum\nolimits_{q=1}^{M} \sum\nolimits_{n=1}^{N} a_{i,n}^{p,q} \sin(\theta_n^q - \theta_i^p), \tag{5.1}$$

for any $p \in \mathcal{T}_M := 1, \ldots, m$ and any $i \in \mathcal{T}_N := 1, \ldots, n$, where $\theta_i^p \in \mathbb{S}^1$ and $\omega_i^p \geq 0$ represent the phase and natural frequency of the ith oscillator in the pth community, respectively. Here, the uniform coupling strength of all the edges in the complete graph of the pth community is denoted by $K^p > 0$, which we refer to as the *intra-community* coupling strengths. The coupling strengths $a_{i,n}^{p,q}$, which we call the *inter-community* coupling strengths, satisfy $a_{i,n}^{p,q} > 0$ if $i \neq n$ and there is a connection between the ith oscillator in the pth community and the jth oscillator in the qth community, and $a_{i,n}^{p,q} = 0$ otherwise. We define the inter-community coupling matrices by $A^{p,q} := [a_{i,j}^{p,q}]_{N\times N} \in \mathbb{R}^{N\times N}$, and each satisfies $A^{p,q} = A^{q,p}$.

Remark 5.1. *Our analysis later on applies to the case when each community has a different network topology and even when the numbers of oscillators in the communities are different. However, for the sake of notational simplicity, we assume that each community is connected by a uniformly weighted complete network.*

The Kuramoto oscillator network model (5.1) is used in [146] to study synchronization phenomena of the human brain. Along this line of research and motivated by brain research data, we focus on studying the widely observed but still not well understood phenomenon for networks of communities of Kuramoto oscillators, the so called *partial phase cohesiveness*, in which some but not all of the oscillators have close phases. To facilitate the discussion of some properties of interest for a subset of communities in the network, we use $\mathcal{T}_r = \{1, \dots, r\}$, $1 \le r \le M$, to denote a set of chosen communities with the aim to investigate how phase cohesiveness can occur among these r communities. We then define the following set to capture the situation when the oscillators in the communities in $\mathcal{T}_r$ reach phase cohesiveness.

Definition 5.1. *Let $\theta \in \mathbb{T}^{MN}$ be a vector composed of the phases of all N oscillators in all M communities. Then, for a given $\mathcal{T}_r$ and $\varphi \in [0, \pi]$, define the partial phase cohesiveness set:*

$$\mathcal{S}_\infty(\varphi) := \left\{ \theta \in \mathbb{T}^{MN} : \max_{i,j \in \mathcal{T}_N, k,l \in \mathcal{T}_r} |\theta_i^k - \theta_j^l| \le \varphi \right\}. \tag{5.2}$$

Note that φ describes a level of phase cohesiveness since it is the maximum pairwise phase difference of the oscillators in $\mathcal{T}_r$. The smaller φ is, the more cohesive the phases are. All the phases in $\mathcal{T}_r$ are identical when $\varphi = 0$, which is called *partial phase synchronization*, and this can only happen when all the oscillators have the same natural frequency. In this chapter, we allow the natural frequencies to be different, and are only interested in the cases when phase differences in $\mathcal{T}_r$ are small enough. We say that partial phase cohesiveness can take place in $\mathcal{T}_r$ if the solution $\theta : \mathbb{R}_{\ge 0} \to \mathbb{T}^{MN}$ to the system (5.1) asymptotically converges to this set $\mathcal{S}_\infty(\varphi)$ for some $\varphi \in [0, \pi/2)$. We call the particular case when $\mathcal{T}_r = \mathcal{T}_M$ *complete* phase cohesiveness, which is also called practical phase synchronization in [78]. In the rest of the chapter, we study the partial phase cohesiveness by investigating how a solution $\theta(t)$ can asymptotically converge to the set $\mathcal{S}_\infty(\varphi)$ and also by estimating the value of φ.

Let $\mathcal{G}_r = (\mathcal{V}_r, \mathcal{E}_r, Z)$ denote the subgraph composed of the nodes in the communities contained in $\mathcal{T}_r$ and the edges connecting pairs of them. The weighted adjacency matrix of this subgraph $Z := [z_{ij}]_{Nr \times Nr} \in \mathbb{R}^{Nr \times Nr}$ is then given by

$$Z := \begin{bmatrix} K^1 C & A^{1,2} & \cdots & A^{1,r} \\ A^{1,2} & K^2 C & \cdots & A^{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ A^{1,r} & A^{2,r} & \cdots & K^r C \end{bmatrix}, \tag{5.3}$$

where $C = [c_{ij}]_{N \times N} \in \mathbb{R}^{N \times N}$ is the adjacency matrix of a complete graph with N nodes, where $c_{ij} = 1$ for $i \ne j$, and $c_{ij} = 0$ otherwise (recall that $A^{p,q}$ is symmetric). Let $D := \operatorname{diag}(Z \mathbf{1}_{Nr})$, then the Laplacian matrix of the graph $\mathcal{G}_r$ is $L := D - Z$. Let $\lambda_2(L)$ denote the second smallest eigenvalue of L, which is always referred to as the *algebraic connectivity* [110].

Let $\theta^p := [\theta_1^p, \dots, \theta_N^p]^\top$, $\omega^p := [\omega_1^p, \dots, \omega_N^p]^\top$ for all $p \in \mathcal{T}_M$. As we are only interested in the behavior of the oscillator in $\mathcal{G}_r$, we define $x := [\theta^{1\top}, \dots, \theta^{r\top}]^\top$, and $\varpi := [\omega^{1\top}, \dots, \omega^{r\top}]^\top$. For $i \in \mathbb{N}$, we define $\mu(i) := \lceil i/N \rceil$ and $\rho(i) := i - N \cdot \lfloor i/N \rfloor$. By using these new notations, from (1), the dynamics of the oscillators on $\mathcal{G}_r$ can be rewritten as

$$\begin{aligned}\dot{x}_i = \varpi_i + &\sum\nolimits_{n=1}^{Nr} z_{i,n} \sin(x_n - x_i) \\ &+ \sum\nolimits_{q=r+1}^{M} \sum\nolimits_{n=1}^{N} a_{\rho(i),n}^{\mu(i),q} \sin(\theta_n^q - x_i),\end{aligned} \tag{5.4}$$

where $i \in \mathcal{T}_{Nr}$. The first summation term describes the interactions among the oscillators within the subset of communities $\mathcal{T}_r$, and the second one represents the interactions from the outside of $\mathcal{T}_r$ to the oscillators in $\mathcal{T}_r$. In order to study the phase cohesiveness of the oscillators in $\mathcal{G}_r$, we then look into the dynamics of pairwise phase differences, given by

$$\begin{aligned}\dot{x}_i - \dot{x}_j =& \varpi_i - \varpi_j \\ &+ \sum_{n=1}^{Nr} \left(z_{i,n} \sin\left(x_n - x_i\right) - z_{jn} \sin\left(x_n - x_j\right)\right) + u_{ij}, i, j \in \mathcal{T}_{Nr},\end{aligned} \tag{5.5}$$

where

$$u_{ij} := \sum_{q=r+1}^{M} \sum_{n=1}^{N} \Big(a_{\rho(i),n}^{\mu(i),q} \sin(\theta_n^q - x_i) - a_{\rho(j),n}^{\mu(j),q} \sin(\theta_n^q - x_j) \Big).$$

Let $\mathbf{u}_r := [u_{ij}]_{i<j} \in \mathbb{R}^{Nr(Nr-1)/2}$. The incremental dynamics (5.5) play crucial roles in what follows. In the next two sections, we study partial phase cohesiveness in $\mathcal{G}_r$ with the help of (5.5) using the incremental 2-norm or ∞-norm (which will be introduced subsequently). To analyze phase cohesiveness, the techniques of ultimate boundedness theorem [157, Theorem 4.18] will be employed.

5.3 Incremental 2-Norm

In this section, we introduce the incremental 2-norm, and use it as a metric to study partial phase cohesiveness. According to Definition 5.1, we observe that a partially phase cohesive solution across $\mathcal{T}_r$ should satisfy $|x_i - x_j| \le \varphi$ for all $i, j \in \mathcal{T}_{Nr}$. A quadratic Lyapunov function has been widely used to study phase cohesiveness even when the graph is not complete [78, 145, 149, 151], which is defined by

$$W(x) := \frac{1}{2} \|B_c^\top x\|_2^2, \tag{5.6}$$

where $B_c \in \mathbb{R}^{Nr \times (Nr(Nr-1)/2)}$ is the incidence matrix of the complete graph. It is also known as the incremental 2-norm metric of phase cohesiveness. For a given $\gamma \in [0, \pi)$, define

$$\mathcal{S}_2(\gamma) := \left\{\theta \in \mathbb{T}^{MN} : \|B_c^\top x\|_2 \leq \gamma\right\}. \tag{5.7}$$

Note that $\mathcal{S}_2(\gamma) \subseteq \mathcal{S}_\infty(\gamma)$ for any given $\gamma \in [0, \pi)$. Different from the existing results that apply to complete cohesiveness taking place among all the oscillators in the networks [78, 145, 149, 151], we have studied partial phase cohesiveness in [104] using the incremental 2-norm metric. Compared to [104] , we consider more general inter-community coupling structures as stated in Section 5.2.

Let $\hat{B}_c = |B_c|$ be the element-wise absolute value of the incidence matrix B_c. Let $d_i^{\mathrm{ex}} = \sum_{m=r+1}^{M} \sum_{n=1}^{N} a_{\rho(i),n}^{\mu(i),m}$ for all $i \in \mathcal{T}_{Nr}$, and denote $D^{\mathrm{ex}} := [d_1^{\mathrm{ex}}, \dots, d_{Nr}^{\mathrm{ex}}]^\top$. Now let us provide our first result on partial phase cohesiveness on incremental 2-norm. A similar result can be found in [150, Theorem 4.4]. Difference from it, we consider a two-level network, i.e., communities of oscillators, and study the partial phase cohesiveness.

Theorem 5.1. *Assume that the algebraic connectivity of $\mathcal{G}_r$ is greater than the critical value specified by*

$$\lambda_2(L) > \|B_c^\top \varpi\|_2 + \|\hat{B}_c^\top D^{\mathrm{ex}}\|_2. \tag{5.8}$$

Then, each of the following equations

$$\lambda_2(L)\sin(\gamma_s) - \|\hat{B}_c^\top D^{\mathrm{ex}}\|_2 = \|B_c^\top \varpi\|_2, \tag{5.9}$$

$$(\pi/2)\lambda_2(L)\,\mathrm{sinc}(\gamma_m) - \|\hat{B}_c^\top D^{\mathrm{ex}}\|_2 = \|B_c^\top \varpi\|_2, \tag{5.10}$$

has a unique solution, $\gamma_s \in [0, \pi/2)$ and $\gamma_m \in (\pi/2, \pi]$, respectively, where $\mathrm{sinc}(\eta) = \sin(\eta)/\eta$ *for any $\eta \in \mathbb{S}^1$. Furthermore, the following statements hold:*

(i) for any $\gamma \in [\gamma_s, \gamma_m]$, $\mathcal{S}_2(\gamma)$ is a positively invariant set of the system (5.1)*;*

(ii) for any $\gamma \in [\gamma_s, \gamma_m)$, the solution to (5.1) *starting from any $\theta(0) \in \mathcal{S}_2(\gamma)$ converges to the set $\mathcal{S}_2(\gamma_s)$.*

Proof. First, let us show the existence and uniqueness of the solutions to (5.9) and (5.10). The equality (5.9) can be written as $\sin(\gamma_s) = (\|\hat{B}_c^\top D^{\mathrm{ex}}\|_2 + \|B_c^\top \varpi\|_2)/\lambda_2(L) < 1$, which apparently has a unique solution in $[0, \pi/2)$, which is given by

$$\gamma_s = \arcsin\left(\frac{\|\hat{B}_c^\top D^{\mathrm{ex}}\|_2 + \|B_c^\top \varpi\|_2}{\lambda_2(L)}\right).$$

To show there is a unique solution to (5.10), define the function of $\gamma \in [\pi/2, \pi]$

$$f(\gamma) = (\pi/2)\lambda_2(L)\ \mathrm{sinc}(\gamma) - \left(\|\hat{B}_c^\top D^{\mathrm{ex}}\|_2 + \|B_c^\top \varpi\|_2\right). \tag{5.11}$$

Since $\mathrm{sinc}(\gamma)$ is an decreasing functions on $[\pi/2, \pi]$, respectively, we know $f(\gamma)$ is monotonically decreasing. Moreover, we observe that

$$f(\pi/2) = \lambda_2(L) - \left(\|\hat{B}_c^\top D^{\mathrm{ex}}\|_2 + \|B_c^\top \varpi\|_2\right) > 0,$$

and $f(\pi) = -\left(\|\hat{B}_c^\top D^{\mathrm{ex}}\|_2 + \|B_c^\top \varpi\|_2\right) < 0$, then one can deduce that there is a unique $\gamma_m \in (\pi/2, \pi)$ such that $f(\gamma_m) = 0$. This means that (5.10) has a unique solution in $(\pi/2, \pi]$.

Next, we show that for any $\gamma \in [\gamma_s, \gamma_m]$, the set $\mathcal{S}_2(\gamma)$ is positively invariant. Choose $W(x)$ in (5.6) as a Lyapunov candidate. Similar to the proof of [151, Theorem 4.6], we take the time derivative of $W(x)$ along the solution to (5.1) and obtain

$$\dot{W}(x) \le x^\top B_c B_c^\top \varpi - \ \mathrm{sinc}(\gamma) N r x^\top B_c\ \mathrm{diag}(\{z_{ij}\}_{i<j}) B_c^\top x + x^\top B_c \mathbf{u}_r.$$

From [150, Lemma 7], it holds that $x^\top B_c\ \mathrm{diag}(\{z_{ij}\}_{i<j}) B_c^\top x \ge \lambda_2(L)\|B_c^\top x\|_2^2/(Nr)$. From the definition of $\mathbf{u}_r$, one can evaluate that $\|\mathbf{u}_r\|_2 \le \|\hat{B}_c^\top D^{\mathrm{ex}}\|_2$. As a consequence, we arrive at

$$\begin{aligned}\dot{W}(x) &\le x^\top B_c B_c^\top \varpi - \lambda_2(L)\ \mathrm{sinc}(\gamma)\|B_c^\top x\|_2^2 + \|B_c^\top x\|_2 \|\hat{B}_c^\top D^{\mathrm{ex}}\|_2 \\ &\le \|B_c^\top x\|_2 \left(\|B_c^\top \varpi\|_2 + \|\hat{B}_c^\top D^{\mathrm{ex}}\|_2 - \lambda_2(L)\ \mathrm{sinc}(\gamma)\|B_c^\top x\|_2\right).\end{aligned}$$

One can obtain that $\dot{W}(x) \le 0$ if $x \in \mathcal{S}_2(\gamma)$ for any $\gamma \in [\gamma_s, \gamma_m]$, which proves that the set $\mathcal{S}_2(\gamma)$ is positively invariant.

Finally, we prove the asymptotic convergence stated in (ii). In fact, one can show that $\dot{W}(x) < 0$ if $x \in \mathcal{S}_2(\gamma)$ with $\gamma \in (\gamma_s, \gamma_m)$. This means that starting from any point in $\mathcal{S}_2(\gamma)$ with $\gamma \in (\gamma_s, \gamma_m)$, the solution converges to $\mathcal{S}_2(\gamma_s)$ asymptotically. Since we have shown in (i) that $S_2(\gamma_s)$ is positively invariant, we know the solution remain in it if starting from it. The proof is complete. $\square$

Suppose there is only 1 oscillator in each community (i.e., $N = 1$), and it hold that $\mathcal{T}_r = \mathcal{T}_M$, $D_o = 0$, Theorem 5.1 reduces to the best-known result on the incremental 2-norm in single level networks [151, Theorem 4.6]. One observes that the established result in Theorem 5.1 is quite restrictive if the number of oscillators is large because we use the incremental 2-norm metric. First, the critical value $\lambda_2(L)$ is quite conservative since the right side of (5.8) depends greatly on the number of oscillators in the network. Second, the region of attraction we have identified in Theorem 5.1(ii) is quite small. To ensure $\|B_c^\top x(0)\|_2 < \gamma < \pi$, the initial phases are required to be nearly identical. In the next section, we use incremental-∞ norm, aiming at obtaining less conservative results.

5.4 Incremental ∞-Norm

In this section, we seek to obtain some less conservative conditions than the ones in the previous section, for partial phase cohesiveness in networks of Kuramoto-oscillator networks described by (5.1). Instead of incremental 2-norm, we employ incremental ∞-norm in what follows.

5.4.1 Main Results

We take the following function as a Lyapunov candidate for partial phase cohesiveness:

$$V(x) = \|B_c^\top x\|_\infty, \tag{5.12}$$

which is also referred to as the incremental ∞-norm metric. It evaluates the maximum of the pairwise phase differences, and thus does not depend on the number of oscillators. Then, one notices that $\mathcal{S}_\infty(\varphi)$ in (5.2) can be rewritten into

$$\mathcal{S}_\infty(\varphi) = \left\{\theta \in \mathbb{T}^{MN} : V(x) = \|B_c^\top x\|_\infty \leq \varphi\right\}. \tag{5.13}$$

To the best of the authors' knowledge, the incremental ∞-norm has not been used to established explicit conditions for phase cohesiveness analysis in weighted complete or non-complete networks, although some implicit conditions ensuring local stability of phase-locked solutions, such as [152, 153], have been obtained. To obtain explicit conditions by using the incremental ∞-norm, it is always required that the oscillators in a network have the same coupling structures (see [78, Theorem 6.6], [144]). The oscillators in a non-complete network always have distinct coupling structures, which makes the analysis quite challenging. To overcome the challenge, we introduce the notion of the *generalized complement* graph as follows, which can be viewed as an extension of the complement graph of an unweighted graph.

Definition 5.2. *Consider the weighted undirected graph $\mathcal{G}$ with the weighted adjacency matrix Z, and let K_m be the maximum coupling strength of its edges. Let A_c denote the unweighted adjacency matrix of the complete graph with the same node set as $\mathcal{G}$. We say $\bar{\mathcal{G}}$ is the* generalized complement *graph of $\mathcal{G}$ if the following two are satisfied: 1) it has the same node set as $\mathcal{G}$; 2) the weighted adjacency matrix is given by $\bar{Z} := K_m A_c - Z$.*

Let K_m be the maximum element in the matrix (5.3), and A_c the unweighted adjacency matrix of the complete graph consisting of the same node set as $\mathcal{G}_r$. Then $\bar{Z} = K_m A_c - Z$ is the weighted adjacency matrix of the generalized complement graph $\bar{\mathcal{G}}_r$. In order to enable the analysis using the incremental ∞-norm, we then rewrite (5.4) into the form taking the difference between the complete graph and the generalized complement graph

$$\dot{x}_i = \varpi_i - K_m \sum_{n=1}^{Nr} \sin(x_i - x_n) + \sum_{n=1}^{Nr} \bar{z}_{i,n} \sin(x_i - x_n) + \sum_{q=r+1}^{M} \sum_{n=1}^{N} a_{\rho(i),n}^{\mu(i),q} \sin(\theta_n^q - x_i),$$

where $i \in \mathcal{T}_{Nr}$. Accordingly, the incremental dynamics (5.5) can be rearranged into

$$\dot{x}_i - \dot{x}_j = \varpi_i - \varpi_j - K_m \sum_{n=1}^{Nr} (\sin(x_i - x_n) - \sin(x_j - x_n)) + \sum_{n=1}^{Nr} (\bar{z}_{in} \sin(x_i - x_n) - \bar{z}_{jn} \sin(x_j - x_n)) + u_{ij}, \tag{5.14}$$

where $i, j \in \mathcal{T}_{Nr}$, and u_{ij} is given by (5.5).

In the incremental 2-norm analysis, the algebraic connectivity plays an important role since it relates to the matrix induced 2-norm. When we proceed with the incremental ∞-norm analysis, the corresponding ideas in terms of the matrix induced ∞-norm are introduced subsequently. Let $\bar{D}_m := \|\bar{Z}\|_\infty$, and call it the *maximum degree* of the generalized complement graph $\bar{\mathcal{G}}_r$. Let $D_s^{\text{in}} := \min_{i=1,\ldots,Nr} \sum_{j=1}^{Nr} z_{ij}$, which we call the *minimum internal* degree of $\mathcal{G}_r$. Likewise, let the maximum external degree be $D_m^{\text{ex}} := \|D^{\text{ex}}\|_\infty$. The following proposition provides a relation between the maximum degree of $\bar{\mathcal{G}}_r$ and minimum internal degree of $\mathcal{G}_r$.

Proposition 5.1. *Given the graph $\mathcal{G}_r$, its minimum degree and the maximum degree of the associated generalized complement graph satisfy $\bar{D}_m = (Nr-1)K_m - D_s^{\text{in}}$.*

Proof. From $\bar{Z} = K_m A_c - Z$, the following holds:

$$\bar{z}_{ij} = \begin{cases} 0, & i = j \\ K_m - z_{ij}, & i \neq j. \end{cases}$$

By taking the summation with respect to j, we have

$$\sum\nolimits_{j=1}^{Nr} \bar{z}_{ij} = (Nr-1)K_m - \sum\nolimits_{j=1}^{Nr} z_{ij},$$

where $z_{ii} = 0$. From the definition of the ∞-norm of the matrix and the fact that all the elements of $\bar{Z}$ and Z are non-negative, it follows that

$$\begin{aligned}\bar{D}_m = \|\bar{Z}\|_\infty &= \max_{i=1,\dots,Nr}\left((Nr-1)K_m - \sum\nolimits_{j=1}^{Nr} z_{ij}\right)\\ &= (Nr-1)K_m - \min_{i=1,\dots,Nr}\sum\nolimits_{j=1}^{Nr} z_{ij}\\ &= (Nr-1)K_m - D_s^{\text{in}}.\end{aligned}$$

The proof is complete. □

Now we provide our main result in this section.

Theorem 5.2. *Suppose that the minimum internal degree D_s^{in} is greater than the critical value specified by*

$$D_s^{\text{in}} > \frac{\|B_c^\top \varpi\|_\infty + 2D_m^{\text{ex}} + (Nr-2)K_m}{2}. \tag{5.15}$$

Then, there exist two solutions, $\varphi_s \in [0, \pi/2)$ and $\varphi_m \in (\pi/2, \pi]$, to the equation $\|B_c^\top \varpi\|_\infty + 2D_m^{\text{ex}} + 2(Nr-1)K_m - 2D_s^{\text{in}} = NrK_m \sin\varphi$, which are given by

$$\varphi_s = \arcsin\left(\frac{\|B_c^\top \varpi\|_\infty + 2D_m^{\text{ex}} + 2(Nr-1)K_m - 2D_s^{\text{in}}}{NrK_m}\right), \tag{5.16}$$

$$\varphi_m = \pi - \varphi_s, \tag{5.17}$$

respectively. Furthermore, the following statements hold:

(i) For any $\varphi \in [\varphi_s, \varphi_m]$, $\mathcal{S}_\infty(\varphi)$ is a positively invariant set of the system (5.1)*;*

(ii) For every initial condition $\theta(0) \in \mathbb{T}^{MN}$ such that $\varphi_s < \|B_c^\top x(0)\|_\infty < \varphi_m$, the solution $\theta(t)$ to (5.1) *converges to $\mathcal{S}_\infty(\varphi_s)$.*

Proof. We first prove (i) by showing that the upper Dini derivative of $V(x(t))$ along the solution to (5.1),

$$D^+V(x(t)) = \limsup_{\tau\to 0^+} \frac{V(x(t+\tau)) - V(x(t))}{\tau},$$

satisfies $D^+V(x(t)) \le 0$ when $V(x(t)) = \varphi$. Define $\mathcal{I}'_M(t) := \{i : x_i(t) = \max_{j\in\mathcal{V}_r} x_j(t)\}$ and $\mathcal{I}'_S(t) := \{i : x_i(t) = \min_{j\in\mathcal{V}_r} x_j(t)\}$. Then one can rewrite (5.12) into

$$V(x(t)) = |x_p(t) - x_q(t)|,\ \forall p \in \mathcal{I}'_M(t), \forall q \in \mathcal{I}'_S(t).$$

Let $\mathcal{I}_M(t) := \{i : \dot{x}_i(t) = \max_{j\in\mathcal{I}'_M} \dot{x}_j(t)\}$ and $\mathcal{I}_S(t) := \{i : \dot{x}_i(t) = \min_{j\in\mathcal{I}'_S} \dot{x}_j(t)\}$. Then the upper Dini Derivative is

$$D^+V(x(t)) = \dot{x}_m(t) - \dot{x}_s(t)$$

for all $m \in \mathcal{I}_M(t)$ and $s \in \mathcal{I}_S(t)$. It follows from (5.14) that

$$\begin{aligned} D^+V(x(t)) =& \dot{x}_m - \dot{x}_s \\ =& \varpi_m - \varpi_s - K_m \sum_{n=1}^{Nr} (\sin(x_m - x_n) - \sin(x_s - x_n)) \\ &+ \sum_{n=1}^{Nr} (\bar{z}_{mn} \sin(x_m - x_n) - \bar{z}_{sn} \sin(x_s - x_n)) + u_{ms} \end{aligned}$$

By using the trigonometric identity $\sin(x) - \sin(y) = 2 \sin \frac{x-y}{2} \cos \frac{x+y}{2}$, we have

$$\begin{aligned} D^+V(x(t)) =& \varpi_m - \varpi_s \\ &- 2K_m \sum_{n=1}^{Nr} \sin\left(\frac{x_m - x_s}{2}\right) \cos\left(\frac{x_m - x_n}{2} - \frac{x_n - x_s}{2}\right) \\ &+ \sum_{n=1}^{Nr} (\bar{z}_{mn} \sin(x_m - x_n) - \bar{z}_{sn} \sin(x_s - x_n)) + u_{ms}. \end{aligned}$$

Since for any $\varphi \in [0, \pi]$, $V(x(t)) = \varphi$ implies that $x_m(t) - x_s(t) = \varphi$, it follows that

$$-\frac{\varphi}{2} \leq \frac{x_m(t) - x_j(t)}{2} - \frac{x_j(t) - x_s(t)}{2} \leq \frac{\varphi}{2}.$$

Consequently, from the double-angle formula $\sin(\varphi) = 2 \sin(\varphi/2) \cos(\varphi/2)$, it holds that

$$\begin{aligned} D^+V(x(t)) \leq& \varpi_m - \varpi_s - NrK_m \sin(\varphi) \\ &+ \sum_{n=1}^{Nr} (\bar{z}_{mn} \sin(x_m - x_n) - \bar{z}_{sn} \sin(x_s - x_n)) + u_{ms}. \end{aligned}$$

Recalling the definitions of $\bar{D}_m$ and D_m^{ex}, one knows that

$$\left| \sum\nolimits_{n=1}^{Nr} (\bar{z}_{mn} \sin(x_m - x_n) - \bar{z}_{sn} \sin(x_s - x_n)) \right| \leq 2\bar{D}_m$$

and $|u_{ms}| \leq 2D_m^{\text{ex}}$. As a consequence, from $\varpi_m - \varpi_s \leq \|B_c^\top \varpi\|_\infty$ and Proposition 5.1, we have

$$D^+V(x(t)) \leq \varpi_m - \varpi_s - NrK_m \sin(\varphi) + 2\bar{D}_m + 2D_m^{\text{ex}} \leq g(\varphi), \tag{5.18}$$

where

$$\begin{aligned} g(y) := \|B_c^\top \varpi\|_\infty &- NrK_m \sin(y) \\ &+ 2\big((Nr - 1)K_m - D_s^{\text{in}}\big) + 2D_m^{\text{ex}}. \end{aligned}$$

Now, we aim to find a subinterval in $[0, \pi]$ such that $g(\varphi) \leq 0$ for any φ in it. If the condition (5.15) holds, then $g(\pi/2) < 0$ and thus there exists such a subinterval around $\varphi = \pi/2$. Moreover, $\sin(y)$ is an increasing and decreasing function in $[0, \pi/2]$ and $[\pi/2, \pi]$, respectively. Then there always exist two points $y_1 \in [0, \pi/2), y_2 \in (\pi/2, \pi]$ such that $g(y_1) = g(y_2) = 0$. These two points y_1 and y_2 are nothing but φ_s in (5.16) and φ_m in (5.17), respectively. In summary, for any $\varphi \in [\varphi_s, \varphi_m]$, $D^+V(x(t)) \leq 0$ when $V(x(t)) = \varphi$, which implies that $\mathcal{S}_\infty(\varphi)$ is positively invariant.

Next, we prove (ii). Given $x(0)$, it follows from (5.18) that for any t there exists $\gamma(t)$ satisfying $\gamma(t) = V(x(t))$ such that

$$\begin{aligned} D^+V(x(t)) \leq \|B_c^\top \varpi\|_\infty - NrK_m \sin(\gamma(t)) \\ + 2\left((Nr-1)K_m - D_s^{\text{in}}\right) + 2D_m^{\text{ex}}. \end{aligned} \tag{5.19}$$

Recalling that the initial condition satisfies that $\varphi_s < \|B_c^\top x(0)\|_\infty < \varphi_m$, one knows that $\varphi_s < \gamma(0) < \varphi_m$. Then the right side of (5.19) is negative, and thus the strict inequality $D^+(V(x(0))) < 0$ holds. From $t = 0$ on, $D^+(V(x(0))) < 0$ for all t such that $\varphi_s < \gamma(t) < \varphi_m$, and $D^+(V(x(0))) \leq 0$ if $\gamma(t) = \varphi_s$. One can then conclude that $\theta(t)$ converges to $\mathcal{S}_\infty(\varphi_s)$. □

The following proposition provides a necessary condition for K_m such that (5.15) can be satisfied.

Proposition 5.2. *Suppose that D_s^{in} satisfies the condition* (5.15)*, then K_m satisfies the following inequality*

$$K_m > \frac{\|B_c^\top \varpi\|_\infty + 2D_m^{\text{ex}}}{Nr}. \tag{5.20}$$

Proof. If the condition (5.15) is satisfied, we have

$$\|B_c^\top \varpi\|_\infty + 2D_m^{\text{ex}} + (Nr-2)K_m < 2D_s^{\text{in}}.$$

One notices that $D_s^{\text{in}} \leq (Nr-1)K_m$ since there are at most $Nr - 1$ edges connecting each node, and the coupling strength of each of them is at most K_m. It then follows that

$$\|B_c^\top \varpi\|_\infty + 2D_m^{\text{ex}} + (Nr-2)K_m < 2(Nr-1)K_m,$$

which implies $K_m > \left(\|B_c^\top \varpi\|_\infty + 2D_m^{\text{ex}}\right)/Nr$. □

In the study of synchronization or phase cohesiveness, the network is usually required to be connected. The following proposition shows that the condition (5.15) implies the connectedness of the graph $\mathcal{G}_r$ since from the condition (5.15) the minimum internal degree satisfies $D_s^{\text{in}} > (Nr-2)K_m/2$.

Proposition 5.3. *Consider a graph $\mathcal{G}$ consisting of n nodes. Let K be the maximum coupling strength of its edges. Suppose the minimum degree of the nodes satisfies $D_s > (n-2)K/2$, and then the graph $\mathcal{G}$ is connected.*

Proof. We prove this proposition by contradiction. We assume that the graph is not connected, and let i^*, j^* be any two nodes that belongs to two isolated connected components $\mathcal{G}_{i^*}, \mathcal{G}_{j^*}$, respectively. Let the numbers of nodes that are connected to i^*, j^* be n_{i^*} and n_{j^*}, respectively. The degree of i^*, denoted by D_{i^*}, satisfies

$$D_s \leq D_{i^*} \leq n_{i^*} K,$$

It follows from the assumption $D_s > (n-2)K/2$ that $n_{i^*} > (n-2)/2$. which implies that the number of nodes in $\mathcal{G}_{i^*}$ is strictly greater than $n_{i^*} + 1 = n/2$. Likewise, one can show the number of nodes in $\mathcal{G}_{j^*}$ is strictly greater than $n_{j^*} + 1 = n/2$. Then the total number of nodes in these two isolated connected components is strictly greater $n_{i^*} + n_{j^*} + 2 = n$, which implies the number of node in the graph $\mathcal{G}$ is greater than n. This is a contradiction, and thus the network $\mathcal{G}$ is connected. $\square$

5.4.2 Comparisons with Existing results

We first compare the results in Theorems 5.1 and 5.2. It is worth mentioning that the condition in Theorem 5.2 is less dependent on the number of nodes Nr than that in Theorem 5.1 in most cases. In sharp contrast to $\|B_c^\top \varpi\|_2$ and $\|\hat{B}_c^\top D^{\mathrm{ex}}\|_2$ in (5.8), both $\|B_c^\top \varpi\|_\infty$ and D_m^{ex} in (5.15) are independent of Nr. Specifically, if we take δ_s, δ_m to be the smallest and largest elements in $|B_c^\top \varpi|$, respectively, it holds that $\delta_s \sqrt{Nr(Nr-1)/2} \leq \|B_c^\top \varpi\|_2 \leq \delta_m \sqrt{Nr(Nr-1)/2}$. A similar inequality holds for $\|\hat{B}_c^\top D^{\mathrm{ex}}\|_2$. Then, one can observe that $\|B_c^\top \varpi\|_2 + \|\hat{B}_c^\top D^{\mathrm{ex}}\|_2$ in (5.8) can be much larger than $(Nr-2)K_m/2$ in (5.15) if Nr is large. More importantly, $\mathcal{S}_\infty(\varphi)$ is much larger than $\mathcal{S}_2(\varphi)$ for the same φ, which implies that the domain of attraction we estimated in Theorem 5.2 is much larger than that in Theorem 5.1. Therefore, the convergence to a partially phase cohesive solution can be guaranteed by Theorem 2 even if the initial phases are not nearly identical.

On the other hand, the condition (5.8) can be less conservative than (5.15), but one would require the natural frequencies to be quite homogeneous, and meanwhile the external connections to be very weak in comparison with K_m. In addition, it can be observed from Proposition 5.3 that each node in $\mathcal{G}_r$ is required to have more than $(Nr-2)/2$ neighbors from the condition (5.15). In this sense, the condition (5.8) is less conservative since it only requires $\mathcal{G}_r$ to be connected.

The following corollary provides a sufficient condition that is independent of the network scale for the partial phase cohesiveness in a dense non-complete subnetwork

$\mathcal{G}_r$. It is certainly less conservative than its counterpart based on the incremental 2-norm.

Corollary 5.1. *Suppose each node in $\mathcal{G}_r$ is connected by at least n_e edges, where $n_e > (Nr-2)/2$, and all the edges have the same weight K. Assume that*

$$K > \frac{\|B_c^\top \varpi\|_\infty + 2D_m^{\text{ex}}}{2n_e - (Nr-2)}, \tag{5.21}$$

then the statements (i) and (ii) in Theorem 5.2 hold.

The proof follows straightforwardly by letting $D_s^{\text{in}} = n_e K$ and $K_m = K$. Since $2n_e - (Nr-2) \geq 1$, any K satisfying $K > \|B_c^\top \varpi\|_\infty + 2D_m^{\text{ex}}$ satisfies the condition (5.21) for any Nr.

Next, we compare our results with the previously-known works in the literature [144,151]. Since in the existing results, researchers usually deal with one-level networks, and study the complete phase cohesiveness, we assume, in what follows, that there is only one oscillator in each community in our two-level network, and let the set $\mathcal{T}_r$ in which we want to synchronize the oscillators be the entire community set $\mathcal{T}_M$. Then we obtain the following two corollaries.

Corollary 5.2. *Given an undirected graph $\mathcal{G}$, assume that the following condition is satisfied*

$$D_s^{\text{in}} > \frac{\|B_c^\top \varpi\|_\infty + (M-2)K_m}{2}, \tag{5.22}$$

then the solutions to the equation $\|B_c^\top \varpi\|_\infty + 2D_m^{\text{ex}} + 2(Nr-1)K_m - 2D_s^{\text{in}} = NrK_m \sin\varphi$, $\varphi_s \in [0, \pi/2)$ and $\varphi_m \in (\pi/2, \pi]$, are given by

$$\varphi_s = \arcsin\left(\frac{\|B_c^\top \varpi\|_\infty + 2(M-1)K_m - 2D_s^{\text{in}}}{MK_m}\right),$$
$$\varphi_m = \pi - \varphi_s.$$

Furthermore, the following two statements hold:

(i) for any $\varphi \in [\varphi_s, \varphi_m]$, the set $\mathcal{S}_\infty(\varphi)$ is positively invariant;

(ii) for every initial condition $x(0)$ such that $\varphi_s < |B_c^\top x(0)\|_\infty < \varphi_m$, the solution $\theta(t)$ converges to $\mathcal{S}_\infty(\varphi_s)$ asymptotically.

This corollary follows from Theorem 5.2 by letting $N = 1$, $r = M$ and $D_m^{\text{ex}} = 0$. In this case, $K_m = \max_{i,j \in \mathcal{T}_M} a_{ij}$ is the maximum coupling strength in $\mathcal{G}$. Compared to the best-known result on the incremental 2-norm [151, Theorem 4.6], the result established in Corollary 5.2 is often less conservative. The explanation is similar to what we provide when we compare Theorem 5.2 with Theorem 5.1. Assuming the network is complete, we obtain the following corollary.

Corollary 5.3. *Suppose the graph $\mathcal{G}$ is complete, and the coupling strength is K/M. Assume that the coupling strength satisfies $K > \|B_c^\top \varpi\|_\infty$. Then, φ_s and φ_m become*

$$\varphi_s = \arcsin\left(\frac{\|B_c^\top \varpi\|_\infty}{K}\right), \qquad \varphi_m = \pi - \varphi_s.$$

Furthermore, the statement (i) and (ii) in Corollary 5.2 hold.

This result is actually identical to the well-known one found in [144, Theorem 4.1], which presents phase cohesiveness on complete graphs with arbitrary distributions of natural frequencies.

5.5 Numerical Examples

In this section, we provide two examples to show the validity of the obtained results (see Example 1), and also to show their applicability to brain networks (see Example 2). We first introduce the order parameter as a measure of phase cohesiveness [62], which is defined by $re^{i\psi} = \frac{1}{n}\sum_{i=1}^{n} e^{i\theta_j}$. The value of r ranges from 0 to 1. The greater the r is, the higher the degree of phase cohesiveness becomes. If $r = 1$, the phases are completely synchronized; on the other hand, if $r = 0$, the phases are evenly spaced on the unit circle $\mathbb{S}^1$.

Example 5.1: We consider a small two-level network consisting of 6 communities described in Fig. 5.1(a). Each community consists of 5 oscillators coupled by a complete graph. We assume that the oscillators between every two adjacent communities are interconnected in a way shown in Fig. 5.1(b). The inter-community coupling strengths are given beside the edges in Fig. 5.1(a). Denote $\omega = [{\omega^1}^\top, \dots, {\omega^6}^\top]^\top$, and let $\omega_1 = 0.5$ rad/s and $\omega_i = \omega_1 + 0.1(i-1)$ for all $i = 2, \dots, 30$. Let the intra-community coupling strengths be $K^2 = K^3 = 2.9$, and $K^1 = K^4 = K^5 = K^6 = 0.01$. One can check that the condition (5.15) is satisfied for the candidate regions of partial phase cohesiveness in the red rectangular, i.e., $\mathcal{T}_r = \{2, 3\}$. The evolution of the incremental ∞-norm of the oscillators' phases in $\mathcal{T}_r$ is plotted in Fig. 5.1(c), from which one can observe that starting from a value less than φ_m, $\|B_c^\top x(t)\|_\infty$ eventually converges to a value less than φ_s. One can then conclude that phase cohesiveness takes place between the communities 2, 3. On the other hand, it can be seen in Fig. 5.1(d) that the value of r, which measures the level of synchrony, remains small, which means that the other communities in the network are always incoherent. These observations validate our obtained results on partial phase cohesiveness in Theorem 5.2. Moreover,

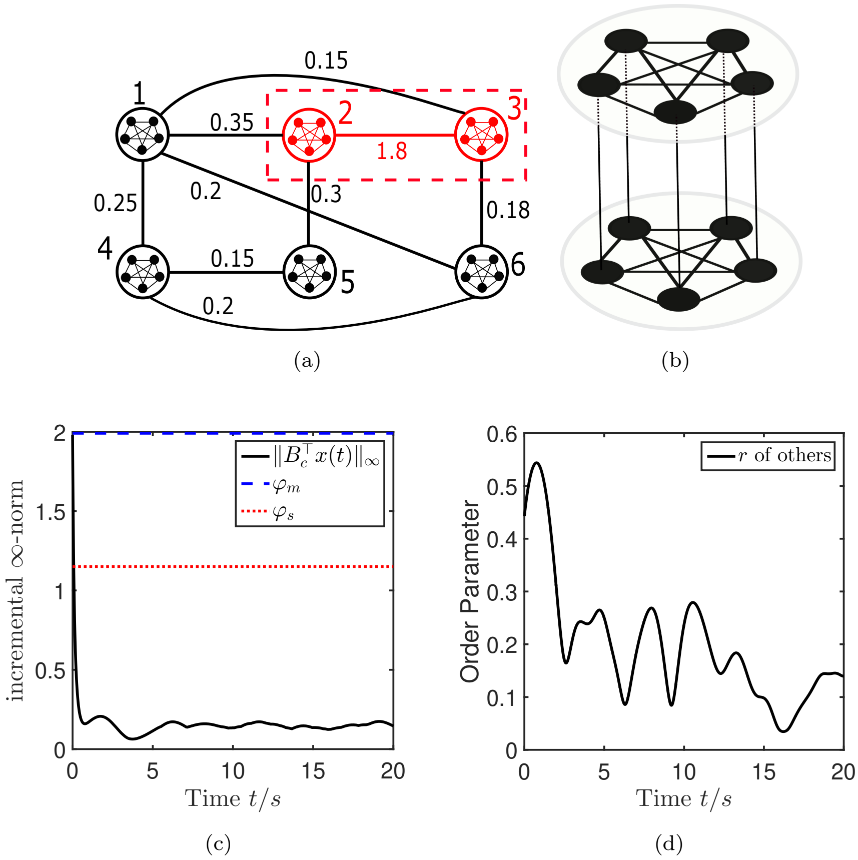


Figure 5.1: (a) The network structure considered in Example 5. 1; (b) the interconnection structure: each oscillator in a community is connected to exact one oscillator in another; (c) the trajectory of $\|B_c^\top x(t)\|_\infty$, where $x = [\theta_j^p]_{10\times 1}, j = \mathcal{T}_5, p = 2, 3$; (d) the magnitude r of the order parameter evaluated on other regions $(1, 4, 5$ and $6)$.

calculating the algebraic connectivity of the subgraph in the red rectangular, we obtain $\lambda_2(L) = 5.6$, which is not sufficient to satisfy the condition (5.8) in Theorem 5.1. Consistent with what we have claimed earlier, the results in Theorem 5.2 can be sharper than those in Theorem 5.1.

Example 5.2: In this example, we investigate partial phase cohesiveness in the human brain with the help of an anatomical network consisting of 66 cortical regions. The coupling strengths between regions are described by a weighted adjacency matrix $A = [a_{ij}]_{66\times 66}$ whose elements represent axonal fiber densities computed by means

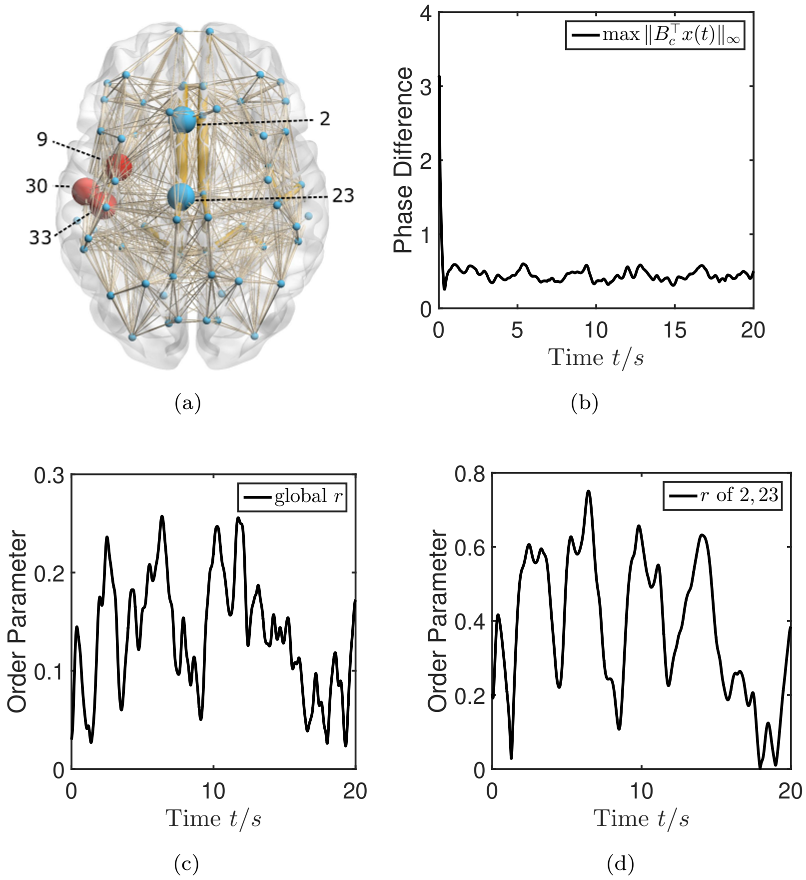


Figure 5.2: (a) the anatomical brain network visualized by BrainNet Viewer [158], edges only of weights larger than 0.15 are shown for clarity; (b) the maximum phase difference (absolute value) of the oscillators in $9, 30, 33$, where $x = [\theta_j^p]_{30\times 1}, j \in \mathcal{T}_{10}, p = 9, 30, 33$; (c) the magnitude r of the global order parameter; (d) the magnitude r evaluated on the regions 2 and 23.

of diffusion tensor imaging (DTI). This matrix is the average of the normalized anatomical networks obtained from 17 subjects [155]. From our earlier analysis, strong regional connections play an essential role in forming partial phase cohesiveness. We identify some candidate regions by selecting the connections of strengths greater than 20 (visualized by the large size edges in Fig. 5.2(a)). In particular, we consider two subsets of the brain regions $\{9, 30, 33\}$ and $\{2, 23\}$, (see the red and blue nodes in Fig.

5.2(a)), and investigate whether phase cohesiveness can occur among them.

We use the model in which each of the 66 regions consists of 10 oscillators coupled by a complete graph with the coupling strength $K^p, p = 1, \dots, 66$, and any two adjacent regions are connected by 10 randomly generated edges. The weights of the 10 edges connecting regions i and j are assigned randomly, and sum up to a_{ij}. The natural frequencies of all the oscillators are drawn from a normal distribution with the mean 13π rad/s (6.5 Hz) and the standard deviation 1.5π. Let the intra-community coupling strengths $K^p = 8$ for $p = 9, 30, 33$, and $K^p = 0.1$ for all the other p's. Thus, we have obtained a two-level network from the anatomical brain network. For this two-level network, we obtain some simulation results in Fig. 5.2(b), 5.2(c) and 5.2(d). One can observe from Fig. 5.2(b) that the regions $9, 30, 33$ eventually become phase cohesive, although the whole brain remains quite incoherent (see Fig. 5.2(c), where the mean value of r is approximately 0.15). This observation indicates that strong regional connections can be the cause of partial phase cohesiveness. On the other hand, one observes from Fig. 5.2(d) that without strong intra-community coupling strengths phase cohesiveness does not take place between the regions 2 and 23 (the blue large nodes in Fig. 5.2(a)), although they have a strong inter-region connection, $a_{2,23} = 52.8023$. This means that intra-community coupling strengths could play an important role in selecting regions to be synchronized.

From our theoretical results and simulations, we believe that there are at least two factors leading to partial brain synchronization. One factor relies on the anatomical properties of the brain network. The second factor depends on local changes in coupling strength. We hypothesize in this chapter that strong inter-regional coupling is one of the anatomical properties that allow for synchrony among brain regions. Then, selective synchronization of a subset of those strongly connected regions is achieved by increasing the intra-community coupling strengths on the target regions, which can give rise to various synchrony patterns. Other properties of the anatomical brain network such as symmetries studied in [96] and [108], can be a topic of future work.

5.6 Concluding Remarks

We have studied partial phase cohesiveness, instead of global synchronization, of Kuramoto oscillators coupled by two-level networks in this chapter. Sufficient conditions in the forms of algebraic connectivity and nodal degree have been obtained by using the incremental 2-norm and ∞-norm, respectively. The notion of generalized complement graphs that we introduced provides a much better tool than those in the literature to estimate the region of attraction and ultimate level of phase cohesiveness when the network is weighted complete or non-complete. However, the disadvantage of this method is that the number of edges connecting each node has a noticeable lower bound. The simulations we have performed provides some insight into understanding the partial synchrony observed in the human brain.

6 New Criteria for Partial Stability of Nonlinear Systems

We have studied partial synchronization among a set of directly connected oscillators in the previous chapter. From what has been observed in the brain, partial synchronization can also emerge among brain regions that have no direct links. We are also interested in studying this interesting phenomenon, termed remote synchronization. To study remote synchronization, one often needs to prove the partial stability of a nonlinear system. In this chapter, we develop some new criteria for partial stability of nonlinear systems. These new criteria will become very important theoretical tools in the next chapter to study remote synchronization in star networks.

6.1 Introduction

Partial stability describes the behavior of a dynamical system in which only a given part of its state variables, instead of all, are stable. The earliest study on partial stability dates back to a century ago in Lyapunov's seminal work in 1892, some comprehensive and well known results of which were documented by Vorotnikov in his book [99]. Different from classic full-state stability theory which usually deals with the stability of point-wise equilibria, partial stability is more associated with the stability of motions lying in a subspace [100].

A lot of engineering problems, such as spacecraft stabilization by rotating masses [99], inertial navigation systems [159], combustion systems [160], and power systems [161], can be analyzed by partial stability theory. It is also related to a wide range of theoretic topics including Lotka-Volterm predator-prey models [162], output regulation [163], and synchronization [108, 164]. For example, in the study of synchronization problems of coupled oscillators, only the pairwise state errors are desired to be stable, while individual states can be periodic, unbounded or even chaotic [165]. Moreover, when designing an observer one usually requires the error between the constructed observer and the system to be asymptotically stable, while the system itself, regarded as a reference, does not need to be stable. Interestingly, the stability of time-varying systems can be treated as partial stability of autonomous systems since the time t can be taken as an additional unbounded variable [166]. In the above mentioned problems, one frequently encounters the need to study stability of invariant manifolds [167], sets [168], limit cycles [169]. Partial stability theory provides a unified and powerful framework to study them.

As it turns out later in the next chapter, the existing criteria for partial stability can not be directly applied to remote synchronization analysis in many circumstances. Therefore, there is a great need for further developing new criteria for partial stability, and that is exactly the aim of this chapter.

Outline

The remainder of this chapter is structured as follows. In Section 6.2, we develop some new Lyapunov criteria for partial asymptotic and exponential stability of nonlinear systems without requiring the time derivative of the constructed Lyapunov functions to be negative definite. Some further new criteria for partial exponential stability of a particular class of slow-fast systems are provided in Section 6.3. Some concluding remarks appear in Section 6.4.

6.2 New Lyapunov Criteria for Partial Stability

The traditional Lyapunov theory is not directly applicable to partial stability analysis since a partially stable system is not stable in the standard sense. By generalizing the hypotheses on the traditional Lyapunov functions, researchers have established some sufficient conditions for partial stability analysis [99–101]. Similar to the classical Lyapunov theory, in order to show partial asymptotic or exponential stability, the time derivative of a Lyapunov function candidate is required to be negative definite [99–101]. However, as one will see later in this section, it is not always easy to construct such a Lyapunov function. For full-state asymptotic stability, the requirement for negative definiteness can be relaxed under additional assumptions. A well known result is to apply LaSalle's invariance principle [157, Chap. 4]. Another method is to show that a Lyapunov function candidate decreases after a finite time [170]. Aeyels and Peuteman have further relaxed the requirement of the latter method by allowing the time derivative to be positive [120]. A Lyapunov criterion using similar ideas has also been obtained for stochastic discrete-time systems [102].

Inspired by these works, in this section we have obtained new sufficient conditions for partial asymptotic or exponential stability of autonomous nonlinear systems. We first consider the case when the time derivative of a Lyapunov function candidate is negative semi-definite and show that partial asymptotic or exponential stability is guaranteed if the value of the candidate decreases after a finite time. Next, we further show that the requirement of negative semi-definiteness can be relaxed by imposing Lipshitz-like properties for the vector field of the system instead. Our obtained criteria enlarge the class of allowable functions that can be used in the analysis of partial stability of nonlinear systems. Compared with those Lyapunov criteria for full-state stability in [120, 170], theoretical analysis is more involved when it comes to partial stability in this section.

6.2.1 System Dynamics

In this subsection, we first formally define partial stability and briefly review some existing results. We then provide a motivating toy example.

Consider the autonomous system described by

$$\dot{x} = f_1(x, y), \tag{6.1a}$$

$$\dot{y} = f_2(x, y), \tag{6.1b}$$

where $x \in \mathcal{D}, y \in \mathbb{R}^m$ (here $\mathcal{D}$ is a domain in $\mathbb{R}^n$). The map $f_1 : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and satisfies $f_1(0, y) = 0$ for any y, and $f_2 : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}^m$ is also locally Lipschitz. Furthermore, we assume that the solution to (6.1), i.e.,

$$s(t) := \mathrm{col}(x(t), y(t)) \tag{6.2}$$

exists for all $t \geq 0$. Obviously, $s(0)$ represents the initial condition.

In this section, we are interested in studying partial stability. Let us first introduce some definitions of uniform partial stability which we will rely on in what follows.

Definition 6.1 ([100, Chap. 4])**.** *The partial equilibrium point $x = 0$ of the system* (6.1) *is*

(i) x-stable *uniformly in y if, for every $\varepsilon > 0$ and any $y \in \mathbb{R}^m$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| < \varepsilon$ for all $t \geq 0$.*

(ii) asymptotically x-stable *uniformly in y if it is* stable *uniformly in y, and there exists $\delta > 0$ such that $\|x(0)\| < \delta$ implies that* $\lim_{t\to\infty} \|x(t)\| = 0$ *for any $y(0) \in \mathbb{R}^m$.*

(iii) exponentially x-stable *uniformly in y if there exist $c_1, c_2, \delta > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| \leq c_1 \|x(0)\| e^{-c_2 t}, \forall t \geq 0$ for any $y(0) \in \mathbb{R}^m$.*

Note that when we refer to the cases (i), (ii) or (iii), we also say that $x = 0$ is partially stable, asymptotically stable or exponentially stable, respectively. Some sufficient conditions for partial stability using Lyapunov methods can be found in [99, 100]. To be self-contained, we present them in the following lemma and discuss the possibility of relaxing some conditions. Note that, in what follows, for a given continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}$ we denote

$$\dot{V}(x, y) := \frac{\partial V(x, y)}{\partial x} f_1(x, y) + \frac{\partial V(x, y)}{\partial y} f_2(x, y)$$

for notational simplicity.

Lemma 6.1 ([100, Chap. 4]). *Let $V : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \le V(x, y) \le \alpha_2(\|x\|), \tag{6.3}$$

$$\dot{V}(x, y) \le -\gamma(\|x\|), \tag{6.4}$$

for any $(x, y) \in \mathcal{D} \times \mathbb{R}^m$, where α_1, α_2 and γ are class $\mathcal{K}$ functions. Then $x = 0$ of the system (6.1) *is asymptotically x-stable uniformly in y. If there is $p \in \mathbb{N}$ such that V satisfies*

$$\beta_1\|x\|^p \le V(x, y) \le \beta_2\|x\|^p, \tag{6.5}$$

$$\dot{V}(x, y) \le -\beta_3\|x\|^p, \tag{6.6}$$

for any $(x, y) \in \mathcal{D} \times \mathbb{R}^m$, where $\beta_1, \beta_2, \beta_3 > 0$, then $x = 0$ of the system (6.1) *is exponentially x-stable uniformly in y.*

It is also shown by the converse theorems in [100, Chap. 4] that there always exists a Lyapunov function such that (6.3) and (6.4) (or (6.5) and (6.6), respectively) are satisfied provided that $x = 0$ is asymptotically (exponentially, respectively) x-stable uniformly in y. However, it is not always easy to construct a Lyapunov function candidate such that the condition (6.4) or (6.6) is satisfied. Let us illustrate this point by a toy example.

Example 6.1. *Consider the following system*

$$\dot{x} = -\frac{1}{5}x - ax\sin y, \tag{6.7}$$

$$\dot{y} = 3b - b\sin x, \tag{6.8}$$

where $x, y \in \mathbb{R}$, and $a, b > 0$. Note that $x = 0$ is exponentially x-stable uniformly in y when b is sufficiently large. In order to prove this, choose $V(x, y) = \frac{1}{2}x^2$ as a

Lyapunov function candidate, then its time derivative is

$$\dot{V} = -\left(\frac{1}{5} + a \sin y\right) x^2.$$

According to Lemma 6.1, one can show the uniform partial exponential stability when $a < \frac{1}{5}$, since there holds that $\dot{V} \leq -(\frac{1}{5} - a)x^2$. However, if $a \geq \frac{1}{5}$, the negative definite property of $\dot{V}$ is no longer guaranteed, implying that the considered Lyapunov function candidate is not an appropriate one. △

This motivates us to further develop Lyapunov theory for partial stability. When it comes to full-state stability, the existing results have shown that the requirement for classic Lyapunov theory can be relaxed by allowing the time derivative of Lyapunov function candidates to be negative semi-definite or even positive [120, 170–172]. Inspired by these ideas, we establish some Lyapunov criteria for the analysis of partial stability, without requiring the time derivatives to be negative definite. This could create more freedom to construct allowable functions in partial stability analysis. As it turns out later, using the same Lyapunov function candidate in Example 6.1, one is able to show the uniform partial exponential stability of $x = 0$ even when $a \geq \frac{1}{5}$ according to the alternative criteria set out in the next section.

6.2.2 Partial Asymptotic and Exponential Stability

In this subsection, we aim at further developing Lyapunov theory to enlarge choices of allowable functions that can be used to analyze partial stability of autonomous nonlinear systems. We first provide two criteria for partial asymptotic and exponential stability in Theorems 6.1 and 6.2, respectively. Unlike what is required in Lemma 6.1, we only assume that the time derivative of a Lyapunov function candidate is negative semi-definite. Moreover, we further relax the requirement of negative semi-definiteness in Theorems 6.3 and 6.4.

Let us first provide a new criterion for partial asymptotic stability of the system (6.1).

Theorem 6.1. *Let $V : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \leq V(x, y) \leq \alpha_2(\|x\|), \tag{6.9}$$

$$\dot{V}(x, y) \leq 0, \tag{6.10}$$

for any $(x, y) \in \mathcal{D} \times \mathbb{R}^m$. Furthermore, suppose that for any $s(0) \in \mathcal{D}' \times \mathbb{R}^m$ with an open set $\mathcal{D}' \subset \mathcal{D}$, there exists $T = T(s(0)) > 0$ that is bounded from above so that

$$\int_0^T \dot{V}(x(\tau), y(\tau)) d\tau \leq -\gamma(\|x(0)\|), \tag{6.11}$$

where α_1, α_2 and γ are class $\mathcal{K}$ functions. Then $x = 0$ of the system (6.1) *is asymptotically x-stable uniformly in y.*

Proof. We first prove the uniform stability. It follows from the inequality (6.10) that $V(x(t), y(t)) \leq V(x(0), y(0))$. Taking into the inequalities (6.9) into account, one obtains $\alpha_1(\|x(t)\|) \leq V(x(t), y(t)) \leq V(x(0), y(0)) \leq \alpha_2(\|x(0)\|)$. This implies that $\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(0)\|))$. It is clear that for any $\varepsilon > 0$ satisfying $\mathcal{B}_\varepsilon \subset \mathcal{D}$, there exists $\delta = \alpha_2^{-1}(\alpha_1(\varepsilon))$ such that $x(0) \in \mathcal{B}_\delta$ and $y(0) \in \mathbb{R}^m$ implies that $x(t) \in \mathcal{B}_\varepsilon$ for all $t \geq 0$.

It remains to show the convergence by finding $\delta_1 > 0$ such that for any $s(0) \in \mathcal{B}_{\delta_1} \times \mathbb{R}^m$ and $\varepsilon_1 > 0$, there exists $T(\varepsilon_1) > 0$ for which $x(t) \in \mathcal{B}_{\varepsilon_1}$, $\forall t \geq T(\varepsilon_1)$. Let ε_2 be such that $\mathcal{B}_{\varepsilon_2} \subset \mathcal{D}'$, and let $\delta_1 = \alpha_2^{-1}(\alpha_1(\varepsilon_2))$. Then, from the analysis above, $x(0) \in \mathcal{B}_{\delta_1}$ implies that $x(t) \in \mathcal{B}_{\varepsilon_2}$ for all $t \geq 0$. We point out that $T(\varepsilon_1) = 0$ if $\varepsilon_1 \geq \varepsilon_2$ for any $x(0) \in \mathcal{B}_{\delta_1}$ and $y(0) \in \mathbb{R}^m$.

We then consider the other case when $\varepsilon_1 < \varepsilon_2$, and show that for any $x(0) \in \mathcal{B}_{\delta_1}$, there exists a finite $T(\varepsilon_1)$ such that $x(t) \in \mathcal{B}_{\delta_1'}$, where $\delta_1' = \alpha_2^{-1}(\alpha_1(\varepsilon_1))$, which implies that $x(t)$ will stay in $\mathcal{B}_{\varepsilon_1}$ for all $t \geq T(\varepsilon_1)$. From the choice of δ_1, for any $x_0 \in \mathcal{B}_{\delta_1} \times \mathbb{R}^m$, $x(t) \in \mathcal{D}'$ is guaranteed. Resetting the initial condition x_0, it follows from (6.11) that there exists a sequence $\{T_i, i \in \mathbb{N}_0\}$, (note that $T_0 = 0$), such that

$$\int_{\sum_{i=0}^{k} T_i}^{\sum_{i=0}^{k+1} T_i} \dot{V}\big(x(\tau), y(\tau)\big) d\tau \leq -\gamma\Big(\|x(\sum\nolimits_{i=0}^{k} T_i)\|\Big), \tag{6.12}$$

for any $k \in \mathbb{N}_0$. By using this inequality, we show that there exists a finite $k' \in \mathbb{N}_0$ such that $\|x(\sum_{i=0}^{k'+1} T_i)\| < \delta_1' = \alpha_2^{-1}(\alpha_1(\varepsilon_1))$. Then, from the continuity of the solution, the proof is completed by choosing $T(\varepsilon_1) = \sum_{i=0}^{k'+1} T_i$.

We prove $\|x(\sum_{i=0}^{k'+1} T_i)\| < \delta_1'$ by contradiction. Suppose that for all k', $\|x(\sum_{i=0}^{k'} T_i)\| \geq \delta_1'$. Then, it follows from the property of the class $\mathcal{K}$ function that $-\gamma(\|x(\sum_{i=0}^{k'} T_i)\|) \leq -\gamma(\delta_1')$.

Moreover, from (6.10) and the first inequality of (6.9), we have

$$\begin{aligned} &V\Big(x\big(\sum\nolimits_{i=0}^{k_1} T_i\big), y\big(\sum\nolimits_{i=0}^{k_1} T_i\big)\Big) \\ &\qquad\qquad \geq V\Big(x\big(\sum\nolimits_{i=0}^{k'} T_i\big), y\big(\sum\nolimits_{i=0}^{k'} T_i\big)\Big) \geq \alpha_1(\delta_1'), \end{aligned}$$

for any $0 \leq k_1 \leq k'$. From the second inequality of (6.9), it follows that $\|x(\sum_{i=0}^{k_1} T_i)\| \geq \alpha_2^{-1}(\alpha_1(\delta_1'))$ and consequently $-\gamma(\|x(\sum_{i=0}^{k_1} T_i)\|) \leq -\gamma(\alpha_2^{-1}(\alpha_1(\delta_1')))$ for all $k_1 \leq k'$. Summing up all the right and left sides of (6.12) from $k = 0$ to $k = k'$, we have

$$\begin{aligned} &V\Big(x(\sum\nolimits_{i=0}^{k'+1} T_i), y(\sum\nolimits_{i=0}^{k'+1} T_i)\Big) - V(x(0), y(0)) \\ &\qquad\qquad \leq -\gamma(\delta_1') - k'\gamma(\alpha_2^{-1}(\alpha_1(\delta_1'))), \end{aligned}$$

which implies, with (6.9),

$$\alpha_1\big(\|x(\sum_{i=0}^{k'+1} T_i)\|\big) \leq \alpha_2(\delta_1) - \gamma(\delta_1') - k'\gamma(\alpha_2^{-1}(\alpha_1(\delta_1'))).$$

Then, for sufficiently large finite k', it follows that $\alpha_2(\delta_1) - \gamma(\delta_1') - k'\gamma(\alpha_2^{-1}(\alpha_1(\delta_1'))) \leq \alpha_1(\delta_1)$, and consequently $\|x(\sum_{i=0}^{k'+1} T_i)\| \leq \delta_1$, which is a contradiction. The proof is complete. □

The following theorem provides a new Lyapunov criterion for partial exponential stability of the system (6.1).

Theorem 6.2. *Let $V : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function satisfying*

$$\beta_1\|x\|^p \leq V(x,y) \leq \beta_2\|x\|^p, \tag{6.13}$$

$$\dot{V}(x,y) \leq 0, \tag{6.14}$$

for any $(x,y) \in \mathcal{D} \times \mathbb{R}^m$. Furthermore, suppose that for any $s(0) \in \mathcal{D}' \times \mathbb{R}^m$ with an open set $\mathcal{D}' \subset \mathcal{D}$, there exists $T = T(s(0)) > 0$ that is bounded from above so that

$$\int_0^T \dot{V}(x(\tau), y(\tau))d\tau \leq -\beta_3\|x(0)\|^p, \tag{6.15}$$

where $\beta_1, \beta_2, \beta_3 > 0$, then $x = 0$ of the system (6.1) *is exponentially x-stable uniformly in y.*

Proof. Consider the ball $\bar{\mathcal{B}}_\varepsilon \subset \mathcal{D}'$. From the proof of stability in Theorem 6.1, it can be guaranteed that $s(t) \in \mathcal{B}_\varepsilon \times \mathbb{R}^m, \forall t \geq 0$, if the initial condition satisfies $x(0) \in \mathcal{B}_\delta \times \mathbb{R}^m$ with $\delta = \alpha_2^{-1}(\alpha_1(\varepsilon))$. Then the inequality (6.15) is ensured for all $t \geq 0$ if $x(0) \in \mathcal{B}_\delta \times \mathbb{R}^m$. According to the inequality (6.15), there exists a sequence $\{T_i, i \in \mathbb{N}_0\}$, (note that $T_0 = 0$), such that for any $k \in \mathbb{N}_0$ it holds that

$$\begin{aligned} &V\Big(x\big(\sum_{i=0}^{k+1} T_i\big), y\big(\sum_{i=0}^{k+1} T_i\big)\Big) - \\ &V\Big(x\big(\sum_{i=0}^{k} T_i\big), y\big(\sum_{i=0}^{k} T_i\big)\Big) \leq -\beta_3\|x(\sum_{i=0}^{k} T_i)\|^p. \end{aligned}$$

From the inequality (6.13), there holds that

$$\|x(\sum_{i=0}^{k} T_i)\|^p \geq \frac{1}{\beta_2} V\left(x(\sum_{i=0}^{k} T_i), y(\sum_{i=0}^{k} T_i)\right).$$

It then follows that

$$\begin{aligned} V\Big(&x\big(\sum_{i=0}^{k+1} T_i\big), y\big(\sum_{i=0}^{k+1} T_i\big)\Big) \\ &\leq (1 - \frac{\beta_3}{\beta_2}) V\Big(x\big(\sum_{i=0}^{k} T_i\big), y\big(\sum_{i=0}^{k} T_i\big)\Big), \end{aligned}$$

for any $k \in \mathbb{N}_0$, where $0 \le 1 - \beta_3/\beta_2 < 1$. For any $t \ge 0$, there is a $k^* \ge 0$ such that $t \in [\sum_{i=0}^{k^*} T_i, \sum_{i=0}^{k^*+1} T_i)$, then there holds that

$$\begin{aligned} V(x(t), y(t)) &\le V\Big(x\big(\sum_{i=0}^{k^*} T_i\big), y\big(\sum_{i=0}^{k^*} T_i\big)\Big) \\ &\le \big(1 - \beta_3/\beta_2\big)^{k^*} V(x(0), y(0)), \end{aligned}$$

where the inequality (6.14) has been used. From inequality (6.13), one knows that $V(x(t), y(t)) \ge \beta_1 \|x(t)\|^p$ and $V(x(0), x(0)) \le \beta_2 \|x(0)\|^p$, which yields

$$\|x(t)\| \le (\beta_2/\beta_1)^{1/p} (1 - \beta_2/\beta_1)^{k^*/p} \|x(0)\|. \tag{6.16}$$

Let $\lfloor y \rfloor$ be the largest integer that is less than or equal to the scaler y. Let $\hat{T}$ be the upper bound of those T's in (6.15), i.e., $T_i \le \hat{T}$ for any i, then there holds that $k^* \ge \lfloor t/\hat{T} \rfloor \ge t/\hat{T} - 1$. Substituting this inequality into (6.16) we have

$$\|x(t)\| \le (\beta_2/\beta_1)^{1/p} (1 - \beta_2/\beta_1)^{(t/\hat{T}-1)/p} \|x(0)\|.$$

Let $c_1 = (\beta_2/\beta_1)^{1/p}(1-\beta_2/\beta_1)^{-1/p}, c_2 = -1/(p\hat{T}) \cdot \ln(1-\beta_2/\beta_1)$, the above inequality can be rewritten as $\|x(t)\| \le c_1 e^{-c_2 t} \|x(0)\|$, which proves the partial exponential stability of $x = 0$ uniformly in y. □

In Theorems 6.1 and 6.2, we have made the assumption that the time derivative of the Lyapunov function $\dot{V}(x, y)$ is negative semi-definite. When a similar assumption is made for full-state stability problems, Lassalle's invariance principle is usually used to prove asymptotic stability [157, Chap. 4]. Analogous conditions to (6.11) can be found in [170, 171], where full-state asymptotic stability has been studied. Using analogous ideas to those results, we have established some criteria for partial asymptotic and exponential stability. Moreover, greatly inspired by the results on full-state asymptotic and exponential stability in [120, 172], we next show that this negative semi-definite condition can be further relaxed, while still guaranteeing the asymptotic or exponential x-stability uniformly in y. Before providing those results, let us first make the following assumption.

Assumption 6.1. *We assume that the map f_1 satisfies the following conditions*

(a) for any $x \in \mathcal{D}, y \in \mathbb{R}^m$, it holds that $\|f_1(x', y) - f_1(x'', y)\| \le L_1 \|x' - x''\|$ for all $x', x'' \in \mathcal{B}_r(x)$, where $r = r(x) > 0$, and $L_1 = L_1(x) > 0$ is finite.

(b) there exists $K > 0$ such that for any $x \in \mathcal{D}$ and $y', y'' \in \mathbb{R}^m$, there holds that $\|f_1(x, y') - f_1(x, y'')\| \le K\|x\|$.

Note that Assumption 6.1 is similar to but a bit stronger than the locally Lipschitz condition. One can check that (6.7) in Example 6.1 satisfies this assumption. We show that the following two lemmas hold if Assumption 6.1 is satisfied.

Lemma 6.2. *Suppose that Assumption 6.1 is satisfied. Let $\mathcal{U}$ be any compact convex set contained in $\mathcal{D}$. Then there exist constants $L > 0$ and $K > 0$ such that for any $(x_1, y_1), (x_2, y_2) \in \mathcal{U} \times \mathbb{R}^m$, it holds that*

$$\|f_1(x_1, y_1) - f_1(x_2, y_2)\| \leq L\|x_1 - x_2\| + K\|x_1\|. \tag{6.17}$$

Proof. We first show that if (a) of Assumption 6.1 is satisfied, for any $x, y_1 \in \mathcal{U}$ and $y \in \mathbb{R}^m$, there holds that

$$\|f_1(x_1, y) - f_1(x_2, y)\| \leq L\|x_1 - x_2\|. \tag{6.18}$$

Denote $z = \mathrm{col}(x_1, y_1)$ and $z' = \mathrm{col}(x_2, y_2)$. Since $\mathcal{U}$ is compact, it can be covered by a finite number of neighborhoods. In other words, there exist a finite integer k and $a_i, r_i, i = 1, \dots, k$, such that

$$\mathcal{U} \subset \bar{\mathcal{B}}_{r_1}(a_1) \cup \bar{\mathcal{B}}_{r_2}(a_2) \cup \cdots \cup \bar{\mathcal{B}}_{r_k}(a_k). \tag{6.19}$$

Draw a line that connects z and z', denoted by l, and we know that all the points on l belong to $\mathcal{U} \times \{y\}$ since $\mathcal{U}$ is convex. Then there exists a subset of those neighborhoods in (6.19), say $\bar{\mathcal{B}}_{r_{k_j}}(a_{k_j}), j = 1, \dots, m$, such that

$$l \subset \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_m,$$

where $\mathcal{N}_j := \bar{\mathcal{B}}_{r_{k_j}}(a_{k_j}) \times \{y\}$. Without loss of generality, we assume that starting from z the line l passes through $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_m$ in sequence. Let $b^{(1)}, b^{(2)}, \dots, b^{(m-1)}$ be a sequence of points on the line l such that $b^{(i)} \in l \cap \mathcal{N}_i \cap \mathcal{N}_{i+1}$ for any i. If Assumption 6.1 is satisfied, it follows that

$$\begin{aligned}
&\|f_1(z) - f_1(b^{(1)})\| \leq L'_1\|x_1 - b_1^{(1)}\|, \\
&\|f_1(b^{(i)}) - f_1(b^{(i+1)})\| \leq L'_{i+1}\|b^{(i)} - b_1^{(i+1)}\|, \\
&\|f_1(b^{(m-1)}) - f_1(z')\| \leq L'_m\|b_1^{(m-1)} - x_2\|,
\end{aligned}$$

where $i = 1, \dots, m-2$. The sum of the terms on the left hand side of the inequalities is greater than or equal to $\|f_1(z) - f_1(z')\|$ by using the triangle inequality. The sum of all the right hand side of the inequalities is

$$\begin{aligned}
&L\Big(\|x - b_1^{(1)}\| + \sum\nolimits_{i=1}^{m-2} \|b_1^{(i)} - b_1^{(i+1)}\| + \|b_1^{(m-1)} - y_1\|\Big) \\
&= L\|x_1 - x_2\|,
\end{aligned}$$

where $L := \max\{L'_i\}$, and the last inequality follows from the fact that $x, b_1^{(1)}, \dots, b_1^{(m-1)}, y_1$ are all on the line l.

We next show the inequality (6.17). For any $x, y \in \mathcal{D} \times \mathbb{R}^m$, there holds that

$$\begin{aligned}
&\|f_1(x_1, y_1) - f_1(x_2, y_2)\| \\
&= \|f_1(x_1, y_1) - f_1(x_1, y_2) + f_1(x_1, y_2) - f_1(x_2, y_2)\| \\
&\leq \|f_1(x_1, y_1) - f_1(x_1, y_2)\| + \|f_1(x_1, y_2) - f_1(x_2, y_2)\|.
\end{aligned}$$

If (b) of Assumption 6.1 is satisfied, one obtains that $\|f_1(x_1, y_1) - f_1(x_2, y_2)\| \leq L\|x_1 - x_2\| + K\|x_1\|$ by invoking the inequality (6.18). □

Lemma 6.3. *Under Assumption 6.1, for any $\varepsilon > 0$ and $T > 0$, there exists a $\delta = \delta(\varepsilon, T) > 0$ such that for any initial condition $s(0) \in \mathcal{B}_\delta(0) \times \mathbb{R}^m$, it holds that $s(t) \in \mathcal{B}_\varepsilon(0) \times \mathbb{R}^m$ for all $t \in [0, T]$.*

Proof. Let $\mathcal{U}$ of Lemma 6.2 be $\mathcal{B}_\delta(0)$, then one knows that for any $(x_1, y_1), (x_2, y_2) \in \mathcal{U} \times \mathbb{R}^m$, there exist $L', K' > 0$ such that

$$\|f_1(x_1, y_1) - f_1(x_2, y_2)\| \leq L'\|x_1 - x_2\| + K'\|x_1\|.$$

Let $x_2 = 0$, we have $\|f_1(x_1, y_1)\| \leq L\|x_1\|$ with $L := L' + K'$. Following similar steps to those in Lemma 1 of [120], one can prove that for any $\varepsilon > 0$, one can ensure that $x(t) \in \mathcal{B}_\varepsilon(0)$ for any $t \in [0, T]$ by taking $\delta = \varepsilon e^{-LT}$. □

We are now ready to provide another result on partial asymptotic stability, where the negative semi-definiteness of $\dot{V}$ is no longer required.

Theorem 6.3. *Suppose Assumption 6.1 is satisfied. Let $V : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function satisfying*

$$\alpha_1(\|x\|) \leq V(x, y) \leq \alpha_2(\|x\|), \tag{6.20}$$

for any $(x, y) \in \mathcal{D} \times \mathbb{R}^m$. Furthermore, suppose that for any $s(0) \in \mathcal{D}' \times \mathbb{R}^m$ with an open set $\mathcal{D}' \subset \mathcal{D}$, there exists $T = T(s(0)) > 0$ that is bounded from above so that

$$\int_0^T \dot{V}(x(\tau), y(\tau))dt \leq -\gamma(\|x(0)\|), \tag{6.21}$$

where α_1, α_2 and γ are class $\mathcal{K}$ functions. Then $x = 0$ of the system (6.1) is asymptotically x-stable.

Proof. We first prove the stability of $x = 0$ uniformly in y. Consider a closed ball $\bar{\mathcal{B}}_\varepsilon(0) \subset \mathcal{D}'$, and we show that there exists $\delta > 0$ such that $s(0) \in \mathcal{B}_\delta(0) \times \mathbb{R}^m$ implies that $s(t) \in \mathcal{B}_\varepsilon(0) \times \mathbb{R}^m$ for all $t \geq 0$. According to Lemmas 6.2 and 6.3, there exists

$L > 0$ such that $\|f_1(x,y)\| \leq L\|x\|$ for any $(x,y) \in \bar{\mathcal{B}}_\varepsilon(0) \times \mathbb{R}^m$. The solution of (6.1) satisfies that

$$\|x(t)\| \leq \|x(t_0)\|e^{L(t-t_0)}$$

for any $t \geq t_0$ such that $x(t) \in \bar{\mathcal{B}}_\varepsilon(0)$. Let $\hat{T}$ be the upper bound of those T's in (6.21). Following similar steps to those in Theorem 1 of [120], one can show that for any $\varepsilon > 0$, $\|x(t)\| < \varepsilon$ for any $t \in [t_0, t_0 + \hat{T}]$ if $\|x(t_0)\| < \alpha_2^{-1}(\alpha_1(\varepsilon e^{-L\hat{T}}))$. Consider any initial condition satisfying $\|x(0)\| < \delta := \alpha_2^{-1}(\alpha_1(\varepsilon e^{-L\hat{T}}))$, from (6.21) there is T_1, $0 < T_1 \leq \hat{T}$, such that

$$\int_0^{T_1} \dot{V}(x,y)dt \leq -\gamma(\|x(0)\|).$$

It follows that

$$V(x(T_1), y(T_1)) < V(x(0), y(0)) \leq \alpha_2(\|x(0)\|) < \alpha_2(\delta) = \alpha_1(\varepsilon e^{-L\hat{T}}).$$

Since $V(x(T_1), y(T_1)) \geq \alpha_1(\|x(T_1)\|)$, one then obtains $\|x(T_1)\| < \varepsilon e^{-L\hat{T}}$. Moreover, there holds that $x(t) < \varepsilon$ for any $t \in [0, T_1]$ since $T_1 \leq \hat{T}$. We use (6.21) again by resetting the initial condition to $x(T_1)$, and then there also exists $T_2 > 0$ such that $\int_{T_1}^{T_1+T_2} \dot{V}(x,y)dt \leq -\gamma(\|x(T_1)\|)$. Likewise, one can see that $x(t) < \varepsilon$ for all $t \in [T_1, T_1 + T_2]$ and $\|x(T_1 + T_2)\| < \varepsilon e^{-L\hat{T}}$. By simply repeating the same process, it holds that $x(t) < \varepsilon$ for all $t \in [T_i, T_i + T_{i+1}]$ for all nonnegative integer i (note that $T_0 := 0$). For any $t \geq 0$, there exists a nonnegative integer m such that $t \leq \sum_{i=0}^m T_i$, which implies that $x(t) < \varepsilon$ for all $t \geq 0$. In other words, for any $\varepsilon > 0$, one can ensure that (i) of Definition 6.1 is satisfied by taking $\delta = \alpha_2^{-1}(\alpha_1(\varepsilon e^{-L\hat{T}}))$, which proves the partial stability of $x = 0$ uniformly in y.

The asymptotic convergence of x to 0 uniformly in y can be proven following the same lines as those in the proof of Theorem 6.1, which is omitted here. Then the partial asymptotic stability of $x = 0$ is proven. □

Furthermore, the next theorem provides a new criterion for partial exponential stability of the system (6.1) without requiring the negative semi-definiteness of $\dot{V}$.

Theorem 6.4. *Suppose Assumption 6.1 is satisfied. Let $V : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function satisfying*

$$\beta_1\|x\|^p \leq V(x,y) \leq \beta_2\|x\|^p, \tag{6.22}$$

for any $(x,y) \in \mathcal{D} \times \mathbb{R}^m$. Furthermore, suppose that for any $s(0) \in \mathcal{D}' \times \mathbb{R}^m$ with an open set $\mathcal{D}' \subset \mathcal{D}$, there exists $T = T(s(0)) > 0$ that is bounded from above so that

$$\int_0^T \dot{V}(x(\tau), y(\tau))d\tau \leq -\beta_3\|x(0)\|^p, \tag{6.23}$$

where β_1, β_2 *and* β_3 *are positive scalers. Then* $x = 0$ *of the system* (6.1) *is exponentially* x*-stable uniformly in* y*.*

Proof. Consider a closed ball $\bar{\mathcal{B}}_\varepsilon(0) \subset \mathcal{D}'$. Let $\hat{T}$ be the upper bound of those T's in (6.23). From the proof of Theorem 6.3, $s(t) \in \mathcal{B}_\varepsilon(0) \times \mathbb{R}^m$ for all $t \geq 0$ if the initial condition satisfies $s(0) \in \mathcal{B}_\delta(0) \times \mathbb{R}^m$ with $\delta = \alpha_2^{-1}(\alpha_1(\varepsilon e^{-L\hat{T}}))$. We next consider such an initial condition, ensuring that the inequality (6.23) holds for all $t \geq 0$. We show the exponential convergence subsequently.

Similar to the proof of Theorem 6.2, one knows that there exists a sequence $\{T_i, i \in \mathbb{N}_0\}$, (note that $T_0 = 0$), such that for any $k \in \mathbb{N}_0$,

$$\begin{aligned} &V\Big(x\big(\sum\nolimits_{i=0}^{k+1} T_i\big), y\big(\sum\nolimits_{i=0}^{k+1} T_i\big)\Big) \\ &\quad \leq (1 - \frac{\beta_3}{\beta_2}) V\Big(x(\sum\nolimits_{i=0}^{k} T_i), y(\sum\nolimits_{i=0}^{k} T_i)\Big), \end{aligned}$$

where $0 \leq 1 - \beta_3/\beta_2 < 1$. By iteration and by using the inequality (6.22), we have

$$\|(x(\sum\nolimits_{i=0}^{k} T_i)\| \leq (\beta_2/\beta_1)^{1/p}(1 - \beta_2/\beta_1)^{k/p}\|x(0)\|, \tag{6.24}$$

for any $k \in \mathbb{N}_0$. For any $t \geq 0$, there is a $k^* \in \mathbb{N}_0$ such that $t \in [\sum_{i=1}^{k^*} T_i, \sum_{i=1}^{k^*+1} T_i)$. From the analysis in the proof of Theorem 6.3, one knows that

$$\|x(t)\| \leq e^{L\hat{T}} \left\| (x\left(\sum_{i=0}^{k^*} T_i\right) \right\|.$$

It then follows from (6.24) that

$$\|x(t)\| \leq e^{L\hat{T}}(\beta_2/\beta_1)^{1/p}(1 - \beta_2/\beta_1)^{k^*/p}\|x(0)\|.$$

Recall that $k^* \geq \lfloor t/\hat{T} \rfloor \geq t/\hat{T} - 1$. Substituting it into the above inequality and letting $c_1 = e^{L\hat{T}}(\beta_2/\beta_1)^{1/p}(1 - \beta_2/\beta_1)^{-1/p}, c_2 = -1/(p\hat{T}) \cdot \ln(1 - \beta_2/\beta_1)$, one obtains that $\|x(t)\| \leq c_1\|x(0)\|e^{-c_2 t}$, which completes the proof. □

6.2.3 Examples

In this subsection, we first look back at Example 6.1, and prove the partial exponential stability of $x = 0$ uniformly in y with the help of the obtained results in the previous section, which the existing criterion in Lemma 6.1 fails to show using the same Lyapunov function candidate. We then perform a simulation to illustrate the idea behind our results.

Revisit Example 6.1

It is not hard to see that for any initial condition $s(0) \in \mathbb{R}^2$, the solution $s(t)$ to the system (6.7) and (6.8) exists for all $t \geq 0$. Let $\phi(t, s(0)) = (\phi_1(t, s(0)), \phi_2(t, s(0)))^\top$ be the solution to (6.7) and (6.8) that starts at $x(0)$. Then we have

$$\dot{x} = -\frac{1}{5}x - ax\sin(\phi_2(t, s(0))). \tag{6.25}$$

Choose $V(x, y) = x^2/2$ as a Lyapunov function candidate, and we next show the existence of $T = T(s(0)) > 0$ for any $s(0) \in \mathbb{R}^2$ such that

$$\int_0^T \dot{V}(x(\tau), y(\tau))d\tau \leq -\beta_3\|x\|^2, \qquad \beta_3 > 0, \tag{6.26}$$

with the help of (6.25). Let $A(t, s(0)) = -1/5 - a\sin(\phi_2(t, s(0)))$, then (6.25) can be rewritten as $\dot{x} = A(t, s(0))x$. Let $\Phi(t, s(0))$ be the state transition matrix from time 0 to t with initial condition $s(0)$, then one knows that $\phi_1(t, s(0)) = \Phi(t, s(0))x(0)$. For notational convenience, we denote $A(t, s(0))$ and $\Phi(t, s(0))$ simply by $A(t)$ and $\Phi(t)$, respectively, without causing any ambiguity. Since $A(t)$ is a scalar, the transition matrix is given by $\Phi(t) = \exp(\int_0^t A(\tau)d\tau)$, which, in turn, can be expressed in the form of power series as follows:

$$\Phi(t) = e^{\int_0^t A(\tau)d\tau} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\int_0^t A(\tau)d\tau\right)^k. \tag{6.27}$$

We observe that $\|A(t)\| \leq 1/5 + a := L$. Following similar steps as Section III in [120], we let $\Phi(t) = 1 + \Gamma$ with Γ denoting the summation term. Taking into account that $\|A(t)\| \leq L$, it follows from (6.27) that

$$\|\Phi(t)\| \leq 1 + \|\Gamma\| \leq 1 + \sum_{k=1}^{\infty} \frac{1}{k!}(Lt)^k.$$

It can be seen that the rightmost side of the above inequalities is the Taylor series for the exponential function e^{Lt} at 0. Then one knows that $\|\Phi(t)\| \leq 1 + \|\Gamma(t)\| \leq e^{Lt}$, which implies that $\|\Gamma(t)\| \leq e^{Lt} - 1$ for any $t \geq 0$.

The time derivative of V is $\dot{V} = \dot{x} \cdot x$. Integrating it from 0 to T along (6.25) can be expressed by

$$\int_0^T \dot{V}(x(\tau), y(\tau))d\tau = \int_0^T \dot{\phi}_1(\tau, s(0)) \cdot \phi_1(\tau, s(0))d\tau.$$

Since $\dot{\phi}_1(\tau, s(0)) = A(\tau)\phi_1(\tau, s(0))$ and $\phi_1(\tau, s(0)) = \Phi(\tau)x(0)$, we have

$$\int_0^T \dot{V}(x(\tau), y(\tau))d\tau = x^2(0)\int_0^T \Phi^2(\tau)A(\tau)d\tau. \tag{6.28}$$

We next estimate the integral on the right side by substituting $\Phi(\tau) = 1 + \Gamma(\tau)$ into it. There then holds that

$$\int_0^T \Phi^2(\tau)A(\tau)d\tau = \int_0^T (1+\Gamma(\tau))^2 A(\tau)d\tau$$
$$= \underbrace{\int_0^T A(\tau)d\tau}_{I_1} + \underbrace{2\int_0^T \Gamma(\tau)A(\tau)d\tau + \int_0^T \Gamma^2(\tau)A(\tau)d\tau}_{I_2}.$$

We then show that there exists a finite $T > 0$, which is dependent of the initial condition $s(0)$, such that the above integral is negative. For the integral I_1, we have

$$I_1 = -\int_0^T 1/5 + a\sin(\phi_2(\tau, s(0)))d\tau.$$

Let T be such that $\phi_2(T, s(0)) - \phi_2(0, s(0)) = 2\pi$ for the considered initial condition $s(0)$. From (6.8), it is observed that $2b \leq \dot{y} \leq 4b$, which implies that for any $s(0) \in \mathbb{R}^2$, T is finite since it satisfies $\pi/(2b) \leq T \leq \pi/b$. Implementing this T, we have $I_1 = -T/5$. Substituting $\|\Gamma(\tau)\| \leq e^{L\tau} - 1$ and $\|A(\tau)\| \leq L$ into I_2 we have

$$\|I_2\| \leq 2L\int_0^T (e^{L\tau}-1)d\tau + L\int_0^T (e^{L\tau}-1)^2 d\tau$$
$$= L\int_0^T (e^{2L\tau}-1)d\tau = \frac{1}{2}(e^{2LT}-1) - LT.$$

Consequently, we have

$$\int_0^T \Phi^2(\tau)A(\tau)d\tau \leq -\frac{1}{5}T - LT + \frac{1}{2}(e^{2LT}-1) := p(T).$$

Substituting it into (6.28) we arrive at

$$\int_0^T \dot{V}(x(\tau), y(\tau))d\tau \leq p(T)x^2(0). \tag{6.29}$$

It remains to show that there exists a positive solution to the inequality $p(z) < 0$. The first and second derivatives of p are $\dot{p}(z) = -1/5 - L + Le^{2Lz}$ and $\ddot{p}(z) = 2L^2e^{2Lz}$. It can be seen that $\dot{p}(z) = -1/5$ when $z = 0$ and $\ddot{p}(z) > 0$ for all z. Thus there is $z' > 0$ such that $p(z)$ is decreasing when $z \in [0, z']$ and increasing when $z > z'$. Since $p(z) = 0$ when $z = 0$, one knows that there is $z^* > 0$ such that $p(z) < 0$ for $z \in (0, z^*)$ and $p(z) > 0$ for $z > z^*$. In other words, there is $T^* > 0$ such that $p(T) < 0$ if $T < T^*$. Since $T \leq \pi/b$, one knows that the inequality $T < T^*$ is ensured if

$$b > \frac{\pi}{2T^*}. \tag{6.30}$$

For any given b such that (6.30) is satisfied, we know that $\pi/(2b) \leq T \leq \pi/b$ for any initial condition $s(0) \in \mathbb{R}^2$. Then there certainly exists a $\beta_3 > 0$ such that $p(T) \leq -\beta_3$. Subsequently, the inequality (6.29) becomes $\int_0^T \dot{V}(x(\tau), y(\tau))d\tau \leq -\beta_3 \|x(0)\|^2$, which yields the partial exponential stability of $x = 0$ uniformly in y according to the criterion in Theorem 6.4.

Remark 6.1. *Note that the parameter a can be arbitrary ($\dot{x}$ is allowed to be positive), and the partial exponential stability of the system is still guaranteed if the parameter b is sufficiently large. The key idea behind the analysis is that the considered Lyapunov function can be increasing due to a large value of the parameter a, but a sufficiently large parameter b ensures that it is always decreasing in average, which can give rise to stability and convergence. Moreover, the term $-ax\sin(y)$ can also be regarded as a fast time-varying perturbation for a large parameter b, thus averaging techniques shown in [157, Chap. 10], [172] and [173] might be also applied to the analysis.*

A Simulation

We consider a spring-mass-damper system shown in Fig. 6.1. Let x be the position of the mass, and subsequently $\dot{x} := y$ be the velocity. Instead of a constant damper, we assume that the damping coefficient b is dependent of the velocity y in a way described by the following dynamics of this system

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -\frac{1}{m}kx - \frac{1}{m}(a + \sin z)y, \\
\dot{z} &= c - \sin y,
\end{aligned}$$

where $k, a > 0$ and $c > 1$. Note that the damping coefficient $b = a + \sin z$. Let $m = 2, k = 1, a = 0.2$, and denote $w = (x, y)^\top$. Choose

$$V(x, y, z) = \frac{1}{2}m(x^2 + y^2)$$

as a Lyapunov function candidate. After some simple calculation, its time derivative is

$$\dot{V} = -(a + \sin z)y^2.$$

Using existing results given in Lemma 6.1, one fails to prove the partial stability of $w = 0$ since the matrix Q depends on z, and is thus not always negative definite. However, we are able to prove that $w = 0$ is asymptotically w-stable uniformly in z using the new criterion we established in Theorem 6.3. Following similar steps to Section

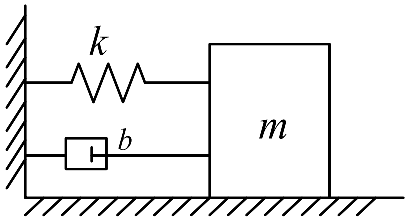


Figure 6.1: A spring-mass-damper system

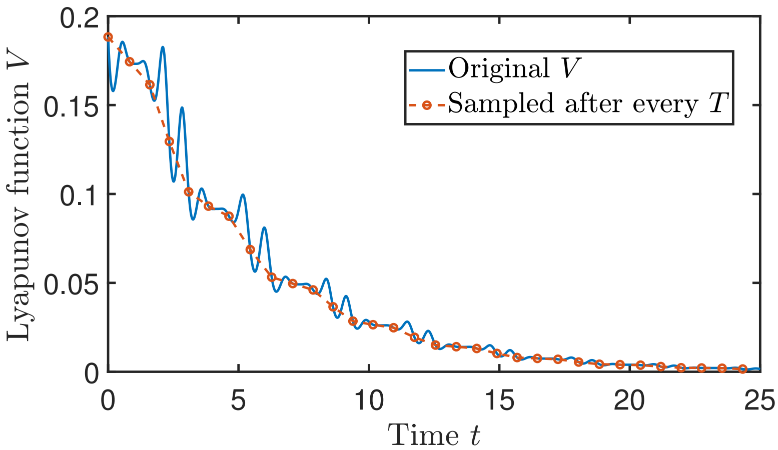


Figure 6.2: The trajectory of the Lyapunov function

6.2.3, one is able to show that for any initial condition $s(0) = \mathrm{col}(x(0), y(0), z(0)) \in \mathbb{R}^3$, there exists $T = T(s(0))$ satisfying $2\pi/(c+1) \leq T \leq 2\pi/(c-1)$ such that the inequality (6.23) holds, provided that c is sufficiently large. One possible way to identify such a T is just letting T be such that $z(T) - z(0) = 2\pi$. Let $c = 8$, and we simulate the spring-mass-damper system. The trajectory of the Lyapunov function candidate $V(x, y, z)$ is plotted in Fig. 6.2. One observes that V is not monotonically decreasing. However, if we sample it whenever z increases by 2π, the sample points are always decreasing, which is illustrated by the red dots in Fig. 6.2. These observations show that $y = 0$ is asymptotically stable, although $z \to \infty$ as $t \to \infty$, as long as the Lyapunov function decreases after a finite time, which validates our obtained results.

6.3 Partial Exponential Stability via Periodic Averaging

As a further extension of the previous section, in this section, we study exponential partial stability of a class of slow-fast systems, wherein the fast variable is a scalar. Various practical systems can be modeled by this class of slow-fast systems, such as semiconductor lasers [174], and mixed-mode oscillations in chemical systems [175], where the fast scalar variables are the photon density and a chemical concentration, respectively. In particular, fast time-varying systems can always be modeled in this way since the time variable t can be taken as the fast scalar [166]. However, existing criteria on partial stability [99–101] are not directly applicable to the analysis of these systems. Therefore, we aim at further developing new criteria to study exponential partial stability of the considered class of slow-fast systems.

In classic stability analysis of fast time-varying systems, averaging methods are widely used to establish criteria for full-state exponential stability [157, Chap. 10] [172] and also asymptotic stability [173]. Inspired by these works, we utilize the periodic averaging techniques and establish some criteria for partial exponential stability of the considered class of slow-fast systems. Unlike what is usually done in standard averaging, we construct an averaged system by averaging the original one over the fast scalar, which is in general different from the time variable. We show that partial exponential stability of the averaged system implies partial exponential stability of the original one. In contrast to the criteria we proposed in the previous section, this one using averaging methods is easier to test. Compared to the existing criteria for full-state exponential stability [157, 166, 173], the analysis in our case is much more challenging since some state variables are unstable. To construct the proof, we also develop a new converse Lyapunov theorem and some perturbation theories for partially exponentially stable systems. Compared to the converse Lyapunov theorem in [100, Theorem 4.4], we present two bounds for the partial derivatives of the Lyapunov function with respect to the stable and unstable states, respectively. Moreover, the obtained perturbation theories are the first-known ones for partial exponential stability analysis, although their counterparts [100, Chap. 9 and 10] for full-state stability have been widely used to analyze perturbed systems.

6.3.1 A Slow-Fast System

A wide range of systems exhibit multi-timescale dynamics, and among them many have a fast changing variable that is scalar. This motivates us to study a class of slow-fast systems in this section, of which dynamics are described by

$$\dot{x} = f_1(x, y, z), \tag{6.31a}$$

$$\dot{y} = f_2(x, y, z), \tag{6.31b}$$

$$\varepsilon \dot{z} = f_3(x, y, z), \tag{6.31c}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m, z \in \mathbb{R}$, and $\varepsilon > 0$ is a small constant. That is, x, y are the states of slow dynamics, and z is the state of fast dynamics. All the maps, $f_1 : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$, $f_2 : \mathbb{R}^{n+m+1} \to \mathbb{R}^m$, $f_3 : \mathbb{R}^{n+m+1} \to \mathbb{R}$, are continuously differentiable, and T-periodic in z, i.e., $f_i(x, y, z + T) = f_i(x, y, z)$ for all $i = 1, 2, 3$. Assume further that the solution to the system (6.31) exists for all $t \geq 0$. Moreover, $x = 0$ is a partial equilibrium point of the system (6.31), i.e., $f_1(0, y, z) = 0$ for any $y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Also, $f_2(0, y, z) = 0$ for any y and z. However, f_3 is not required to satisfy $f_3(0, y, z) = 0$.

The variable z appears in various problems. For instance, it can be the photon density of a semiconductor laser [174], the centroid position of rapidly flying UAVs that execute formation tasks [23], and in particular, the time variable in a time-varying system (where $\varepsilon \dot{z} = 1$). We are interested in studying the partial stability of the system (6.31). Let us first define uniform partial exponential stability, which is similar to that in Definition 6.1, but with an additional variable z.

Definition 6.2 ([100, Chap. 4])**.** *A partial equilibrium point $x = 0$ of the system* (6.31) *is* exponentially x-stable *uniformly in y and z if there exist $c_1, c_2, \delta > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| \leq c_1 \|x(0)\| e^{-c_2 t}$ for any $t \geq 0$ and $(y_0, z_0) \in \mathbb{R}^m \times \mathbb{R}$.*

Note that when we refer to this definition, we also say that $x = 0$ of the system (6.31) is partially exponentially stable or the system (6.31) is partially exponentially stable with respect to x. Some efforts have been made to study partial stability of nonlinear systems, [99], [100, Chap. 4], [101]. Although these results do not explicitly utilize the slow-fast structure, it is possible to apply them to slow-fast systems. Some Lyapunov criteria have been established in [100, Chap. 4], which are presented in Lemma 6.1. However, it is not always easy to verify partial stability by using such criteria. As a motivating example, we consider the following academic but suggestive model.

Example 6.2. *Consider a nonlinear system whose dynamics are described by*

$$\begin{aligned}
\dot{x} &= -x - 0.2x \sin y - 2x \cos z, \\
\dot{y} &= 2x \cos y + x \sin z, \\
\varepsilon \dot{z} &= 3 - \sin x + \cos y.
\end{aligned}$$

As will be shown later, for sufficiently small $\varepsilon > 0$, it is possible to prove that the partial equilibrium point $x = 0$ is exponentially stable uniformly in y and z. However, it is difficult to construct an appropriate Lyapunov function using the criteria in Lemma 6.1. For example, we choose $V = x^2$ as a Lyapunov function candidate. Its time derivative is $\dot{V} = -2(1 + 0.2 \sin y + 2 \cos z)x^2$, which can be positive for some y and z, while it is required by [100, Theorem 1] to be negative for any $x \neq 0$, y, and z in order to show the partial exponential stability. $\triangle$

Motivated by the above example, in the next subsection we aim at further developing Lyapunov theory for partial stability analysis of slow-fast systems.

6.3.2 Partial Stability of Slow-Fast Dynamics

In this subsection, our goal is to provide a new Lyapunov criterion for partial stability of slow-fast systems. Our analysis consists of several steps. We first construct reduced slow dynamics. Under some practically reasonable assumptions, the partial stability of the constructed slow dynamics and the original slow-fast system are shown to be equivalent. That is, analysis reduces to the partial stability analysis of the constructed slow dynamics. Moreover, since the original slow-fast system is periodic, the constructed one is also periodic.

Next, in order to study the partial stability of the constructed slow dynamics, we use an averaging method. For periodic systems, averaging methods have been widely used to establish criteria for the standard full-state exponential stability [157, Chap. 10] [172] and also asymptotic stability [173]. Inspired by these works, we will develop a new criterion for partial stability of the fast periodic dynamics via averaging. According to our new criterion, if the averaged system is partially exponentially stable, then the slow dynamics and consequently the original periodic slow-fast system is partially exponentially stable for sufficiently small $\varepsilon > 0$. It is worth emphasizing that compared with the case of full-state stability, partial stability analysis is much more challenging since some states are not stable.

Slow Dynamics

As the first step, we construct a reduced slow dynamics studied in the following subsections. One important fact is that the partial stability of the constructed slow dynamics is equivalent to that of the original slow-fast system (6.31) under the assumption blow.

Assume that for the fast subsystem:

$$f_3(x, y, z) \geq \alpha, \qquad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m, z \in \mathbb{R}, \tag{6.32}$$

or $f_3(x, y, z) \leq -\alpha$, where $\alpha > 0$. Note that we only consider the first inequality since these two inequalities are essentially the same. This assumption (6.32) is naturally satisfied for some practical problems such as vibration suppression of rotating machinery where f_3 is the angular velocity [176], spin stabilization of spacecrafts where f_3 describes the spin rate [177].

The assumption (6.32) implies that $t \mapsto z(t)$ can be interpreted as a change of time (recall that z is a scalar). In fact, (6.32) implies that for any given initial state $(x(0), y(0), z(0))$ of the slow-fast system (6.31), a part of the solution $z(t)$ is a strictly increasing function of t. That is, $t \mapsto z(t)$ is a global diffeomorphism from $[0, \infty)$ to $[0, \infty)$. In the new time axis $z(t)$, the slow-fast system becomes

$$\begin{aligned} \frac{dx(t)}{dz(t)} &= \frac{dx(t)}{dt}\frac{dt}{dz(t)} = \varepsilon \frac{f_1(x(t), y(t), z(t))}{f_3(x(t), y(t), z(t))}, \\ \frac{dy(t)}{dz(t)} &= \frac{dy(t)}{dt}\frac{dt}{dz(t)} = \varepsilon \frac{f_2(x(t), y(t), z(t))}{f_3(x(t), y(t), z(t))}, \\ \frac{dz(t)}{dz(t)} &= 1, \end{aligned}$$

The first two subsystems can be viewed as time-varying systems with the new time variable $z(t)$. Note that, since $t \mapsto z(t)$ is a global diffeomorphism, the partial stability with respect to x of the first two time-varying subsystems in the new time axis is equivalent to the partial stability with respect to x of the system (6.31) in the original time axis. Therefore, hereafter we focus on the first two time-varying subsystems in the new time axis. For the sake of simplicity of description, the first two time-varying subsystems are described by

$$\frac{dx}{dz} = \varepsilon h_1(x, y, z), \tag{6.33a}$$

$$\frac{dy}{dz} = \varepsilon h_2(x, y, z), \tag{6.33b}$$

where $h_1 = f_1/f_3$, and $h_2 = f_2/f_3$. From the properties of f_1, f_2, and f_3, it follows that both $h_1(0, y, z) = 0$ and $h_2(0, y, z) = 0$ for any $y \in \mathbb{R}^m$ and $z \in \mathbb{R}$, and the solution to the system (6.33) exists for all $z \geq 0$. Moreover, the constructed slow dynamics (6.33) is again T-periodic in z.

Partial Stability Conditions via Averaging

In order to study partial stability with respect to x of the constructed periodic slow dynamics (6.33), we use an averaging technique for periodic systems. The averaged system obtained from the slow dynamics (6.33) can be used for the partial stability analysis of the slow dynamics (6.33) before averaging. From the discussion in the previous subsection, the partial stability of the slow dynamics (6.33) is equivalent to that of the original slow-fast system (6.31).

Since the slow dynamics (6.33) is T-periodic in z, it is possible to apply an averaging method. Its partially averaged system is given by

$$\frac{dw}{dz} = \varepsilon h_{\mathrm{av}}(w, v), \tag{6.34a}$$
$$\frac{dv}{dz} = \varepsilon h_2(w, v, z), \tag{6.34b}$$

where the function h_{av} is defined by

$$h_{\mathrm{av}}(w, v) = \frac{1}{T}\int_0^T h_1(w, v, \tau)d\tau, \tag{6.35}$$

where $h_{\mathrm{av}}(0, v) = 0$ for any $v \in \mathbb{R}^m$ from $h_1(0, v, z) = 0$. Note that only the dynamics of w is averaged with respect to z.

In fact, if the averaged system (6.34) is partially exponentially stable with respect to w, then the periodic slow system (6.33) is partially exponentially stable with respect to x for sufficiently small $\varepsilon > 0$. This implies that partial exponential stability of the original slow-fast system (6.31) can be verified by using the averaged system (6.33). This fact is stated formally as follows, which is one of the main results in this section.

Theorem 6.5. *Suppose that $w = 0$ of the averaged system* (6.34) *is partially exponentially stable uniformly in v, i.e., there exists $\delta > 0$ such that for any $z_0 \in \mathbb{R}$ and $w(0) \in \mathcal{B}_\delta$,*

$$\|w(z)\| \le k\|w(0)\|e^{-\lambda(z-z_0)}, \qquad k, \lambda > 0, \forall z \ge z_0. \tag{6.36}$$

Assume that there are $L_1, L_2 > 0$ such that for any $x \in \mathcal{B}_\delta, y \in \mathbb{R}^m, z \in \mathbb{R}$, the functions h_1 and h_2 in (6.33) *satisfy*

$$\left\|\frac{\partial h_1}{\partial x}(x, y, z)\right\| \le L_1, \qquad \left\|\frac{\partial h_2}{\partial x}(x, y, z)\right\| \le L_2. \tag{6.37}$$

Then, there exists $\varepsilon_1 > 0$ such that, for any $\varepsilon < \varepsilon_1$, a partial equilibrium point $x = 0$ of the system (6.33) *is exponentially stable uniformly in y. As a consequence, for any $\varepsilon < \varepsilon_1$, a partial equilibrium point $x = 0$ of the system* (6.31) *is exponentially stable uniformly in y and z.* △

The following subsections are dedicated to proving this theorem. Before providing the proof, we illustrate its utility. First, let us look back at Example 6.2, and see how the obtained results can be applied.

Continuation of Example 6.2: As $3 - \sin x + \cos y \geq 1$ for any x, y, the property (6.32) holds. Then, one can construct the averaged system (6.34) of the system in Example 6.2 as

$$\begin{aligned}\frac{dw}{dz} &= \varepsilon\frac{-w - 0.2w\sin v}{3 - \sin w + \cos v},\\ \frac{dv}{dz} &= \varepsilon\frac{2w\cos v + w\sin z}{3 - \sin w + \cos v}.\end{aligned}$$

Choose a Lyapunov function candidate as $V(w, v, z) = w^2$. Then, it holds that

$$\frac{dV}{dz} = -2\varepsilon \cdot \frac{1 + 0.2\sin v}{3 - \sin w + \cos v}w^2 \leq -\frac{8}{15}\varepsilon w^2.$$

According to [100, Theorem 1], $w = 0$ of the averaged system is partially exponentially stable. From Theorem 6.5, one can conclude that $x = 0$ of the original system in Example 6.2 is partially exponentially stable if $\varepsilon > 0$ is sufficiently small. △

By using averaging techniques, Theorem 6.5 provides a new way to study partially stability of slow-fast systems for which the existing criteria is difficult to apply. As its another application, we can cover the conventional criteria [157, Chap. 10] and [172] for exponential stability of fast time-varying systems. Consider the following system with respect to x,

$$\dot{x} = f_1(x, z), \tag{6.38}$$

$$\varepsilon\dot{z} = f_3(x, z), \tag{6.39}$$

where f_1 and f_3 satisfy all the assumption made for system (6.31). The difference from (6.31) is that there is no variable y. To study partial stability with respect to x, we apply the change of time-axis, $t \to z$. Then, we have

$$\frac{dx}{dz} = \varepsilon h_1(x, z). \tag{6.40}$$

Next, compute the averaged system of the fast subsystem

$$\frac{dw}{dz} = \varepsilon\hat{h}_{\text{av}}(w), \tag{6.41}$$

where the function $\hat{h}_{\text{av}}$ is defined by

$$\hat{h}_{\text{av}}(w) = \frac{1}{T}\int_0^T \hat{h}_1(w, \tau)d\tau, \qquad \hat{h}_1(w, z) = \frac{f_1(w, z)}{f_3(w, z)} \tag{6.42}$$

As expected, if the averaged system (6.41) is exponentially stable, then the partial stability of (6.38) is ensured as long as $\varepsilon > 0$ is sufficiently small, which is formally stated in the following corollary. If $f_3(x, z) = 1$ for all $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$, this corollary reduces to the criteria in [157, Chap. 10] and [172] for exponential stability of fast time-varying systems.

Corollary 6.1. *Suppose that $w = 0$ is exponentially stable for the averaged system* (6.41)*. Assume that there is $L > 0$ such that for any $x \in \mathcal{B}_\delta, z \in \mathbb{R}$ the function h_1 in* (6.42) *satisfies*

$$\left\| \frac{\partial h_1}{\partial x}(x, z) \right\| \leq L. \tag{6.43}$$

Then, there exists $\varepsilon_1 > 0$ such that, for any $\varepsilon < \varepsilon_1$, a partial equilibrium $x = 0$ of the system (6.38) *is partially exponentially stable uniformly in z.* △

In next section, we show how our results on partial exponential stability can be applied to remote synchronization in a simple network of Kuramoto oscillators. Before that in the following subsections, we construct the proof of Theorem 6.5.

6.3.3 A converse Lyapunov Theorem and Some Perturbation Theorems

In the following subsections, our objective is to prove Theorem 6.5, that is to show the partial exponential stability of the averaged system (6.34) implies that of the periodic slow system (6.33). Recall that partial stability of the slow system (6.33) is equivalent to that of the original slow-fast system (6.31) under (6.32).

For conventional full-state exponential stability analysis, the original system is regarded as a perturbed system of the averaged one. As long as the perturbation characterized by ε is sufficiently small, the exponential stability of the original system is ensured [172], [157, Chap. 10]. Similar ideas are used in this section for partial exponential stability analysis. Instead of full-state stability, we only require the averaged system (6.34) to be partially exponentially stable.

A New Converse Lyapunov Theorem

In order to show the partial exponential stability of the periodic slow system (6.33), we use Lyapunov theory. First, we construct a Lyapunov function for a partially exponentially stable averaged system. Then, by using this Lyapunov function, we show the partial exponential stability of the periodic slow system if the perturbation is sufficiently small. This subsubsection is dedicated to constructing a Lyapunov function. That is, we provide a new converse Lyapunov theorems for partial stability.

As a generalized form of (6.34), we consider the following time-varying systems in this section

$$\frac{dw}{dz} = \varphi_1(w, v, z), \tag{6.44a}$$

$$\frac{dv}{dz} = \varphi_2(w, v, z), \tag{6.44b}$$

where $w \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, $z \in \mathbb{R}$, and the functions, $\varphi_1 : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$, $\varphi_2 : \mathbb{R}^{n+m+1} \to \mathbb{R}^m$ are continuously differentiable. Moreover, it holds that $\varphi_1(0, v, z) = 0$ and $\varphi_2(0, v, z) = 0$ for any $v \in \mathbb{R}^m$. We further assume that for any z_0 the solution to the system (6.44) exists for all $z \geq z_0$.

Now, we provide a converse theorem for exponential partial stability of the system (6.44), which is directly applicable to the averaged system (6.34).

Theorem 6.6. *Suppose that $w = 0$ is partially exponentially stable uniformly in v for the system* (6.44)*, i.e., there exists $\delta > 0$ such that for any $z_0 \in \mathbb{R}$ and $w(0) \in \mathcal{B}_\delta$,*

$$\|w(z)\| \leq k\|w(0)\|e^{-\lambda(z-z_0)}, \qquad k, \lambda > 0, \forall z \geq z_0. \tag{6.45}$$

Also, assume that there are $L_1, L_2 > 0$ such that

$$\left\|\frac{\partial \varphi_1}{\partial w}(w, v, z)\right\| \leq L_1, \qquad \left\|\frac{\partial \varphi_2}{\partial w}(w, v, z)\right\| \leq L_2, \tag{6.46}$$

for any $w \in \mathcal{B}_\delta, v \in \mathbb{R}^m, z \in \mathbb{R}$. Then, there exists a function $V : \mathcal{B}_\delta \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ that satisfies the following inequalities:

$$c_1\|w\|^2 \leq V(w, v, z) \leq c_2\|w\|^2, \tag{6.47}$$

$$\frac{\partial V}{\partial z} + \frac{\partial V}{\partial w}\varphi_1(w, v, z) + \frac{\partial V}{\partial v}\varphi_2(w, v, z) \leq -c_3\|w\|^2, \tag{6.48}$$

$$\left\|\frac{\partial V}{\partial w}\right\| \leq c_4\|w\|, \tag{6.49}$$

$$\left\|\frac{\partial V}{\partial v}\right\| \leq c_5\|w\|, \tag{6.50}$$

for some positive constants c_1, c_2, c_3, c_4 and c_5. $\triangle$

The same uniform boundedness assumptions on the partial derivatives of the functions φ_1 and φ_2 are actually made in Theorem 6.6 and Theorem 4.4 of [100]. Unlike the theorem in [100], we work on time-varying systems. Moreover, we obtain additional two bounds for the partial derivative of V, namely (6.49) and (6.50) by assuming $\varphi_2(0, v, z) = 0$ for any v and z. Thus, our proof is more involved, which can be found as follows. Before proving Theorem 6.6, we first present an intermediate result.

Proposition 6.1. *Consider a continuously differentiable function $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, which satisfies $h(0, v) = 0$ for any $v \in \mathbb{R}^m$. Suppose that there exists a connected set $\mathcal{D} \subset \mathbb{R}^n$ containing the origin $x = 0$ such that*

$$\left\|\frac{\partial h}{\partial w}(w, v)\right\| \leq l_1, \qquad \forall w \in \mathcal{D}, v \in \mathbb{R}^m,$$

for a positive constant l_1. Then, there exists $l_2 > 0$ such that

$$\left\| \frac{\partial h}{\partial v}(w, v) \right\| \leq l_2 \|w\|, \qquad \forall w \in \mathcal{D}, v \in \mathbb{R}^m.$$

Proof. Due to the continuous differentiability of h, it follows from the mean value theorem that for any $w \in \mathcal{D}$, and $v, \delta \in \mathbb{R}^m$, there exists $0 \leq \lambda \leq 1$ such that

$$h_i(w, v + \delta) - h_i(w, v) = \frac{\partial h_i}{\partial v}(w, v + \lambda\delta) \cdot \delta.$$

Since $\|\partial h/\partial w\| \leq l_1$ and $h(0, v) = 0$, we have

$$\begin{aligned} &\|h_i(w, v + \delta) - h_i(w, v)\| \\ &\leq \|h_i(w, v + \delta) - h_i(0, v + \delta)\| + \|h_i(w, v) - h_i(0, v)\| \\ &\leq 2l_1\|w\|, \end{aligned} \tag{6.51}$$

where the last inequality has used the fact that $\|\partial h_i/\partial w\| \leq \|\partial h/\partial v\| \leq l_1$. We then observe that the inequality

$$\left\| \frac{\partial h_i}{\partial v}(w, v + \lambda\delta) \cdot \delta \right\| \leq 2l_1\|w\|$$

holds for any $w \in \mathcal{D}$, and $v, \delta \in \mathbb{R}^m$, which implies that for any i there exists $l_1' > 0$ such that $\|\partial h_i/\partial v\| \leq l_1'\|w\|$. As a consequence, there is $l_2 > 0$ so that $\|\partial h/\partial v\| \leq l_2\|w\|$. □

We are now ready to prove Theorem 6.6.

Proof of Theorem 6.6. Let $\phi_1(\tau; w, v, z)$ and $\phi_2(\tau; w, v, z)$ denote the solution to the system (6.44a) and (6.44b) that starts at (w, v, z), respectively; note that $\phi_1(z; w, v, z) = x$ and $\phi_2(z; w, v, z) = y$. Let

$$V(w, v, z) = \int_z^{z+\delta} \|\phi_1(\tau; w, v, z)\|_2^2 d\tau. \tag{6.52}$$

Following similar steps as those in Theorem 4.14 of [157] and Theorem 4.4 of [100], one can show that

$$\frac{1}{2L_1}(1 - e^{-2L_1\delta})\|w\|_2^2 \leq V(w, v, z) \leq \frac{k^2}{2\lambda}(1 - e^{-2\lambda\delta})\|w\|_2^2,$$

and $\dot{V}(w, v) = -(1 - k^2 e^{-2\lambda\delta})\|w\|_2^2$, which proves the inequalities (6.47) and (6.48).

We next prove the inequalities (6.49) and (6.50). Let $\phi = \text{col}(\phi_1, \phi_2), \varphi = \text{col}(\varphi_1, \varphi_2)$, and for $k = 1, 2$, denote

$$\phi_{k,w}(\tau; w, v, z) = \frac{\partial}{\partial w}\phi_k(\tau; w, v, z).$$

and $\phi'_w = \partial\phi/\partial w$. Note that $\phi_1(\tau; w, v, z)$ and $\phi_2(\tau; w, v, z)$ satisfy

$$\phi_1(\tau; w, v, z) = x + \int_z^\tau \varphi_1(\phi_1(s; w, v, z), \phi_2(s; w, v, z))ds,$$
$$\phi_2(\tau; w, v, z) = y + \int_z^\tau \varphi_2(\phi_2(s; w, v, z), \phi_2(s; w, v, z))ds.$$

Then, the partial derivative ϕ'_w is

$$\phi'_w(\tau; w, v, z) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \int_z^\tau \frac{\partial\varphi}{\partial\phi}\phi'_x(s; w, v, z)ds. \tag{6.53}$$

Recall that φ_1 and φ_2 are continuously differentiable, and satisfy $\varphi_1(0, v, z) = 0$ and $\varphi_2(0, v, z) = 0$ for any v and z, and $\|\partial\varphi_1/\partial w\|_2 \leq L_1$ and $\|\partial\varphi_2/\partial w\|_2 \leq L_2$ for any $w \in \mathcal{B}_\delta$ and $v \in \mathbb{R}^m, t \in \mathbb{R}$. It then follows from Proposition 6.1 that $\|\partial\varphi_1/\partial v\|_2 \leq L'_1$ and $\|\partial\varphi_2/\partial v\|_2 \leq L'_2$ for some positive constants L'_1 and L'_2, since $\|w\|$ is upper bounded by some constant in a compact set $\mathcal{B}_\delta$. Equivalently, there exists $L > 0$ such that

$$\left\|\frac{\partial\varphi}{\partial X}\right\|_2 \leq L, \qquad \forall X \in \mathcal{B}_\delta \times \mathbb{R}^m, \tag{6.54}$$

where $X = \text{col}(w, v)$. Consequently, $\|\partial\varphi/\partial\phi\|_2 \leq L$, and it then follows from (6.53) that

$$\|\phi'_w(\tau; w, v, z)\|_2 \leq 1 + L\int_z^\tau \|\phi'_w(s; , w, v, z)\|_2 ds,$$

which implies that $\|\phi'_w(\tau; w, v, z)\|_2 \leq e^{L(\tau-z)}$ by Grönwall's lemma [100, Lemma 2.2]. Since $\|\phi_{1,w}\|_2 \leq \|\phi_{1,w}, \phi_{2,w}\|_2 = \|\phi'_w(\tau; w, v, z)\|_2$, it holds that

$$\|\phi_{1,w}(\tau; w, v, z)\|_2 \leq e^{L(\tau-z)}. \tag{6.55}$$

From (6.45), it holds that $\|\phi_1(\tau; w, v, z)\| \leq K\|w\|_2 e^{-\lambda(\tau-z)}$. Then, the partial derivative $\partial V/\partial w$ satisfies

$$\begin{aligned}\left\|\frac{\partial V}{\partial w}\right\|_2 &= \left\|\int_z^{z+\delta} 2\phi_1^\top(\tau; w, v, z)\phi_{1,w}(\tau; w, v, z)d\tau\right\|_2 \\ &\leq \int_z^{z+\delta} 2\,\|\phi_1(\tau; w, v, z)\| \cdot \|\phi_{1,w}(\tau; w, v, z)\|_2 d\tau \\ &\leq \int_z^{z+\delta} 2k\|w\|_2 e^{-\lambda(\tau-z)}e^{L(\tau-z)}d\tau = c_4\|w\|_2,\end{aligned}$$

with $c_4 = 2k(1 - e^{-(\lambda - L)\delta})/(\lambda - L)$, which proves the inequality (6.49).

Following the similar line, one can show there exists $c_5 > 0$ such that (6.50) are satisfied, which completes the proof. $\square$

Some Perturbation Theorems

In the previous subsubsection, we have constructed a converse Lyapunov theorem. By applying this to a partially exponentially stable averaged system (6.34), one can construct a Lyapunov function satisfying all conditions in Theorem 6.6. By using this Lyapunov function, we consider to study the partial exponential stability of the periodic slow dynamics (6.33). As typically done in averaging methods, we consider periodic slow dynamics (6.33) as a perturbed system of its averaged system (6.34). Then, we conclude the partial exponential stability of the periodic slow dynamics (6.33) by using the Lyapunov function for its averaged system (6.34).

To this end, we study the following perturbed system of the system (6.44) in the previous subsubsection:

$$\frac{dw_p}{dz} = \varphi_1(w_p, v_p, z) + g_1(w_p, v_p, z), \tag{6.56a}$$

$$\frac{dv_p}{dz} = \varphi_2(w_p, v_p, z) + g_2(w_p, v_p, z), \tag{6.56b}$$

where $g_1 : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ and $g_2 : \mathbb{R}^{n+m+1} \to \mathbb{R}^m$ are piecewise continuous in z and locally Lipschitz in (w_p, v_p). Particularly, we assume that the perturbation terms satisfy the bounds

$$\|g_1(w_p, v_p, z)\| \le \gamma_1(z)\|w_p\| + \psi_1(z), \tag{6.57}$$

$$\|g_2(w_p, v_p, z)\| \le \gamma_2(z)\|w_p\| + \psi_2(z), \tag{6.58}$$

where $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}$ are nonnegative and continuous for all $z \in \mathbb{R}$, and $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ are nonnegative, continuous and bounded for all $z \in \mathbb{R}$.

The following theorem presents some results on the asymptotic behavior of the perturbed system (6.56) when the the nominal system (6.44) has a partially exponentially stable equilibrium point $w_p = 0$.

Theorem 6.7. *Suppose that the nominal system* (6.44) *satisfies all the assumptions in Theorem 6.6. Also, assume that the perturbation terms* $g_1(w_p, v_p, z)$ *and* $g_2(w_p, v_p, z)$ *are respectively bounds as in* (6.57) *and* (6.58) *for* γ_1, γ_2 *and* ψ_1, ψ_2 *satisfying the following inequalities*

$$c_4 \int_{z_0}^{z} \gamma_1(\tau) d\tau + c_5 \int_{z_0}^{z} \gamma_2(\tau) d\tau \le \kappa(z - z_0) + \eta, \tag{6.59}$$

where

$$0 \leq \kappa < \frac{c_1 c_3}{c_2}, \qquad \eta \geq 0; \tag{6.60}$$

and

$$c_4 \psi_1(z) + c_5 \psi_2(z) < \frac{2c_1 k_1 \delta}{k_2}, \qquad \forall z \geq z_0, \tag{6.61}$$

where

$$k_1 = \frac{c_3}{2c_2} - \frac{\kappa}{2c_1}, \qquad k_2 = \exp\left(\frac{\eta}{2c_1}\right). \tag{6.62}$$

Then, the solution to the perturbed (6.56) *satisfies*

$$\begin{aligned}\|w_p(z)\| \leq k_2 \sqrt{\frac{c_2}{c_1}} \|w_p(z_0)\| e^{-k_1(z-z_0)} \\ + \frac{k_2}{2c_1} \int_{z_0}^{z} e^{-k_1(z-\tau)} \psi(\tau) d\tau, \qquad \forall z \geq z_0.\end{aligned}$$

for any initial time $z_0 \in \mathbb{R}$ *and any initial state* $w_p(z_0) \in \mathbb{R}^n$ *and* $v_p(z_0) \in \mathbb{R}^m$ *such that*

$$\|w_p(z_0)\| < \frac{\delta}{k_2} \sqrt{\frac{c_1}{c_2}}. \tag{6.63}$$

△

A similar result is found in [157, Lemma 9.4], where the nominal system is assumed to be exponentially stable. With some perturbation, the asymptotic behavior of the full state is reported there. In contrast, the nominal system is only assumed to be partially exponentially stable in Theorem 6.7, and we show that the asymptotic behavior of a part of the states w_p for the perturbed system follows some specific rule, without concerning how the other part of states, v_p, is evolving. The proof is based on the constructed Lyapunov function in Theorem 6.6; for more details, see as follows.

Proof. From the assumption for the nominal system (6.44), there is a Lyapunov function V satisfying all conditions in Theorem 6.6. Here, we use this V to estimate the convergence speed of the perturbed system (6.56).

For any $w_p \in \mathcal{B}_\delta$ and $v_p \in \mathbb{R}^m, t \in \mathbb{R}$, the time derivative of the Lyapunov function

$V(x, y, t)$ along the trajectories of the perturbed system (6.56) satisfies

$$\begin{aligned}\frac{dV}{dz} =& \frac{\partial V}{\partial z} + \frac{\partial V}{\partial w_p}\left(\varphi_1(w_p, v_p, z) + g_1(w_p, v_p, z)\right) \\ &+ \frac{\partial V}{\partial v_p}\left(\varphi_2(w_p, v_p, z) + g_2(w_p, v_p, z)\right) \\ \leq& - c_3\|w_p\|^2 + \left(c_4\gamma_1(z) + c_5\gamma_2(z)\right)\|w_p\|^2 \\ &+ \left(c_4\psi_1(z) + c_5\psi_2(z)\right)\|w_p\|,\end{aligned}$$

where the last inequality follows from the inequalities (6.48), (6.49), and (6.50). For simplicity, denote $\gamma(z) = c_4\gamma_1(z) + c_5\gamma_2(z)$ and $\psi(z) = c_4\psi_1(z) + c_5\psi_2(z)$. By the inequality (6.47) one can obtain an upper bound for $\dot{V}$ given by

$$\frac{dV}{dz} \leq -\left[\frac{c_3}{c_2} - \frac{1}{c_1}\gamma(z)\right]V + \frac{1}{\sqrt{c_1}}\psi(z)\sqrt{V}.$$

Let $W(w_p, v_p, z) = \sqrt{V(w_p, v_p, z)}$, and when $V \neq 0$ its time derivative satisfies

$$\frac{dW}{dz} = \frac{dV/dz}{2\sqrt{V}} \leq -\frac{1}{2}\left[\frac{c_3}{c_2} - \frac{1}{c_1}\gamma(z)\right]W + \frac{1}{2\sqrt{c_1}}\psi(z). \tag{6.64}$$

When $V = 0$, following the same idea as the proof of Lemma 9.4 in [157] one can show that the Dini derivative of W satisfies

$$D^+W \leq \frac{1}{2\sqrt{c_1}}\psi(z),$$

which implies that D^+W satisfies (6.64) for all values of V. Using the comparison lemma [157, Lemma 3.4], one can show that $W(z)$ satisfies the following inequality

$$W(z) \leq \Phi(z, z_0)W(z_0) + \frac{1}{2\sqrt{c_1}}\int_{z_0}^{z}\Phi(z, \tau)\psi(\tau)d\tau, \tag{6.65}$$

where the transition function is

$$\Phi(z, z_0) = \exp\left(-\frac{c_3}{2c_2}(z - z_0) + \frac{1}{2c_1}\int_{z_0}^{z}\gamma(\tau)d\tau\right). \tag{6.66}$$

Substituting (6.59) and (6.60) into (6.66), we have

$$\Phi(z, z_0) \leq k_2 e^{-k_1(z - z_0)}, \tag{6.67}$$

with k_1 and k_2 given by (6.62).

From the inequality (6.47), we obtain $\sqrt{c_1}\|w_p\| \leq W \leq \sqrt{c_2}\|w_p\|$. Then, for any $z \geq z_0$ such that $\|w_p(z)\| \in \mathcal{B}_\delta$, it follows from the inequalities (6.65) and (6.67) that

$$\|w_p(z)\| \leq k_2\sqrt{\frac{c_2}{c_1}}\|w_p(z_0)\|e^{-k_1(z-z_0)} + \frac{k_2}{2c_1}\int_{z_0}^{z} e^{-k_1(z-\tau)}\psi(\tau)d\tau, \tag{6.68}$$

for any $z \geq z_0$ such that $\|w_p(z)\| \in \mathcal{B}_\delta$. Under the assumption (6.61), if the initial condition satisfies (6.63), it then holds that

$$\|w_p(z)\| < \delta e^{-k_1(z-z_0)} + \delta(1 - e^{-k_1(z-z_0)}) = \delta, \forall z \geq z_0,$$

which ensures that the inequality (6.68) holds for any $z \geq z_0$. The proof is complete. □

We next consider a particular case where g_1 and g_2 in (6.56) are vanishing perturbations, i.e., $\psi_1(z)$ and $\psi_2(z)$ in (6.57) satisfy $\psi_1(z) = \psi_2(z) = 0$, and obtain the next corollary. In fact, this corollary is used to prove Theorem 6.5.

Corollary 6.2. *Suppose that the nominal system* (6.44) *satisfies all the assumptions in Theorem 6.6. Also, assume that the perturbation terms $g_1(w_p, v_p, z)$ and $g_2(w_p, v_p, z)$ are respectively bounded by $\|g_1(w_p, v_p, z)\| \leq \gamma_1(z)\|w_p\|$ and $\|g_2(w_p, v_p, z)\| \leq \gamma_2(z)\|w_p\|$ for γ_1 and γ_2 satisfying* (6.59) *and* (6.60)*, i.e. $\psi_1(\cdot) = 0$ and $\psi_2(\cdot) = 0$. Then, $w_p = 0$ is partially exponentially stable uniformly in v_p for the system* (6.56)*. Moreover, the solution to* (6.56) *satisfies*

$$\|w_p(z)\| \leq k_2\sqrt{\frac{c_2}{c_1}}\|w_p(z_0)\|e^{-k_1(z-z_0)}, \qquad \forall z \geq z_0.$$

for any initial time $z_0 \in \mathbb{R}$ and any initial condition $w_p(z_0) \in \mathbb{R}^n$ and $v_p(z_0) \in \mathbb{R}^m$ satisfying (6.63)*.*

In the next subsection, we show how the results obtained in the previous and this subsubsections enable us to use averaging techniques to study the partial stability of a periodic slow dynamics (6.33) from its averaged system (6.34).

6.3.4 Proof of Theorem 6.5

We first present an intermediate result.

Proposition 6.2. *Consider function $u(w_p, v_p, z)$ defined in* (6.72)*. For any $w_p \in \mathcal{B}_\delta$, $v_p \in \mathbb{R}^m$, $z \in \mathbb{R}$, $\|u(w_p, v_p, z)\|$, $\|\partial u/\partial w_p\|$, and $\|\partial u/\partial v_p\|$ are all bounded.*

Proof. First, we prove that $\|u(w_p, v_p, z)\|$ is bounded. One can observe that $u(w_p, v_p, z)$ is T-periodic in z since $\Delta(w_p, v_p, z)$ is. For any $z \geq 0$, there exists a nonnegative integer N_1 and z' satisfying $0 \leq z' < T$. such that $z = N_1T + z'$. Then, using (6.73) we have

$$\begin{aligned}\int_0^z \Delta(w_p, v_p, \tau)d\tau &= \int_0^{N_1T} \Delta(w_p, v_p, \tau)d\tau + \int_0^{z'} \Delta(w_p, v_p, \tau)d\tau,\\ &= \int_0^{z'} \Delta(w_p, v_p, \tau)d\tau.\end{aligned}$$

Next, the partial derivative of Δ with respect to w satisfies

$$\begin{aligned}&\left\|\frac{\partial \Delta}{\partial w_p}(w_p, v_p, z)\right\|\\ &= \left\|\frac{\partial h_1}{\partial w_p}(w_p, v_p, z) - \frac{1}{T}\int_0^T \frac{\partial h_1}{\partial w_p}(w_p, v_p, \tau)d\tau\right\| \leq 2L_1,\end{aligned}$$

where the triangle inequality and (6.37) have been used. Using this inequality and $\Delta(0, v_p, z) = 0$ to (6.72) yields

$$\begin{aligned}\|u(w_p, v_p, z)\| &\leq \int_0^{z'} \|\Delta(w_p, v_p, \tau) - \Delta(0, v_p, \tau)\|d\tau\\ &\leq 2z'L_1\|w_p\| \leq 2TL_1\|w_p\|. \end{aligned} \tag{6.69}$$

For any $w_p \in \mathcal{B}_\delta$ and $v_p \in \mathbb{R}^m, z \in \mathbb{R}$, it is clear that $\|u(w_p, v_p, z)\| \leq 2TL_1\delta$.

Second, we prove $\|\partial u/\partial w_p\|$ is bounded. Since $\partial u/\partial w_p$ is T-periodic in z and satisfies $\int_0^T \frac{\partial u}{\partial w_p}(w_p, v_p, \tau)d\tau = 0$, for any $z \geq 0$, there exists a nonnegative integer N_2 and z'' satisfying $0 \leq z'' \leq T$ such that $z = N_2T + z''$ such that

$$\frac{\partial u}{\partial w_p}(w_p, v_p, z) = \int_0^{z''} \frac{\partial \Delta}{\partial w_p}(w_p, v_p, \tau)d\tau,$$

which implies that for any $w_p \in \mathcal{B}_\delta$, $v_p \in \mathbb{R}^m$, and $z \in \mathbb{R}$,

$$\begin{aligned}\left\|\frac{\partial u}{\partial w_p}(w_p, v_p, z)\right\| &\leq \int_0^{z''} \left\|\frac{\partial \Delta}{\partial w_p}(w_p, v_p, \tau)\right\| d\tau\\ &\leq 2z''L_1 \leq 2TL_1.\end{aligned}$$

Finally, we prove $\|\partial u/\partial v_p\|$ is bounded. Since $h_1(0, v_p, z) = 0$ for any v_p and z, from Proposition 6.1, there exists $L_1' > 0$ such that for any $w_p \in \mathcal{B}_\delta$, and $v_p \in \mathbb{R}^m, z \in \mathbb{R}$,

$$\left\|\frac{\partial h_1}{\partial v_p}(w_p, v_p, z)\right\| \leq L_1'\|w_p\|. \tag{6.70}$$

Then, the partial derivative of Δ with respect to v_p satisfies

$$\begin{aligned}
&\left\|\frac{\partial \Delta}{\partial v_p}(w_p, v_p, z)\right\| \\
&= \left\|\frac{\partial h_1}{\partial v_p}(w_p, v_p, z) - \frac{1}{T}\int_0^T \frac{\partial h_1}{\partial v_p}(w_p, v_p, \tau)d\tau\right\| \\
&\leq 2L_1'\|w_p\|,
\end{aligned}$$

which implies that $\left\|\frac{\partial u}{\partial v_p}(w_p, v_p, z)\right\| \leq 2TL_1'\|w_p\| \leq 2TL_1'\delta$ for any $w_p \in \mathcal{B}_\delta$. □

Now, we are ready to provide the proof of Theorem 6.5.

Proof of Theorem 6.5. First, in order to describe the original slow system (6.33) as a perturbation of the averaged system (6.34), we substitute the following into (x, y) of the original slow system (6.33):

$$x = w_p + \varepsilon u(w_p, v_p, z), \tag{6.71a}$$

$$y = v_p, \tag{6.71b}$$

where

$$u(w_p, v_p, z) = \int_0^z \Delta(w_p, v_p, \tau)d\tau, \tag{6.72}$$

with $\Delta(w_p, v_p, z) = h_1(w_p, v_p, z) - h_{\mathrm{av}}(w_p, v_p)$. From the definition of h_{av} in (6.35), it holds that

$$\int_0^T \Delta(w_p, v_p, \tau)d\tau = 0. \tag{6.73}$$

After substituting (6.71) into (6.33), we obtain the following

$$\begin{aligned}
\frac{dx}{dz} &= \frac{dw_p}{dz} + \varepsilon\frac{\partial u}{\partial z} + \varepsilon\frac{\partial u}{\partial w_p}\frac{dw_p}{dz} + \varepsilon\frac{\partial u}{\partial v_p}\frac{dv_p}{dz}, \\
\frac{dy}{dz} &= \frac{dv_p}{dz}.
\end{aligned}$$

Substituting (6.33) and (6.71a) into the above equations yields

$$\begin{aligned}
&P(\varepsilon)\begin{bmatrix} \frac{dw_p}{dz} \\ \frac{dv_p}{dz} \end{bmatrix} = \\
&\begin{bmatrix} \varepsilon h_1(w_p + \varepsilon u, v_p, z) - \varepsilon h_1(w_p, v_p, z) + \varepsilon h_{av}(w_p, v_p) \\ \varepsilon h_2(w_p + \varepsilon u, v_p, z) \end{bmatrix},
\end{aligned} \tag{6.74}$$

where

$$P(\varepsilon) = \begin{bmatrix} I + \varepsilon \frac{\partial u}{\partial w_p} & \varepsilon \frac{\partial u}{\partial y} \\ 0 & I \end{bmatrix}.$$

In the following, we show that the obtained dynamics (6.74) can be viewed as a perturbation of the averaged system (6.34). Therefore, Corollary 6.2 can be used in order to show the partial stability of the obtained dynamics from that of the averaged system. Note that (6.71) is not necessarily invertible. Our goal is, however, to show that the partial stability of the obtained dynamics (6.74) implies that of the original slow system (6.33).

Let us represent the obtained dynamics (6.74) by a perturbation of the averaged system (6.34). For $k = 1, 2$, let h_k^i be the ith component of h_k. From mean value theorem, for each $k = 1, 2$ there exists $\lambda_k^i = \lambda_k^i(w_p, v_p, z, \varepsilon) > 0$ such that

$$\begin{aligned} & h_k^i(w_p + \varepsilon u, v_p, z) - h_k^i\,(w_p, v_p, z) \\ & = \frac{\partial h_k^i}{\partial w_p}(w_p + \varepsilon \lambda_k^i u, v_p, z) \cdot \varepsilon u. \end{aligned}$$

Let us denote

$$\begin{aligned} & H_1(w_p, v_p, z, \varepsilon u) \\ & = \left[\frac{\partial h_1^1}{\partial w_p}(w_p + \varepsilon \lambda_1^1 u, v_p, z), \ldots, \frac{\partial h_1^n}{\partial w_p}(w_p + \varepsilon \lambda_1^n u, v_p, z) \right]^\top \\ & H_2(w_p, v_p, z, \varepsilon u) \\ & = \left[\frac{\partial h_2^1}{\partial w_p}(w_p + \varepsilon \lambda_2^1 u, v_p, z), \ldots, \frac{\partial h_2^n}{\partial w_p}(w_p + \varepsilon \lambda_2^n u, v_p, z) \right]^\top. \end{aligned}$$

Then we have

$$h_1(w_p + \varepsilon u, v_p, z) - h_1\,(w_p, v_p, z) = H_1(w_p, v_p, z, \varepsilon u) \cdot \varepsilon u, \tag{6.75}$$

$$h_2(w_p + \varepsilon u, v_p, z) - h_2\,(w_p, v_p, z) = H_2(w_p, v_p, z, \varepsilon u) \cdot \varepsilon u, \tag{6.76}$$

where both $H_1(w_p, v_p, z, \varepsilon u)$ and $H_2(w_p, v_p, z, \varepsilon u)$ are bounded since from (6.37) each $\partial h_k^i / \partial w$ is. Due to the boundedness of $\|\partial u / \partial z\|$, $\|\partial u / \partial w_p\|$, and $\|\partial u / \partial v_p\|$ from Proposition 6.2, it is clear that the matrix $P(\varepsilon)$ is nonsingular for sufficiently small $\varepsilon > 0$, and its inverse can be described as $P^{-1}(\varepsilon) = I + \mathcal{O}(\varepsilon)$ with some $\mathcal{O}(\varepsilon)$. Applying this fact together with the equalities (6.75) and (6.76) to (6.74), one can show that there are bounded $H_1'(w_p, v_p, z, \varepsilon u)$ and $H_2'(w_p, v_p, z, \varepsilon u)$ such that

$$\frac{dw_p}{dz} = \varepsilon h_{av}\,(w_p, v_p) + \varepsilon^2 H_1'(w_p, v_p, z, \varepsilon u) u, \tag{6.77a}$$

$$\frac{dv_p}{dz} = \varepsilon h_2(w_p, v_p, z) + \varepsilon^2 H_2'(w_p, v_p, z, \varepsilon u) u. \tag{6.77b}$$

This is a perturbation of the averaged system (6.34).

Next, we apply Corollary 6.2 to show the partial exponential stability of the averaged system (6.34) implies that of its perturbation (6.77) for sufficiently small $\varepsilon > 0$. From the definition of h_{av}, we have

$$\left\| \frac{\partial h_{\text{av}}}{\partial w_p}(w_p, v_p) \right\| = \left\| \frac{1}{T} \int_0^T \frac{\partial h_1}{\partial w_p}(w_p, v_p, \tau) d\tau \right\| \leq L_1 \tag{6.78}$$

for any $w_p \in \mathcal{B}_\delta, v \in \mathbb{R}^m$. Therefore, the perturbation (6.77) satisfies all the assumptions in Theorem 6.6. To apply Corollary 6.2, it remains to show that the perturbation terms are bounded. Let $b_1 > 0$ and $b_2 > 0$ be constants such that $\|H_1'(w, v, z, \varepsilon u)\| \leq b_1$ and $\|H_2'(w_p, v_p, z, \varepsilon u)\| \leq b_2$. From (6.69) in Appendix, it holds that $\|u(w_p, v_p, s)\| \leq 2TL_1\|w\|$, and then the perturbation terms satisfy

$$\begin{aligned} \|\varepsilon^2 H_1'(w_p, v_p, z, \varepsilon u)u\| &\leq 2\varepsilon^2 b_1 T L_1 \|w_p\|, \\ \|\varepsilon^2 H_2'(w_p, v_p, z, \varepsilon u)u\| &\leq 2\varepsilon^2 b_2 T L_1 \|w_p\|. \end{aligned}$$

Moreover, for sufficiently small $\varepsilon_1 > 0$, one notices that any $\varepsilon < \varepsilon_1$ satisfies both inequalitie (6.59) and (6.60). Therefore, Corollary 6.2 implies that $w_p = 0$ is partially exponentially stable of the perturbation (6.77), i.e., (6.74). In other words, there are $\delta' > 0$ and $k', \lambda' > 0$ such that $w_p(z_0) \in \mathcal{B}_{\delta'}$ implies $\|w_p(z)\| \leq k'\|w_p(z_0)\|e^{-\lambda'(z-z_0)}$, for all $z \geq z_0$.

Finally, we show that the partial exponential stability of the system (6.74) implies that of the slow dynamics (6.33). From (6.71a) and (6.69), one obtains

$$|1 - 2\varepsilon TL_1| \cdot \|w_p(z)\| \leq \|x(z)\| \leq |1 + 2\varepsilon TL_1| \cdot \|w_p(z)\|,$$

for all $z \geq z_0$. Then, it follows that

$$\|x(z)\| \leq k' \frac{|1 + 2\varepsilon TL_1|}{|1 - 2\varepsilon TL_1|} \|x(z_0)\| e^{-\lambda'(z-z_0)}, \qquad \forall z \geq z_0,$$

proving the partial exponential stability of $x = 0$ for the system (6.33) for sufficiently small $\varepsilon > 0$. Finally, one can conclude that $x = 0$ is also partially exponentially stable for the original slow-fast system (6.31) uniformly in y and z under asssumption (6.32).

□

6.4 Concluding Remarks

As a type of partial synchronization, remote synchronization has been widely observed in a healthy brain. One often encounters to prove the partial stability of a nonlinear system when studying remote synchronization. Motivated by this, we have developed some new criteria in this chapter. These new criteria for partial stability analysis are used to prove the stability of remote synchronization in star networks, a task that it is quite difficult, if not possible, to accomplish using the existing criteria.

In Section 6.2, we have shown that asymptotic or exponential stability is ensured if the constructed Lyapunov function decreases after a finite time, without requiring its time derivative to be negative definite. Our obtained criteria enlarge the range of choices of function candidates in analyzing partial stability. As a drawback of these results, it is not always easy to check whether the value of a candidate decreases after a finite time or not. We have further developed some new criteria that are easier to verify for partial exponential stability using periodic techniques in Section 6.3, where a particular class of slow-fast systems with a fast scalar variable is considered. Unlike classic averaging methods, we construct an averaged system by averaging over this fast scalar variable, which is usually different from the time variable. We then show that partial exponential stability of the averaged system implies partial exponential stability of the original one. As some intermediate results, we have also obtained: 1) a converse Lyapunov theorem; and 2) some perturbation theorems that are the first known ones for partially exponentially stable systems.

7 Remote Synchronization in Star Networks of Kuramoto Oscillators

As another type of partial synchronization, *remote synchronization* describes the phenomenon arising in coupled networks of oscillators when two oscillators without a direct connection become synchronized without requiring the intermediate ones on a path linking the two oscillators to also be synchronized with them [164]. In this chapter, we study remote synchronization in a type of characteristic networks, i.e., star networks. The criteria for partial stability of nonlinear systems in the previous chapter will be used for the analysis.

7.1 Introduction

It has been observed that distant cortical regions in the human brain without direct neural links also experience functional correlations [92]. This motivates researchers to study an interesting behavior dubbed remote synchronization. Unlike what has been pointed out in most findings that the coupling strengths in a network are critical for synchronization of coupled oscillators [104, 144, 149], a recent article reveals that morphological symmetry is crucial for remote synchronization [96]. Some nodes located distantly in a network can mirror their functionality between each other. In other words, theoretically, swapping the positions of these nodes will not change the functioning of the overall system.

A star network is a simple paradigm for such networks with morphologically symmetric properties. The peripheral nodes have no direct connection, but obviously play similar roles in the whole network. The node at the center acts as a relay or mediator. As an example, the thalamus is such a relay in neural networks, and it is believed to enable separated cortical areas to be completely synchronized [94, 95]. This observation of robust correlated behavior taking place in distant cortical regions through relaying motivates us to study the stability of remote synchronization in star networks in this chapter. A star network is simple in structure, but capable of characterizing some basic features of remote synchronization, and also provides some idea to understand this phenomenon in more complex networks. Different from [178], we use Kuramoto-Sakaguchi model [63] to describe the dynamics of coupled oscillators in this chapter, and analytically study the stability of remote synchronization. Different from classic the Kuramoto model, there is an additional phase shift term in Kuramoto-Sakaguchi one. This phase shift is usually used to model time delays [89], e.g., synaptic connection delays [93].

The remainder of this chapter is structured as follows. We first consider that the oscillators are coupled by a general directed star graph in Section 7.2. We reveal that the symmetry of outgoing connections from the central oscillator is crucial to shaping different clusters of remote synchronization. On the other hand, the coupling strengths of incoming links to the central oscillator are not required to be symmetric. In Section 7.3, we obtain some sufficient conditions for remote synchronization in star networks with/without the presence of a phase shift. By comparing these conditions, we find that the presence of a phase shift raises the requirement for the coupling strengths to ensure stable remote synchronization. In Section 7.4, we consider a simpler network motif, i.e., a star network with 2 peripheral nodes. This network has been shown to give rise to isochronous synchronization in delay-coupled semiconductor lasers [179], zero-lag synchronization in remote cortical regions of the brain [94]. We introduce a natural frequency detuning to the central oscillator, and investigate how it can actually enhance remote synchronization, making it robust against phase shifts.

7.2 Problem Formulation

Synchronization of distant cortical regions having *no direct links* has been observed in the human brain. The emergence of this phenomenon is sometimes due to a mediator or relay that connecting separated regions, e.g., the thalamus [95]. Motivated by this, we study remote synchronization by considering $n+1$, $n \geq 2$, oscillators, coupled by a star network, which are labeled by $0, 1, \ldots, n$. Let $\mathbf{N} = \{1, \ldots, n\}$ be the set of indices of the peripheral oscillators. The central mediator is labeled by 0. The dynamics of each oscillator are described by

$$\dot{\theta}_0 = \omega_0 + \sum_{i=1}^{n} K_i \sin(\theta_i - \theta_0 - \alpha), \tag{7.1a}$$

$$\dot{\theta}_i = \omega + A_i \sin(\theta_0 - \theta_i - \alpha), i = 1, 2, \ldots, n, \tag{7.1b}$$

where $\theta_i \in \mathbb{S}^1$ is the phase of the ith oscillator, and ω_0 and ω are the natural frequencies of the central and peripheral oscillators, respectively. Here $K_i > 0$ is the coupling strength from the peripheral node i to the central node 0 (for which we refer to as *incoming* (with respect to 0) coupling strengths), and $A_i > 0$ presents the directed coupling strength from the central node 0 and the peripheral node i (for which we refer to as *outgoing* (with respect to 0) coupling strengths). It is worth mentioning that incoming and outgoing couplings are allowed to be different, which means that the underlying star network, denoted by $\mathcal{G}$, is directed. The term α is the phase shift satisfying $\alpha \in [0, \pi/2)$. In the star network considered in this chapter, remote synchronization is the situation where some of the peripheral oscillators are phase synchronized, while the phase of the central mediator 0 connecting them is different. We define remote synchronization formally as follows.

Definition 7.1. *Let $\theta(t) = [\theta_0(t), \dots, \theta_n(t)]^\top \in \mathbb{S}^{n+1}$ be a solution to the system dynamics* (7.1)*. Let $\mathcal{R}$ be a subset of $\mathbf{N}$, whose cardinality satisfies $2 \leq |\mathcal{R}| \leq n$. We say that the solution $\theta(t)$ is remotely synchronized with respect to $\mathcal{R}$ if for every pair of indices $i, j \in \mathcal{R}$ it holds that $\theta_i(t) = \theta_j(t)$ for all $t \geq 0$, but it is not required that $\theta_i(t) = \theta_0(t)$.*

When $\mathcal{R} \subset \mathbf{N}$, we say that $\theta(t)$ is *partially* remotely synchronized; in particular, when $\mathcal{R} = \mathbb{N}$, we say that $\theta(t)$ is *completely* remotely synchronized, for which situation we refer to as remote synchronization for brevity in what follows. A particular case of partially remotely synchronized solution is remote cluster synchronization, which is defined as follows.

Definition 7.2. *Let $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_m\}, 2 \leq m < n$ be a partition of $\mathbf{N}$. The sets $\mathcal{C}_1, \dots, \mathcal{C}_m$ are non-overlapping and satisfy $1 \leq |\mathcal{C}_p| < n$ for all p and $\cup_{p=1}^m \mathcal{C}_p = \mathbf{N}$. A partially remotely synchronized solution $\theta(t)$ to the system dynamics* (7.1) *is said to be remotely clustered with respect to $\mathcal{C}$ if: for any given $\mathcal{C}_p$ and every pair $i, j \in \mathcal{C}_p$ there holds that $\theta_i(t) = \theta_j(t), \forall t \geq 0$; on the other hand, for any given $i \in \mathcal{C}_p, j \in \mathcal{C}_q$ where $p \neq q$, $\theta_i(t) \neq \theta_j(t)$.*

Note that the trivial case when a cluster has only one oscillator is allowed. In fact, remote synchronization behavior for the network (7.1) can be categorized into two depending on being *phase locked* or not. Phase locking is a phenomenon where every pairwise phase difference is a constant, $\theta_i - \theta_j = c_{i,j}$, $\forall i, j$ (the phenomenon when $c_{i,j} = 0$, $\forall i, j$ is especially called phase synchronization). Phase locking is also called frequency synchronization because this is equivalent to that the frequencies of all oscillators are synchronized, $\dot{\theta}_1 = \cdots = \dot{\theta}_n$. In remote synchronization, the frequency of the central oscillator, $\dot{\theta}_0$, is allowed to be different from the peripheral ones, $\dot{\theta}_1, \dots, \dot{\theta}_n$.

In Section 7.3, we exclusively study (partial) remote synchronization when the frequencies of all the oscillators in the network are synchronized, i.e., $\dot{\theta}_0(t) = \dot{\theta}_1(t) = \cdots = \dot{\theta}_n(t) = \omega_{\text{syn}}$ for some $\omega_{\text{syn}} \in \mathbb{R}$. For a given $r \in \mathbb{S}^1$ and an angle $\gamma \in [0, 2\pi]$, let $\text{rot}_\gamma(r)$ be the rotation of r counter-clockwise by the angle γ. For $\theta \in \mathbb{S}^n$, we define an equivalence class $\text{Rot}(\theta) := \{[\text{rot}_\gamma(\theta_1), \dots, \text{rot}_\gamma(\theta_n)]^\top : \gamma \in [0, 2\pi]\}$. Let θ^* be a solution to the equations

$$\omega_0 - \omega_i - \sum_{j=1}^{n} K_i \sin(\theta_j - \theta_0 - \alpha) - A_i \sin(\theta_0 - \theta_i - \alpha) = 0, \tag{7.2}$$

for $i = 1, 2, \dots, n$, which is a solution such that phase synchronization is reached. It is not hard to see that $[\text{rot}_\gamma(\theta_1^*), \dots, \text{rot}_\gamma(\theta_n^*)]^\top$ for any $\gamma \in [0, 2\pi]$ is also a solution to the equations. Consequently, the set $\text{Rot}(\theta^*)$ is said to be a *synchronization manifold* for the dynamics (7.1) [78]. As an extension of the definition of the synchronization manifold in [152], we call $\text{Rot}(\theta^*)$ (partial) *phase locked remote synchronization manifold* if there exists a set ($\mathcal{R} \subset \mathbf{N}$) $\mathcal{R} = \mathbf{N}$ such that $\theta_i^* = \theta_j^*$ for any pair $i, j \in \mathcal{R}$. In order to study the stability of the (partial) phase locked remote synchronization manifold, it suffices to study the stability of θ^*.

In the next section, we investigate how the phase shift affects phase locked remote synchronization in star networks. We start with the assumption that there is no phase shift in Section 7.3.1. The local stability of the remote and cluster synchronization manifolds is studied. In Section 7.3.2, we consider there is a phase shift α and investigate the influence of this phase shift on the stability of the remote synchronization manifold.

7.3 Effects of Phase Shifts on Remote Synchronization

7.3.1 Without a Phase Shift

In this subsection, we consider the case when there is no phase shift, i.e., $\alpha = 0$. We investigate how partial and complete remote synchronization in star networks are formed. We show the important roles that the symmetric outgoing couplings quantified by A_i play in enabling synchronization among oscillators that are not directly connected.

To proceed, define $x_i = \theta_0 - \theta_i$ for $i = 1, 2 \dots, n$. Then the time derivative of x_i is given by

$$\dot{x}_i = \omega_0 - \omega - \sum_{j=1}^{n} K_i \sin(\theta_0 - \theta_j) - A_i \sin(\theta_0 - \theta_i). \tag{7.3}$$

Let $x = [x_1, x_2, \dots, x_n]^\top \in \mathbb{S}^n, \boldsymbol{\omega} = (\omega_0 - \omega)\mathbf{1}_n$, and $\mathbf{sin}x = [\sin x_1, \dots, \sin x_n]^\top$, then the dynamics (7.3) can be represented in a compact form as follows

$$\dot{x} = \boldsymbol{\omega} - T\mathbf{sin}x := f(x), \tag{7.4}$$

where $f(x) = [f_1(x), \dots, f_n(x)]^\top$ and

$$T = \begin{bmatrix} A_1 + K_1 & K_2 & \cdots & K_n \\ K_1 & A_2 + K_2 & \cdots & K_n \\ \vdots & \vdots & \ddots & \vdots \\ K_1 & K_2 & \cdots & A_n + K_n \end{bmatrix}. \tag{7.5}$$

Let x^* be an equilibrium of (7.4), if it exists, i.e., $f(x^*) = 0$. From the definition of x, we observe that x^* corresponds to a (partial) remote synchronization manifold if there exists a set ($\mathcal{R} \subset \mathbf{N}$) $\mathcal{R} = \mathbf{N}$ such that for any $i, j \in \mathcal{R}$, $x_i^* = x_j^*$. In what follows, we show under what conditions on the coupling strengths the equilibrium x^* exists and some (all) of its elements are identical, which gives rise to the corresponding (partial) phase locked remote synchronization of the model (7.1). Towards this end, let us first make an assumption.

Assumption 7.1. *We assume that the coupling strengths satisfy the following inequality*

$$A_i \geq (n-1)K_i, \qquad \forall i \in \mathbf{N}, \tag{7.6}$$

and the corresponding matrix T, given by (7.5)*, satisfies*

$$\|T^{-1}\boldsymbol{\omega}\|_\infty < 1. \tag{7.7}$$

Assumption 7.1 suggests that the strengths of outgoing couplings are much greater than that of incoming ones. By observing that for any i it holds that

$$A_i + K_i - (n-1)K_i \geq (n-1)K_i + K_i - (n-1)K_i = K_i > 0,$$

we know that the matrix T is *column diagonally dominant*. By Gershgorin circle theorem [180, Sec. 6.2], one knows all the eigenvalues of $T^\top$ have positive real parts, which also means that all the eigenvalues of T lie on the right half plane. Thus T is invertible. We are now at a position to present our main result in this section.

Theorem 7.1. *Under Assumption 7.1, there exists a unique locally asymptotically stable equilibrium x^* satisfying $|x_i^*| \in [0, \pi/2)$ for all $i \in \mathbf{N}$ for the dynamics* (7.3)*, which is*

$$x^* = \mathbf{arcsin}(T^{-1}\boldsymbol{\omega}). \tag{7.8}$$

In addition, if there is a pair of distinct $i, j \in \mathbf{N}$ *such that* $A_i = A_j$*, then* $x_i^* = x_j^*$*. This* x^* *corresponds to a partial phase locked remote synchronization manifold, denoted by* $\mathrm{Rot}(\theta^*)$*, for the dynamics* (7.1)*, which implies oscillators* i *and* j *are remotely synchronized.*

Proof. We first show the existence of the equilibrium point x^* satisfying $|x_i^*| \in [0, \pi/2]$ for all $i \in \mathbf{N}$. To obtain the equilibrium point x^*, we then solve $\boldsymbol{\omega} - T\mathbf{sin}x = 0$, i.e., $\mathbf{sin}x = T^{-1}\boldsymbol{\omega}$. From the hypothesis (7.7), one knows that there exists a unique solution to this equation in $[0, \pi/2)$, which is $x^* = \mathbf{arcsin}(T^{-1}\boldsymbol{\omega})$.

We next show the stability of this equilibrium by linearization. Towards this end, we calculate the Jacobian matrix

$$\begin{aligned} J\left(x^*\right) &= -\left.\begin{bmatrix} \frac{\partial}{\partial x_1} f_1 & \cdots & \frac{\partial}{\partial x_n} f_1 \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_n & \cdots & \frac{\partial}{\partial x_n} f_n \end{bmatrix}\right|_{x=x^*} \\ &= -T \operatorname{diag}\left(\cos x_1^*, \ldots, \cos x_n^*\right). \end{aligned} \tag{7.9}$$

Since x^* satisfies $|x_i^*| \in [0, \pi/2)$ for all $i \in \mathbf{N}$, $\cos x_i^* > 0$ for all $i \in \mathbf{N}$. Recalling that the matrix T is column diagonally dominant, by post-multiplying the positive diagonal matrix $D :=\ \operatorname{diag}\left(\cos x_1^*, \ldots, \cos x_n^*\right)$ on T, the matrix TD is also column diagonally dominant since every column is just scaled by a positive number. It follows from Gershgorin circle theorem that all the eigenvalues of TD have positive real parts, which means all the eigenvalues of $J(x^*)$ have negative real parts. Then the equilibrium x_i^* is locally asymptotically stable.

Finally, we show $x_i^* = x_j^*$ if the hypothesis $A_i = A_j$ is satisfied. It is sufficient to show $\sin x_i^* = \sin x_j^*$, since $|x_i^*| < \pi/2$ for all i. Let e_i be the ith column of the identity matrix I_n of order n. It is equivalent to show $(e_i^\top - e_j^\top)\mathbf{sin}x^* = 0$. We observe that

$$(e_i^\top - e_j^\top)T = A_i e_i^\top - A_j e_j^\top.$$

If $A_i = A_j$, it follows straightforwardly that $(e_i^\top - e_j^\top)T = A_i(e_i^\top - e_j^\top)$. By post-multiplying T^{-1} on both sides of this equation, we obtain $(e_i^\top - e_j^\top)T^{-1} = (e_i^\top - e_j^\top)/A_i$. From (7.8) we know $\mathbf{sin}x^* = T^{-1}(\omega_0 - \omega)\mathbf{1}_n$. It then follows that

$$\begin{aligned} (e_i^\top - e_j^\top)\mathbf{sin}x^* &= (e_i^\top - e_j^\top)T^{-1}(\omega_0 - \omega)\mathbf{1}_n \\ &= \frac{\omega_0 - \omega}{A_i}(e_i^\top - e_j^\top)\mathbf{1}_n = 0. \end{aligned}$$

Then one can conclude that if Assumption 7.1 and the hypothesis $A_i = A_j$ are satisfied, the partial phase locked remote synchronization manifold $\mathrm{Rot}(\theta^*)$, in which $\theta_i^* = \theta_j^*$, is locally asymptotically stable. □

Theorem 7.1 shows that the outgoing couplings A_i play essential roles in facilitating remote synchronization. Oscillators that are not directed connected can get phase synchronized just because the directed connections from the central mediator towards them are identical. Authors in [96] show that symmetries in undirected networks are important for remote synchronization. In contrast, we take directions of the couplings into consideration, and show that only the outgoing couplings matter. In order to make two oscillators, say i and j, synchronized, it is not required that the incoming couplings K_i and K_j to be identical. It can be intuitively paraphrased that the mediator at the central position is able to render the oscillators around it synchronized by imposing a common input to them, without requiring the feedback coming back to be identical. This finding shares some similarities with the common-noise-induced synchronization investigated by researchers in physics [181–183]. However, we study network-coupled, not isolated, oscillators and derive conditions on the network to enable synchronization between separated oscillators.

What is worth mentioning, by carefully manipulating the symmetry of the couplings originated from the central node 0, not only synchronization among distant oscillators can be facilitated, but also unnecessary synchronization can be easily prevented. Moreover, interesting patterns of remote synchronization, such as cluster synchronization, can occur. The following corollary provides some sufficient conditions for the existence and stability of remote and cluster synchronization manifold, which follows from Theorem 7.1 straightforwardly.

Corollary 7.1. *Under Assumption 7.1, there is a locally asymptotically stable remote synchronization manifold for the dynamics* (7.1)*, i.e., in which the solution $\theta(t)$ is completely remotely synchronized, if $A_i = A_j$ for every pair $i, j \in \mathbf{N}$; there is a locally asymptotically stable partial remote synchronization manifold for the dynamics* (7.1)*, in which the solution $\theta(t)$ is remotely clustered with respect to $\mathcal{C}$, if there is a partition of $\mathbf{N}$, denote by $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_m\}$, satisfying $|\mathcal{C}_p| \geq 2$ and $\cup_{p=1}^m \mathcal{C}_p = \mathbf{N}$ such that: for any given $\mathcal{C}_p$ and every pair $i, j \in \mathcal{C}_p$ it holds that $A_i = A_j$; on the other hand, for any given $i \in \mathcal{C}_p, j \in \mathcal{C}_q$ where $p \neq q$, $A_i \neq A_j$.*

In the next subsection, we consider the case where there is a phase shift (or phase lag) term α. The model with the presence of a phase shift is known as the Kuramoto-Sakaguchi model [63].

7.3.2 With a Phase Shift

In this section, we consider that there is a phase shift $\alpha \in (0, \pi/2)$. By introducing a phase shift term, it allows the oscillators to get synchronized at a frequency that differs from the simple average of their natural frequencies [184]. This phenomenon has always been observed in many biological systems such as the mammalian intestine and heart cells [185]. Moreover, in neural networks the phase shift α is often used to model delays concerning synaptic connections [93]. To study the remote synchronization of our interest, we simplified the problem by assuming that $A_i = A$ and $K_i = A/n$ for all i. Note that this simplification ensures that the direction of the network is preserved and the condition (7.6) is satisfied, which guarantees the property that the outgoing couplings are much stronger than the incoming ones. Consequently, the dynamics (7.1) become

$$\begin{aligned}\dot{\theta}_0 &= \omega_0 + \frac{A}{n}\sum_{i=1}^{n}\sin(\theta_i - \theta_0 - \alpha);\\ \dot{\theta}_i &= \omega + A\sin(\theta_0 - \theta_i - \alpha), i = 1, 2, \ldots, n,\end{aligned} \tag{7.10}$$

Conditions on the coupling strength A are subsequently obtained to ensure that the dynamics (7.10) admit a locally asymptotically stable remote synchronization manifold. We investigate how these conditions depend on the phase shift α. As frequency synchronization is the footstone for the analysis that follows, let us provide the necessary condition for the existence of a frequency synchronized solution to (7.10) and see how it depends on the phase shift α.

Proposition 7.1. *There is a frequency synchronized solution to the dynamics* (7.10) *only if*

$$A \geq \frac{1}{2\cos\alpha}|\omega_0 - \omega|.$$

Proof. We prove this necessary condition by contradiction. We assume that $A < |\omega_0 - \omega|/2\cos\alpha$ and there is a frequency synchronized solution $\theta^* \in \mathbb{S}^n$. The time derivative of $\theta_0 - \sum_{i=1}^{n}\theta_i/n$ is given by

$$\begin{aligned}\dot{\theta}_0 - \frac{1}{n}\sum_{i=1}^{n}\dot{\theta}_i &= \omega_0 - \omega + \frac{A}{n}\sum_{i=1}^{n}\sin(\theta_i - \theta_0 - \alpha) - \frac{A}{n}\sum_{i=1}^{n}\sin(\theta_0 - \theta_i - \alpha)\\ &= \omega_0 - \omega + \frac{2A}{n}\sum_{i=1}^{n}\sin\left(\theta_i - \theta_0\right)\cos\alpha.\end{aligned}$$

Recalling the hypothesis that there is a frequency synchronized solution θ^*, the right-hand side of this equation satisfies

$$\omega_0 - \omega + \frac{2A}{n}\sum_{i=1}^{n}\sin\left(\theta_i^* - \theta_0\right)\cos\alpha = 0.$$

It follows that $|\sum_{i=1}^{n}\sin\left(\theta_i^* - \theta_0^*\right)| = n|\omega_0 - \omega|/2A\cos\alpha > n$ since $A < |\omega_0 - \omega|/2\cos\alpha$. As $|\sin x| \leq 1$ for any $x \in \mathbb{S}^1$, one knows such a θ^* does not exist, which is a contradiction. Then it can be concluded that there is a frequency synchronized solution to the dynamics (7.10) only if $A \geq |\omega_0 - \omega|/2\cos\alpha$, which completes the proof. $\square$

We observe that when $\alpha = 0$, this necessary condition reduces to $A \geq |\omega_0 - \omega|/2$. Obviously, the existence of the phase shift raises the requirement for the coupling strength A. Next, we show the sufficient conditions on A such that there is a locally asymptotically stable remote synchronization manifold for (7.10). Towards this end, let $y_i = (\theta_0 - \theta_i)/2, y_i \in \mathbb{S}^1$ for $i = 1, 2 \ldots, n$. The time derivative of y_i is

$$\begin{aligned}\dot{y}_i =& \frac{1}{2}(\omega_0 - \omega) + \frac{A}{2n}\sum_{j=1}^{n}\sin(\theta_j - \theta_0 - \alpha) - \frac{1}{2}A\sin(\theta_0 - \theta_i - \alpha)\\ =& \frac{1}{2}(\omega_0 - \omega) - \frac{A}{2n}\sum_{j=1}^{n}\sin(2y_j + \alpha) - \frac{1}{2}A\sin(2y_i - \alpha) := g_i(y), i = 1, 2, \ldots, n.\end{aligned} \tag{7.11}$$

where $y = [y_1, \ldots, y_n]^\top$ and $g(y) = [g_1(y), \ldots, g_n(y)]^\top$. Let us provide the main result in this section.

Theorem 7.2. *There is a unique locally asymptotically stable equilibrium y^* for the dynamics* (7.11) *satisfying $|y_i^*| < \pi/4$ for all i, which is*

$$y^* = \frac{1}{2}\arcsin\left(\frac{\omega_0 - \omega}{2A\cos\alpha}\right)\mathbf{1}_n, \tag{7.12}$$

if the following conditions are satisfied, respectively:

i) when $\omega_0 > \omega$, the coupling strength A satisfies

$$A > \frac{\omega_0 - \omega}{2\cos\alpha}; \tag{7.13}$$

ii) when $\omega_0 < \omega$, the coupling strength A satisfies

$$A > \frac{\omega - \omega_0}{2\cos^2\alpha}. \tag{7.14}$$

This locally asymptotically stable equilibrium y^* for the dynamics (7.11) *corresponds to the locally asymptotically stable remote synchronization manifold for* (7.10).

Proof. We first show the existence of the equilibrium y^*. We observe that

$$\frac{A}{2n}\sum_{j=1}^{n}\sin(2y_j + \alpha) + \frac{1}{2}A\sin(2y_i - \alpha) = \frac{A}{n}\sum_{j=1}^{n}\sin(y_j + y_i)\cos(y_j - y_i + \alpha).$$

It is equivalent to check whether there is a solution y, which satisfies $y_i = y_j$ for all i, j, to the equation $g(y) = 0$. To do this, we have

$$\frac{1}{2}(\omega_0 - \omega) - A \sin 2y_i \cos\alpha = 0. \tag{7.15}$$

Recalling the hypotheses (7.13) and (7.14) both suggesting that $A > (\omega_0 - \omega)/2\cos\alpha$, it is obvious that this equation has a unique solution in $[0, \pi/4)$, which is

$$y_i^* = \frac{1}{2}\arcsin\left(\frac{\omega_0 - \omega}{2A\cos\alpha}\right),$$

then (7.12) follows.

Next, we prove the stability of y^*. To do this, we lineariz the model (7.11) at this equilibrium. Let $J(y) = [J_{ij}] \in \mathbb{R}^{n\times n}$ be the Jacobian Matrix, whose elements are expressed by

$$\begin{aligned} J_{ii} &= \frac{\partial g_i}{\partial y_i} = -\frac{A}{n}\cos(2y_i + \alpha) - A\cos(2y_i - \alpha), \\ J_{ij} &= \frac{\partial g_i}{\partial y_j} = -\frac{A}{n}\cos(2y_i + \alpha). \end{aligned}$$

We then show the Jacobian Matrix $J(y)$ evaluated at the equilibrium y^* is row diagonally dominant in both cases of i) and ii) if the conditions (7.13) and (7.14) are satisfied, respectively. We first study the case when $\omega_0 > \omega$. If condition (7.13) is satisfied, it follows that $0 < 2y_n^* < \pi/2$, which implies that $-\pi/2 < 2y_n^* - \alpha < \pi/2$. Then it holds that $\cos(2y_i^* - \alpha) > 0$. We calculate

$$\begin{aligned} |J_{ii}| - \left|\sum_{j=1, j\neq i}^{n} J_{ij}\right| =& A\cos(2y_i^* - \alpha) + \frac{A}{n}\left|\cos(2y_i^* + \alpha)\right| \\ & - \frac{(n-1)A}{n}\left|\cos(2y_i^* + \alpha)\right| \\ = \frac{2A}{n}\cos(2y_i^* - \alpha) + & \frac{(n-2)A}{n}\left(\cos(2y_i^* - \alpha) - \left|\cos(2y_i^* + \alpha)\right|\right). \end{aligned} \tag{7.16}$$

If $\cos(2y_i^* + \alpha) > 0$, then

$$\begin{aligned} &\cos(2y_i^* - \alpha) - |\cos(2y_i^* + \alpha)| \\ &= \cos(2y_i^* - \alpha) - \cos(2y_i^* + \alpha) = 2\sin 2y^* \sin\alpha > 0. \end{aligned}$$

On the other hand, if $\cos(2y_i^* + \alpha) < 0$, then

$$\begin{aligned} &\cos(2y_i^* - \alpha) - |\cos(2y_i^* + \alpha)| \\ &= \cos(2y_i^* - \alpha) + \cos(2y_i^* + \alpha) = 2\cos 2y_i^* \cos\alpha > 0. \end{aligned}$$

Consequently, from (7.16) it is easy to see $|J_{ii}| - \left|\sum_{j=1,j\neq i}^{n} J_{ij}\right| > 0$. Then the Jacobian matrix $J(y^*)$ is row diagonally dominant. Since the diagonal elements $J_{ii} < 0$, one knows that all the eigenvalues of $J(y^*)$ have negative real parts. The equilibrium of y^* is locally asymptotically stable. Finally, we consider the case when $\omega_0 < \omega$. Recalling that if condition (7.14) is satisfied, it holds that

$$\frac{\omega_0 - \omega}{2A\cos\alpha} > \cos\alpha = -\sin(-\pi/2 + \alpha). \tag{7.17}$$

Since $-1 < -\sin(\pi/2 - \alpha) < 0$, $(\omega_0 - \omega)/2A\cos\alpha < 0$ and arcsin is monotonically increasing in $[-1, 0]$, it follows that

$$\arcsin(-\pi/2 + \alpha) < \arcsin\left(\frac{\omega_0 - \omega}{2A\cos\alpha}\right) < 0.$$

Then it is obvious that $-\pi/2 + \alpha < 2y_i^* < 0$, which implies that $\pi/2 < 2y_i^* - \alpha < 0$. It is easy to see that $\cos(2y_i^* - \alpha) > 0$. Following the same steps as the case when $\omega_0 > \omega$, one can show that the Jacobian matrix $J(y^*)$ is diagonally dominant, which implies that the equilibrium of y^* is locally asymptotically stable. $\square$

Theorem 7.2 provides some sufficient conditions for the existence and local stability of the equilibrium of dynamics (7.11), or equivalently, for the existence and local stability of remote synchronization manifold of (7.10). With the presence of the phase shift α, the requirement of coupling strengths is increased. In fact, the larger the phase shift is, the stronger the coupling is raised, which can be observed from (7.13) and (7.14). Interestingly, comparing (7.14) with (7.13) we observe that the phase shift has a different impact on the coupling strength in the two cases when $\omega_0 > \omega$ and $\omega_0 < \omega$. The latter case is more vulnerable to the phase shift.

7.3.3 Numerical Examples

To validate the results we obtained in Subsection 7.3.1 and 7.3.2, we perform some numerical studies in this section. We consider 7 oscillators coupled by a directed star network illustrated in Fig. 7.1. To measure the levels of synchronization we introduce the two useful functions as follows,

$$h_1(\theta(t)) = \max_{i,j\in\mathbf{N}} |\theta_i(t) - \theta_j(t)|,$$
$$h_2(\theta(t)) = \max_{i\in\mathbf{N}} |\theta_0(t) - \theta_i(t)|,$$

If $h_2 = 0$, the phase difference between any peripheral oscillator and the central one is zero, which implies complete synchronization in the whole network. In particular, if

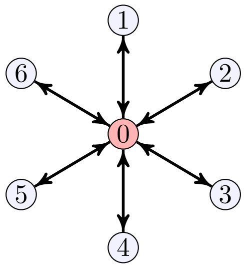


Figure 7.1: The considered star network: central node 0 and peripheral ones $\{1, 2, 3, 4, 5, 6\}$.

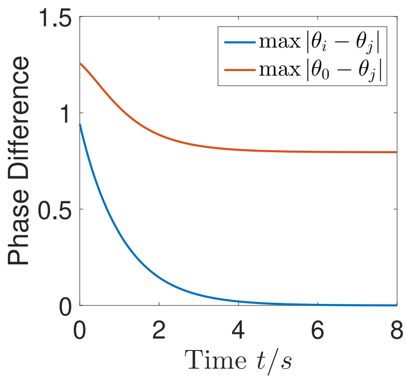
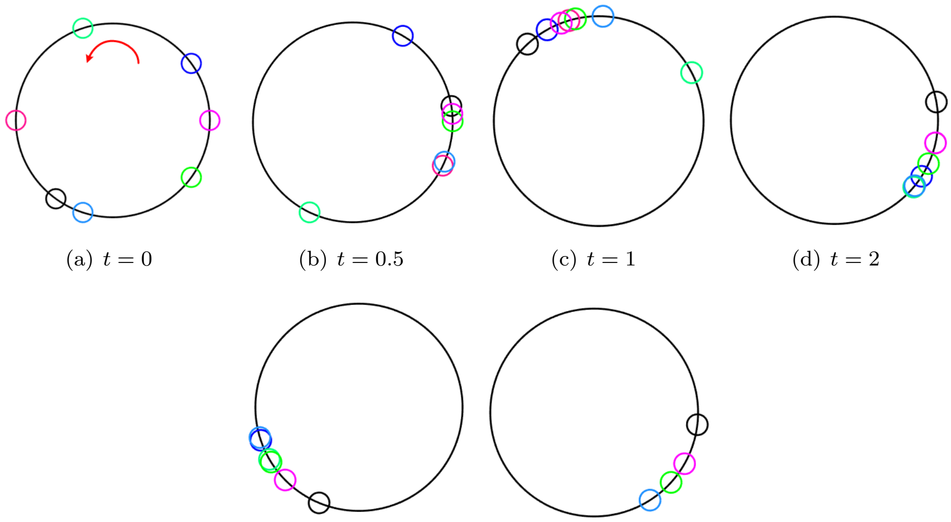


Figure 7.2: Trajectories of the maximum absolute values of the phase differences when $\alpha = 0$: blue represents $h_1 = \max_{i,j \in \mathbf{N}} |\theta_i - \theta_j|$ and red represents $h_2 = \max_{i \in \mathbf{N}} |\theta_0 - \theta_i|$.

$h_1 = 0, h_2 \neq 0$, all the phases of peripheral oscillators are identical remaining central one different, which yields remote synchronization.

We first testify the results obtain in Theorem 7.1. To distinguish the frequencies, let the frequency of each peripheral oscillator be $\omega = 0.8\pi$, and the natural frequency of the central one be $\omega_0 = 1.5\pi$. In order to make complete remote synchronization occur, we let $A_i = 1.4$ for all $i = 1, 2, \ldots, 6$, and let $K_1 = 0.3, K_2 = 0.25, K_3 =$



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$ (d) $t = 2$

(e) $t = 4$ (f) $t = 8$

Figure 7.3: The phases on $\mathbb{S}^1$ at six time instants when $\alpha = 0$: black represents the central oscillator 0; blue represents oscillators 1 and 4; green represents 2 and 5; red represents 3 and 6.

$0.4, K_4 = 0.18, K_5 = 0.2, K_6 = 0.25$. Then the matrix T becomes

$$T = \begin{bmatrix} 1.55 & 0.12 & 0.2 & 0.18 & 0.2 & 0.14 \\ 0.15 & 1.52 & 0.2 & 0.18 & 0.2 & 0.14 \\ 0.15 & 0.12 & 1.6 & 0.18 & 0.2 & 0.14 \\ 0.15 & 0.12 & 0.2 & 1.58 & 0.2 & 0.14 \\ 0.15 & 0.12 & 0.2 & 0.18 & 1.6 & 0.14 \\ 0.15 & 0.12 & 0.2 & 0.18 & 0.2 & 1.54 \end{bmatrix}.$$

It can be verified that T is diagonal dominated and $|T^{-1}\boldsymbol{\omega}| = 0.9201 < 1$, i.e. conditions in Assumption 7.1 are satisfied. Let the initial phases be $\theta(0) = [1.3\pi, 1.2\pi, 1.15\pi, 0.9\pi, 1.2\pi, 1.0\pi, 1.11\pi]^\top$, and then the trajectories of $h_1(\theta(t))$ and $h_2(\theta(t))$ are presented in Fig. 7.2. It can be observed that $h_1(\theta(t))$ converges to zero, while $h_2(\theta(t))$ converges to a constant, suggesting that the peripheral oscillators which are not directly connected achieve phase synchronization, but the ones that have direct connections (the central one with each peripheral one) do not. Next, we show that cluster synchronization is formed if the conditions in Corollary 7.1 are satisfied. Let the outgoing coupling strengths be $A_1 = A_4 = 2.1, A_2 = A_5 = 2.8, A_3 = A_6 = 4.2$, and let the incoming coupling strength be the same as considered above. One can also check Assumption 7.1 is

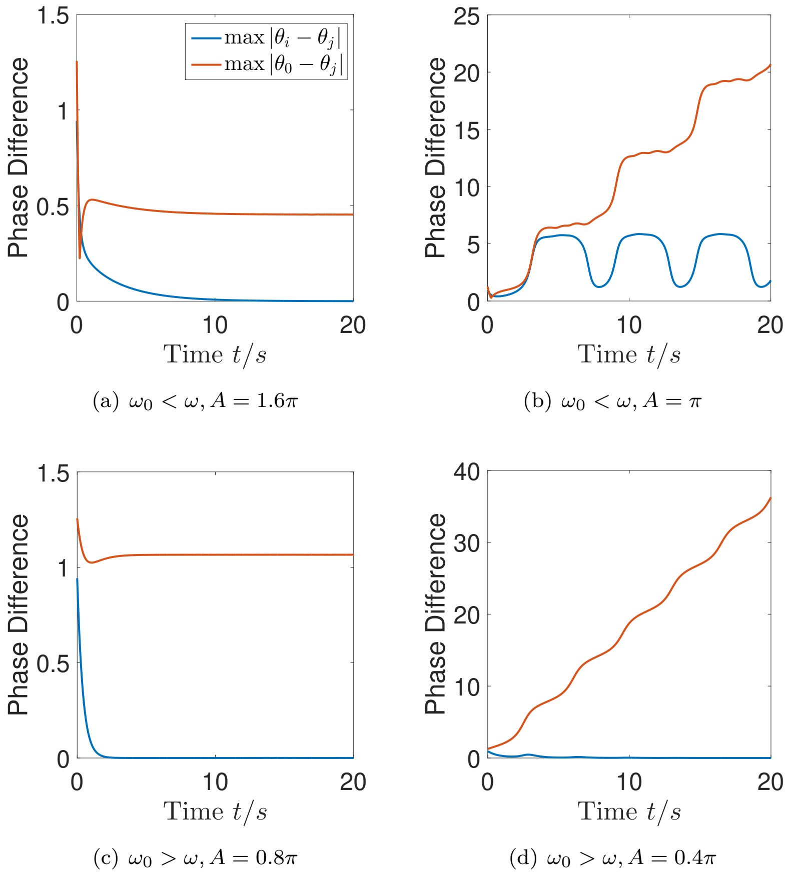


Figure 7.4: Trajectories of the maximum absolute values of the phase differences when $\alpha = \pi/3$: blue represents $h_1 = \max_{i,j\in\mathbf{N}} |\theta_i - \theta_j|$ and red represents $h_2 = \max_{i\in\mathbf{N}} |\theta_0 - \theta_i|$.

satisfied since $|T^{-1}\boldsymbol{\omega}| = 0.7743 < 1$. Let $\theta(0) = [1.3\pi, 0.2\pi, 0.6\pi, 1\pi, 1.4\pi, 1.8\pi, 2\pi]^\top$, and the phases of the oscillators are plotted on the unit circle $\mathbb{S}^1$ at a sequence of time instants (see Fig. 7.3). One can observe that the peripheral oscillators with the same outgoing strength A_i get phase synchronized, forming three clusters (in each of which phases are different from the central one's). This suggests that the symmetry of the outgoing couplings of the peripheral oscillators plays an essential role in facilitating remote synchronization.

Next, we validate the results in Subsection 7.3.2, where there is a phase shift α.

Without loss of generality, let $\alpha = \pi/3$. First, we consider the case when $\omega_0 < \omega$. Let the frequency of each peripheral oscillator be $\omega = 0.8\pi$, and the natural frequency of the central one be $\omega_0 = 0.1\pi$. From the condition (7.14), we calculate the threshold of the coupling strength A, which is $(\omega - \omega_0)/2\cos^2\alpha = 1.4\pi$. Let $A = 1.6\pi > 1.4\pi$, and we plot the absolution value of phase differences $h_1(\theta(t))$ and $h_2(\theta(t))$ in Fig 7.4(a), from which we observe that remote synchronization is achieved. On the contrary, if we let $A = \pi$, it can be seen from Fig. 7.4(b) that remote synchronization does not occur. Finally, we consider the case $\omega_0 > \omega$ by letting $\omega_0 = 1.5\pi, \omega = 0.8\pi$. The threshold given in (7.13) becomes $(\omega - \omega_0)/2\cos\alpha = 0.7\pi$. The trajectories of $h_1(t)$ and $h_2(t)$ when $A = 0.8\pi$ and $A = 0.4\pi$ are presented in Fig. 7.4(c) and 7.4(d), respectively. Shown is Fig. 7.4(c), remote synchronization is achieved. Surprisingly, one can observe from Fig. 7.4(d) that the phase differences among peripheral oscillators approach zero, although the phase differences between the peripheral and the central oscillators are increasing. This implies remote synchronization can also take place without requiring that all the frequencies get synchronized.

7.4 How Natural Frequency Detuning Enhances Remote Synchronization

In this section, we apply the results on partial stability analysis to studying remote synchronization of oscillators. We restrict our attention to remote synchronization in a simpler network motif shown in Fig. 7.5. Unlike the previous section, we further assume that this network is undirected. This network is simple, but has been shown to surprisingly account for the emergence of zero-lag synchronization in remote cortical regions of the brain, even in the presence of large synaptic conduction delays [94]. Experiments have also evidenced that the same network can give rise to isochronous synchronization of delay-coupled semiconductor lasers [179]. The central element 0 in this network plays a critical role in meditating or relaying the dynamics of the peripheral 1 and 2. A recent study reveals that detuning the parameters of the central element from those of the peripheral ones can actually enhance remote synchronization [97]. To study this interesting finding analytically, we employ Kuramoto-Sakaguchi model [63] and detune the natural frequency of the central oscillator.

To investigate the role of natural frequency detuning might play, we make some slight changes on the model (7.1). We assume that the natural frequency of the central oscillator is equal to that of the peripheral ones, i.e., $\omega_0 = \omega$, and introduce a

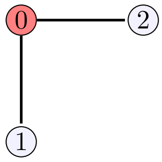


Figure 7.5: A simple network motif: central node 0 and peripherals 1 and 2.

detuning to the central oscillator. Then, the dynamics of the oscillators become

$$\dot{\theta}_i = \omega + A_i \sin(\theta_0 - \theta_i - \alpha), i = 1, 2; \tag{7.18a}$$

$$\dot{\theta}_0 = \omega + \sum_{j=1}^{2} A_j \sin(\theta_j - \theta_0 - \alpha) + u, \tag{7.18b}$$

where $\theta_i \in \mathbb{S}^1$ is the phase of the ith oscillator; $\omega > 0$ is the uniform natural frequency of each oscillator; $A_i > 0$ is the coupling strength between the central node 0 and the peripheral node i; $\alpha \in (0, \pi/2)$ is the phase shift; and $u > 0$ is the natural frequency detuning. Let $\theta = (\theta_0, \theta_1, \theta_2)^\top \in \mathbb{S}^3$. To study the remote synchronization in our considered network, we define a remote synchronization manifold as follows.

Definition 7.3 (Remote Synchronization Manifold)**.** *The remote synchronization manifold is defined by*

$$\mathcal{M} := \left\{\theta \in \mathbb{S}^3 : \theta_1 = \theta_2\right\}.$$

A solution $\theta(t)$ to (7.18) is said to be *remotely synchronized* if it holds that $\theta(t) \in \mathcal{M}$ for all $t \geq 0$. It is shown in [96] that network symmetries are critical to give rise to remote synchronization. In our considered network in Fig. 7.5, we say oscillators 1 and 2 are *symmetric* if $A_1 = A_2$. It can be observed that the requirement $A_1 = A_2$ is necessary for the system (7.18) to have a remote synchronized solution since the equation

$$\dot{\theta}_1 - \dot{\theta}_2 = A_1 \sin(\theta_0 - \theta_1 - \alpha) - A_2 \sin(\theta_0 - \theta_2 - \alpha) = 0$$

has a solution $\theta_1 = \theta_2$ *only if* $A_1 = A_2$. Therefore, we assume that the coupling strengths satisfy

$$A_1 = A_2 = A. \tag{7.19}$$

Therefore, the network in Fig. 7.5 is the simplest symmetric network. In what follows, we study the exponential stability of the remote synchronization manifold $\mathcal{M}$ under assumption (7.19).

Define a δ-neighborhood of $\mathcal{M}$ by $U_\delta = \{\theta \in \mathbb{S}^3 : \text{dist}(\theta, \mathcal{M}) < \delta\}$, where $\text{dist}(\theta, \mathcal{M})$ is the minimum distance from θ to a point on $\mathcal{M}$, that is, $\text{dist}(\theta, \mathcal{M}) = \inf_{y \in \mathcal{M}} \|\theta - y\|$. Let us define the exponential stability of the remote synchronization manifold $\mathcal{M}$.

Definition 7.4. *For the system* (7.18)*, the remote synchronization manifold $\mathcal{M}$ is said to be exponentially stable along the system* (7.18) *if there is $\delta > 0$ such that for any initial phase $\theta(0) \in \mathbb{S}^3$ satisfying $\theta(0) \in U_\delta$ it holds that*

$$\text{dist}(\theta(t), \mathcal{M}) = k \cdot \text{dist}(\theta(0), \mathcal{M}) \cdot e^{-\lambda t}, \qquad \forall t \geq 0,$$

where $k > 0$ and $\lambda > 0$.

Recall that remote synchronization behavior can be categorized into two depending on being *phase locked* or not. The frequency of the central oscillator, $\dot{\theta}_0$, is allowed to be different from the peripheral ones, $\dot{\theta}_1, \dot{\theta}_2$. It is clear that for the system (7.18), if the network is phase locked, it is remotely synchronized. However, the converse is not always true. We will study these two categories of remote synchronization in the next two subsections, where we assume $u = 0$ and $u \neq 0$, respectively, to reveal the role that the natural frequency detuning u plays. The phase locked case is relatively easy to analyze as demonstrated in the next subsection. In contrast, the analysis of the other case is technically involved, but is possible thanks to our results on partial stability established in the previous section.

7.4.1 Natural frequency detuning $u = 0$

In this subsection, we assume that the natural frequency detuning $u = 0$. As we will see later, only the phase locked remote synchronization can appear stably. The Linearization method is sufficient to show the stability of the remote synchronization manifold $\mathcal{M}$. For any $\alpha \in (0, \pi/2)$, there always exists a remotely synchronized solution to (7.18) that is phase locked. To show this, let $x_i := \theta_0 - \theta_i$ for $i = 1, 2$. The time derivative of x_i is

$$\dot{x}_i = \sum_{j=1}^{2} A \sin(-x_j - \alpha) - A \sin(x_i - \alpha). \tag{7.20}$$

Any remotely synchronized solution satisfies $x_1 = x_2$. Solving the equation $\dot{x}_i = 0$ with $x_1 = x_2$ we obtain two isolated equilibrium points of the system (7.20) in the interval $[0, 2\pi]$: 1) $x_1^* = x_2^* = c(\alpha)$; 2) $x_1^* = x_2^* = c'(\alpha)$, where

$$c(\alpha) = -\arctan\left(\frac{\sin\alpha}{3\cos\alpha}\right), \qquad c'(\alpha) = \pi + c(\alpha). \tag{7.21}$$

Note that other equilibrium points outside of $[0, 2\pi]$ are equivalent to these two, and it is thus sufficient to only consider them. Any solution satisfies $\theta_1(t) = \theta_2(t)$ and $\theta_0(t) - \theta_1(t) = c'(\alpha)$ (or $\theta_0(t) - \theta_1(t) = c(\alpha)$) is a phase locked and remotely synchronized solution. To capture this type of remote synchronization, we define $\mathcal{M}_1 := \{\theta \in \mathcal{M} : \theta_0 - \theta_1 = c(\alpha)\}$, $\mathcal{M}_1' := \{\theta \in \mathcal{M} : \theta_0 - \theta_1 = c'(\alpha)\}$, and refer to them as the *phase locked* remote synchronization manifolds. It is not hard to see that they are two positively invariant manifolds of the system (7.18). We show in the following theorem that $\mathcal{M}_1'$ is always unstable, and the phase shift α plays an essential role in determining the stability of $\mathcal{M}_1$.

Theorem 7.3. *Assume that* (7.19) *is satisfied. For any A, the following statements hold:*

1. *if $\alpha < \arctan\left(\sqrt{3}\right)$, there exists a unique exponentially stable remote synchronization manifold in $\mathcal{M}$, that is $\mathcal{M}_1$;*

2. *if $\alpha > \arctan\left(\sqrt{3}\right)$, there does not exist an exponentially stable remote synchronization manifold in $\mathcal{M}$.*

Proof. We prove this theorem by two steps. We first demonstrate that $\mathcal{M}_1$ and $\mathcal{M}_1'$ are the only two positively invariant manifolds in $\mathcal{M}$ for any α by proving that starting from any point in $\mathcal{M}/\mathcal{M}_1'$, the solution to (7.18), $\theta(t)$, converges to $\mathcal{M}_1$ asymptotically. Then, we investigate the stability of $\mathcal{M}_1$ and $\mathcal{M}_1'$ under different assumptions of α.

We start with the first step. When $\theta \in \mathcal{M}$, there holds that $x_1 = x_2$. Then, the dynamics of x_1 and x_2 are described by

$$\dot{x}_i = -2A \sin(x_i + \alpha) - A \sin(x_i - \alpha). \tag{7.22}$$

Note that x_2 has the same dynamics of x_1, and it is thus sufficient to only investigate the asymptotic behavior of x_1. For any initial condition $\theta(0) \in \mathcal{M}/\mathcal{M}_1'$, there hold that $\theta_1(0) = \theta_2(0)$ and $\theta_0(0) - \theta_1(0) \in (-\pi, \pi + c(\alpha)) \cup (\pi + c(\alpha), \pi)$, which means $x_1(0) = x_2(0)$ and $x_1(0) \in (-\pi, \pi + c(\alpha)) \cup (\pi + c(\alpha), \pi)$. When $x_1(0) \in (-\pi, \pi + c(\alpha))$, we choose $V_1 = \frac{1}{2}(x_1 - c(\alpha))^2$ as a Lyapunov candidate. Its time derivative is $\dot{V}_1 = -A(x_1 - c(\alpha))\big(2\sin(x_i + \alpha) + \sin(x_i - \alpha)\big)$, which satisfies $\dot{V} < 0$ for any $x_1 \in (-\pi, \pi + c(\alpha))$ and $\dot{V} = 0$ if $x_1 = c(\alpha)$. Thus, starting from $(-\pi, \pi + c(\alpha))$, $x_1(t)$ converges to $c(\alpha)$ asymptotically. When $x_1(0) \in (\pi + c(\alpha), \pi)$, we choose $V_1 = \frac{1}{2}(x_1 - 2\pi - c(\alpha))^2$ as a Lyapunov candidate. Likewise, one can show starting from $(\pi + c(\alpha), \pi)$, $x_1(t)$ converges to $2\pi + c(\alpha)$ asymptotically. Since $c(\alpha)$ and $2\pi + c(\alpha)$ represent the same point on $\mathbb{S}^1$, the two equilibrium points of (7.22), $x_1 = x_2 = c(\alpha)$ and $x_1 = x_2 = 2\pi + c(\alpha)$, correspond to the same manifold $\mathcal{M}_1$ of $\theta \in \mathbb{S}^3$. Then, there is no positively invariant manifolds in $\mathcal{M}$ other than $\mathcal{M}_1$ and $\mathcal{M}_1'$, since starting from any point in $\mathcal{M}/\mathcal{M}_1'$, $\theta(t)$ converges to $\mathcal{M}_1$.

Second, it remains to study the stability of $\mathcal{M}_1$ and $\mathcal{M}_1'$ for different values of α since they are the only positively invariant manifold in $\mathcal{M}$. The Jacobian matrix of (7.20) evaluated at the $x = (x_1, x_2)^\top$ is

$$J(x) = -A \begin{bmatrix} \cos(x_1+\alpha) + \cos(x_1-\alpha) & \cos(x_2+\alpha) \\ \cos(x_1+\alpha) & \cos(x_2+\alpha) + \cos(x_2-\alpha) \end{bmatrix}.$$

If $\alpha < \arctan(\sqrt{3})$, all the eigenvalues of $J(c(\alpha))$ is negative, which proves that $\mathcal{M}_1$ is exponentially stable; on the other hand, $J(\pi + c(\alpha))$ has a positive eigenvalue, which means $\mathcal{M}_1'$ unstable. Then, there is a unique exponentially stable remote synchronization manifold, that is $\mathcal{M}_1$. Following similar lines, one can show both $\mathcal{M}_1$ and $\mathcal{M}_1'$ are unstable if $\alpha < \arctan(\sqrt{3})$, which proves 2). This implies the remote synchronization manifold $\mathcal{M}$ is unstable if $\alpha > \arctan(\sqrt{3})$. $\square$

Consistent with the findings in [94] and [179], remote synchronization emerges thanks to the central mediating oscillator, and it is exponentially stable for a wide range of phase shift, i.e., $\alpha \in (0, \arctan(\sqrt{3}))$. Nevertheless, an even larger phase shift α out of this range can destabilize the remote synchronization. In the next subsection, we detune the natural frequency of the central oscillator by letting $u \neq 0$, which is similar to the introduction of parameter impurity in [97], and show how a sufficiently large natural frequency detuning can lead to robust remote synchronization that is exponential stable for any phase shift $\alpha \in (0, \pi/2)$.

7.4.2 Natural frequency detuning $u \neq 0$

In this subsection, we consider the case when the natural frequency detuning $u > 0$, and show how it can give rise to robust remote synchronization.

Note that if $u > 3A$, there does not exist a phase locked solution to (7.18). This is because the equations $\dot{x}_i = u + \sum_{j=1}^{2} A\sin(-x_j - \alpha) - A\sin(x_i - \alpha) = 0, i = 1, 2$, do not have a solution. Nevertheless, the remote synchronization can still be exponentially stable for a sufficiently large control input u. In other words, the natural frequency detuning can actually stabilize the remote synchronization, although it makes phase locking impossible. In fact, the remote synchronization manifold $\mathcal{M}$ as a whole becomes exponentially stable for any α with this control input. The following is the main result of this section.

Theorem 7.4. *There is a positive constant $\rho > 3A$ such that for any u satisfying $u > \rho$, the remote synchronization manifold $\mathcal{M}$ is exponentially stable for the system* (7.18) *for any phase shift $\alpha \in (0, \pi/2)$.*

The proof is technically involved and is based on the results on partial stability established in the previous section. Before providing the proof, we first define some variables, and associate the remote synchronization manifold $\mathcal{M}$ with an equivalent set defined on the new variables. We then prove that this set is exponentially stable. Let us define z_1 and z_2 by

$$z_1 := \frac{1}{2}\sum_{j=1}^{2}\cos(\theta_0-\theta_j), \tag{7.23a}$$

$$z_2 := \frac{1}{2}\sum_{j=1}^{2}\sin(\theta_0-\theta_j). \tag{7.23b}$$

Then, it is clear to see $z_1, z_2 \in \mathbb{R}$ satisfy $|z_1| \le 1$ and $|z_2| \le 1$. Note that for any initial condition $\theta(0) \in \mathbb{S}^3$, the unique solution $\theta(t)$ to (7.18) exists for all $t \ge 0$. As a consequence, $z_1(t)$ and $z_2(t)$ exist for all $t \ge 0$. We then define the following unit circle by using z_1 and z_2:

$$\mathcal{L} := \Big\{ z \in \mathbb{R}^2 : z_1^2 + z_2^2 = 1 \Big\}, \tag{7.24}$$

where $z = (z_1, z_2)^\top$. In fact, this set $\mathcal{L}$ has a strong relation with remote synchronization as follows.

Proposition 7.2. *Let $\mathcal{M}$ and $\mathcal{L}$ be defined in Definition 7.3 and* (7.24)*, respectively. The following two statements are equivalent:*

1. *θ belongs to the remote synchronization manifold $\mathcal{M}$.*
2. *$z = (z_1, z_2)^\top$ belongs to the set $\mathcal{L}$.*

Proof. From (7.23), the quadratic sum of z_1 and z_2 is

$$z_1^2 + z_2^2 = \frac{1}{4}\Big(\sum_{j=1}^{2}\cos(\theta_0-\theta_j)\Big)^2 + \frac{1}{4}\Big(\sum_{j=1}^{2}\sin(\theta_0-\theta_j)\Big)^2. \tag{7.25}$$

The right hand side of the equality (7.25) can be simplified to

$$\begin{aligned}
&\frac{1}{4}\Big(\sum_{j=1}^{2}\cos(\theta_0-\theta_j)\Big)^2 + \frac{1}{4}\Big(\sum_{j=1}^{2}\sin(\theta_0-\theta_j)\Big)^2 \\
&= \frac{1}{4}\sum_{j=1}^{2}\big(\cos^2(\theta_0-\theta_j) + \sin^2(\theta_0-\theta_j)\big) \\
&\quad + \frac{2}{4}\cos(\theta_0-\theta_1)\cos(\theta_0-\theta_2) + \frac{2}{4}\sin(\theta_0-\theta_1)\sin(\theta_0-\theta_2) \\
&= \frac{1}{2} + \frac{1}{2}\cos(\theta_1-\theta_2),
\end{aligned} \tag{7.26}$$

where the last equality has used the trigonometric identity $\cos a \cos b + \sin a \sin b = \cos(a-b)$. We first prove that 1) implies 2). If $\theta \in \mathcal{M}$, one obtains $\theta_1 = \theta_2$ from the definition of the remote synchronization manifold $\mathcal{M}$. It follows from (7.26) that the right-hand side of (7.25) equals 1. We then prove that 2) implies 1). If $z \in \mathcal{L}$, from (7.26) we obtain $1/2 + 1/2\cos(\theta_1 - \theta_2) = 1$, which means that $\cos(\theta_1 - \theta_2) = 1$. This, in turn, proves that $\theta \in \mathcal{M}$. The proof is complete. $\square$

Proposition 7.2 provides us an alternative way to study remote synchronization. Any pair of (z_1, z_2) belongs to $\mathcal{L}$ if and only if the corresponding $\theta \in \mathbb{R}^3$ is included in the remote synchronization manifold $\mathcal{M}$. If $\theta_1(0) = \theta_2(0)$, it can be seen from (7.18) that $\theta_1(t) = \theta_2(t)$ for all $t \geq 0$. In other words, $z(0) \in \mathcal{L}$ implies that $z(t) \in \mathcal{L}$ for all $t \geq 0$, which means that the set $\mathcal{L}$ is a positively invariant set of the system (7.18). To show the exponential stability of the remote synchronization manifold, it suffices to show the positively invariant set $\mathcal{L}$ is exponentially stable along the system (7.18) using the distance $\text{dist}(z, \mathcal{L}) = \inf_{y \in \mathcal{L}} \|z - y\|$.

To proceed with the analysis, we represent z_1 and z_2 in the polar coordinates

$$z_1 = r\cos\zeta, \tag{7.27}$$

$$z_2 = r\sin\zeta, \tag{7.28}$$

where with (7.23),

$$r := \frac{1}{2}\sqrt{\Big(\sum_{j=1}^{2}\cos(\theta_0 - \theta_j)\Big)^2 + \Big(\sum_{j=1}^{2}\sin(\theta_0 - \theta_j)\Big)^2}, \tag{7.29}$$

$$\zeta := \arctan\left(\frac{\sum_{j=1}^{2}\sin(\theta_0 - \theta_j)}{\sum_{j=1}^{2}\cos(\theta_0 - \theta_j)}\right). \tag{7.30}$$

It follows from (7.27) and (7.28) that $z_1^2(t) + z_2^2(t) = r^2$. Thus, the distance from $z(t)$ to the circle $\mathcal{L}$, denoted by $\mu(t)$ is

$$\mu(t) := \text{dist}(z(t), \mathcal{L}) = 1 - r(t). \tag{7.31}$$

We first prove a dynamics of $\mu(t)$ in (7.31) and $\zeta(t)$. Then, remote synchronization analysis reduced to partial stability analysis of $\mu(t)$

Proposition 7.3. *The dynamics of $\mu(t)$ in* (7.31) *and $\zeta(t)$ are given by*

$$\frac{d\mu(t)}{dt} = -2A(1 - (1-\mu)^2)\cos(\zeta - \alpha), \tag{7.32a}$$

$$\frac{d\zeta(t)}{dt} = u - A(1-\mu)\big(2\sin(\zeta + \alpha) + \sin(\zeta - \alpha)\big), \tag{7.32b}$$

where $\mu(t) \in [0, 1]$ and $\zeta(t) \in \mathbb{R}$.

Proof. The expression of r in (7.29) can be simplified as

$$r = \frac{1}{2}\sqrt{2 + 2\cos(\theta_1 - \theta_2)} = \sqrt{\cos^2 \frac{\theta_1 - \theta_2}{2}}, \tag{7.33}$$

and then the time derivative of $\mu(t)$ satisfies

$$\frac{d\mu(t)}{dt} = -\frac{dr(t)}{dt} = -\frac{1}{4r}\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2). \tag{7.34}$$

It follows from (7.18a) that

$$\begin{aligned}\dot{\theta}_1 - \dot{\theta}_2 &= A\sin(\theta_0 - \theta_1 - \alpha) - A\sin(\theta_0 - \theta_2 - \alpha)\\ &= -2A\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\theta_0 - \frac{\theta_1 + \theta_2}{2} - \alpha\right).\end{aligned}$$

Substituting this expression of $\dot{\theta}_1 - \dot{\theta}_2$ into (7.34) yields

$$\begin{aligned}\frac{d\mu(t)}{dt} = &- A\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)\\ &\times \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\theta_0 - \frac{\theta_1 + \theta_2}{2} - \alpha\right)\\ = &- A\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)\sum_{j=1}^{2}\cos(\theta_0 - \theta_j - \alpha).\end{aligned} \tag{7.35}$$

We further observe that

$$\begin{aligned}&\sum_{j=1}^{2}\cos(\theta_0 - \theta_j - \alpha)\\ &= \cos\alpha\sum_{j=1}^{2}\cos(\theta_0 - \theta_j) + \sin\alpha\sum_{j=1}^{2}\sin(\theta_0 - \theta_j)\\ &= 2\cos\alpha\cos\zeta + 2\sin\alpha\sin\zeta = 2\cos(\zeta - \alpha),\end{aligned} \tag{7.36}$$

where the second last equality follows from (7.27) and (7.28). Substituting (7.33) and (7.36) into (7.35) one obtains

$$\frac{d\mu(t)}{dt} = -2A(1 - (1-\mu)^2)\cos(\zeta - \alpha),$$

which is nothing but (7.31).

We next derive the time derivative of $\zeta(t)$ given in (7.30). It holds that $\zeta = \arctan(z_2/z_1)$, and then the time derivative of ζ satisfies

$$\frac{d\zeta(t)}{dt} = \frac{1}{z_1^2 + z_2^2}(z_1\dot{z}_2 - z_2\dot{z}_1). \tag{7.37}$$

It follows from (7.27) and (7.28) that

$$
\begin{aligned}
&z_1\dot{z}_2 - z_2\dot{z}_1 \\
&= \frac{1}{4}\sum_{j=1}^{2}\cos(\theta_0-\theta_j)\Big(\sum_{j=1}^{2}\cos(\theta_0-\theta_j)\cdot(\dot{\theta}_0-\dot{\theta}_j)\Big) \\
&+ \frac{1}{4}\sum_{j=1}^{2}\sin(\theta_0-\theta_j)\Big(\sum_{j=1}^{2}\sin(\theta_0-\theta_j)\cdot(\dot{\theta}_0-\dot{\theta}_j)\Big) \\
&= \frac{1}{4}\sum_{j=1}^{2}(\dot{\theta}_0-\dot{\theta}_j) + \frac{1}{4}\sum_{j=1}^{2}\Big(\cos(\theta_0-\theta_j)\cos(\theta_0-\theta_{-j}) \\
&+ \sin(\theta_0-\theta_j)\sin(\theta_0-\theta_{-j})\Big)\cdot(\dot{\theta}_0-\dot{\theta}_{-j}),
\end{aligned}
$$

where $-j$ is defined in a way so that $-j=2$ if $j=1$, and $-j=1$ otherwise. By using the trigonometric identity $\cos\beta_1\cos\beta_2+\sin\beta_1\sin\beta_2=\cos(\beta_1-\beta_2)$, we have

$$
\begin{aligned}
z_1\dot{z}_2 - z_2\dot{z}_1 &= \frac{1}{4}\sum_{j=1}^{2}(\dot{\theta}_0-\dot{\theta}_j) + \frac{1}{4}\cos(\theta_1-\theta_2)\sum_{j=1}^{2}(\dot{\theta}_0-\dot{\theta}_j) \\
&= \frac{1}{2}\cos^2\frac{\theta_1-\theta_2}{2}\sum_{j=1}^{2}(\dot{\theta}_0-\dot{\theta}_j) \\
&= \frac{1}{2}r^2\sum_{j=1}^{2}(\dot{\theta}_0-\dot{\theta}_j).
\end{aligned}
$$

It follows from the system (7.18) that

$$
\begin{aligned}
&\dot{\theta}_0 - \frac{1}{2}(\dot{\theta}_1+\dot{\theta}_2) \\
&= u + A\sum_{j=1}^{2}\sin(\theta_j-\theta_0-\alpha) - \frac{A}{2}\sum_{j=1}^{2}\sin(\theta_0-\theta_j-\alpha) \\
&= u + A\cos\alpha\sum_{j=1}^{2}\sin(\theta_j-\theta_0) - A\sin\alpha\sum_{j=1}^{2}\cos(\theta_j-\theta_0) \\
&\quad - \frac{A}{2}\cos\alpha\sum_{j=1}^{2}\sin(\theta_j-\theta_0) + \frac{A}{2}\sin\alpha\sum_{j=1}^{2}\cos(\theta_j-\theta_0).
\end{aligned}
$$

Substituting (7.23) into the above equation, we obtain

$$
\dot{\theta}_0 - \frac{1}{2}(\dot{\theta}_1+\dot{\theta}_2) = u - 2Ar\sin(\zeta+\alpha) - Ar\sin(\zeta-\alpha).
$$

As a consequence, $z_1\dot{z}_2 - z_2\dot{z}_1 = r^2(u - 2Ar\sin(\zeta+\alpha) - Ar\sin(\zeta-\alpha))$. Using this inequality and the fact $z_1^2 + z_2^2 = r^2$ in (7.37), we obtain

$$\frac{d\zeta(t)}{dt} = u - A(1-\mu)\big(2\sin(\zeta+\alpha) + \sin(\zeta-\alpha)\big),$$

which is nothing but (7.32b). □

We are now ready to provide the proof of Theorem 7.4 based on the results of partial stability obtained in the previous section.

Proof of Theorem 7.4. As we have shown, in order to prove the exponential stability of $\mathcal{M}$, it is sufficient to prove the set $\mathcal{L}$ is exponentially stable along the system (7.18). In other words, we show that $\mu = 0$ of the system (7.32) is exponentially stable uniformly in ζ based on Corollary 6.1. To this end, we confirm that the system satisfies conditions in Corollary 6.1.

First, we confirm the requirements for the system (6.38) as $\mu = x_1$ and $\zeta = z$. One can check $\mu = 0$ is a partial equilibrium of (7.32a), and the system is 2π-periodic in ζ. Also, from the assumption $u > \rho > 3A$, it holds that $d\zeta(t)/dt \neq 0$ for any μ and ζ, i.e. assumption (6.32) holds.

Next, we compute (6.40) by applying the change of time-axis, $t \mapsto \zeta$, and then its averaged system (6.41). The derivative of μ with respect to ζ can be computed as

$$\frac{d\mu}{d\zeta} = \frac{dr}{dt}\Big/\frac{d\zeta}{dt} = \varepsilon f(\mu,\zeta) \tag{7.38}$$

where

$$f(\mu,\zeta) = \frac{-2A(2-\mu)\mu\cos(\zeta-\alpha)}{1 - \frac{A}{u}(1-\mu)\Big(2\sin(\zeta+\alpha) - \sin(\zeta-\alpha)\Big)},$$

and $\varepsilon = 1/u$. Note that for any given u, there is $L > 0$ such that (6.43) holds. Then, its averaged system is

$$\dot{\hat{\mu}} = f_{\mathrm{av}}(\mu) := \int_0^{2\pi} f(\hat{\mu},\tau)d\tau, \tag{7.39}$$

where

$$f_{\mathrm{av}}(\hat{\mu}) = \int_0^{2\pi} f(\hat{\mu},\tau)d\tau = \frac{8\pi(2-\hat{\mu})\hat{\mu}}{(1-\hat{\mu})(5+4\cos 2\alpha)}\cdot g(\hat{\mu}),$$

$$g(\hat{\mu}) = \left(\frac{1}{u} - \frac{1}{\sqrt{u^2 - 5A^2(1-\hat{\mu})^2 - 4A^2(1-\hat{\mu})^2\cos 2\alpha}}\right).$$

According to Corollary 6.1, it remains to check the exponential stability of the averaged system. By the assumption $u > 3A$, it follows that $g(\hat{\mu}) < 0$ and thus $f_{\rm av}(\hat{\mu}) < 0$ for any $0 < \hat{\mu} < 1$. Moreover, for any $\hat{\mu}$ satisfies $0 < \hat{\mu} < \xi < 1$, it holds that

$$f_{\rm av}(\hat{\mu}) < -c\hat{\mu}, \tag{7.40}$$

where the constant c is given by

$$c = \frac{8\pi}{9}\left(\frac{1}{\sqrt{u^2 - 9A^2(1-\xi)^2}} - \frac{1}{u}\right).$$

Choose $V(\hat{\mu}) = \hat{\mu}^2$ as a Lyapunov candidate, and it is easy to see $\dot{V} \leq -c\hat{\mu}^2$, which implies that $\hat{\mu} = 0$ is exponentially stable along the averaged system (7.39) for any $u > 3A$. According to Corollary 1, there exists $\varepsilon^* > 0$ such that if $\varepsilon < \varepsilon^*$, the system (7.32) is partially exponentially stable with respect to μ. As $\varepsilon = 1/u$, it is equivalent to say that there exists $\rho > 3A$ such that if the input $u > \rho$, the system (7.32) is partially exponentially stable respect to μ. Thus, the remote synchronization manifold $\mathcal{M}$ is exponential stable for any phase shift α. □

Similar to the findings in [97,98], Theorem 7.4 analytically shows that by detuning the natural frequency one is able to stabilize the remote synchronization manifold even when the phase shift is quite large. Interestingly, the central oscillator has a different frequency $\dot{\theta}_0$ from the peripheral ones $\dot{\theta}_1$ and $\dot{\theta}_2$ when remote synchronization occurs under the assumption $u > \rho > 3A$.

In fact, what we have proven in Theorem 7.4 is that $\mathcal{L}$ is an exponentially stable limit cycle. Any remotely synchronized solution $\theta(t), t \geq 0$, to (7.18) satisfies $\theta_1(t) = \theta_2(t)$, i.e. $r(t) = 1$ ($\mu(t) = 0$) for all $t \geq 0$. By substituting $\mu(t) = 0$ into (7.32b), we have

$$\dot{\zeta} = u - A\big(2\sin(\zeta + \alpha) + \sin(\zeta - \alpha)\big).$$

Let $S(t, \zeta(0))$ denote the solution at time t to the above equation that starts from $\zeta(0)$, and then it satisfies

$$\begin{aligned} &S(t, \zeta(0)) = \\ &\zeta(0) + \int_0^t u - A\big(2\sin(\zeta(\tau) + \alpha) + \sin(\zeta(\tau) - \alpha)\big)d\tau, \end{aligned}$$

Since $u > 3A$, there is a finite $T = T(\zeta(0)) > 0$ such that $S(t+NT, \zeta(0)) = S(t, \zeta(0)) + 2N\pi$ for any nonnegative integer N. Submitting $r(t) = 1$ and $\zeta(t) = S(t, \zeta(0))$ into (7.27) and (7.28) we have

$$z_1(t) = \cos\left(S(t, \zeta(0))\right), \qquad z_2(t) = \sin\left(S(t, \zeta(0))\right),$$

which implies $z(t) := (z_1(t), z_2(t))^\top$ is T-periodic. Consequently, the set $\mathcal{L}$ in (7.24) satisfies $\mathcal{L} = \{z \in \mathbb{R}^2 : z(t), 0 \le t \le T\}$, which means $\mathcal{L}$ is a *periodic orbit*. Since we have proven in Theorem 7.4 that $\mathcal{L}$ is exponentially stable, the set $\mathcal{L}$ is an exponentially stable *limit cycle* defined on $\mathbb{R}^2$.

Let $v_1 = \dot\theta_1 + \dot\theta_2$ and $v_2 = \dot\theta_0$, and then we can rewrite the set (7.24) into

$$\mathcal{C} := \left\{ (v_1, v_2)^\top \in \mathbb{R}^2 : \frac{(v_1 + v_2 - 3\omega - u)^2}{16A^2 \sin^2 \alpha} + \frac{(v_1 - v_2 - \omega + u)^2}{16A^2 \cos^2 \alpha} = 1 \right\}, \tag{7.41}$$

which is also a limit cycle for the variables v_1 and v_2. Note that v_1 is the sum of the peripheral oscillators' frequencies. One can say the remote synchronization is reached if and only if the frequencies v_1 and v_2 reach the limit cycle $\mathcal{C}$. What we have proven in Theorem 7.4 also implies the exponential stability of the limit cycle $\mathcal{C}$.

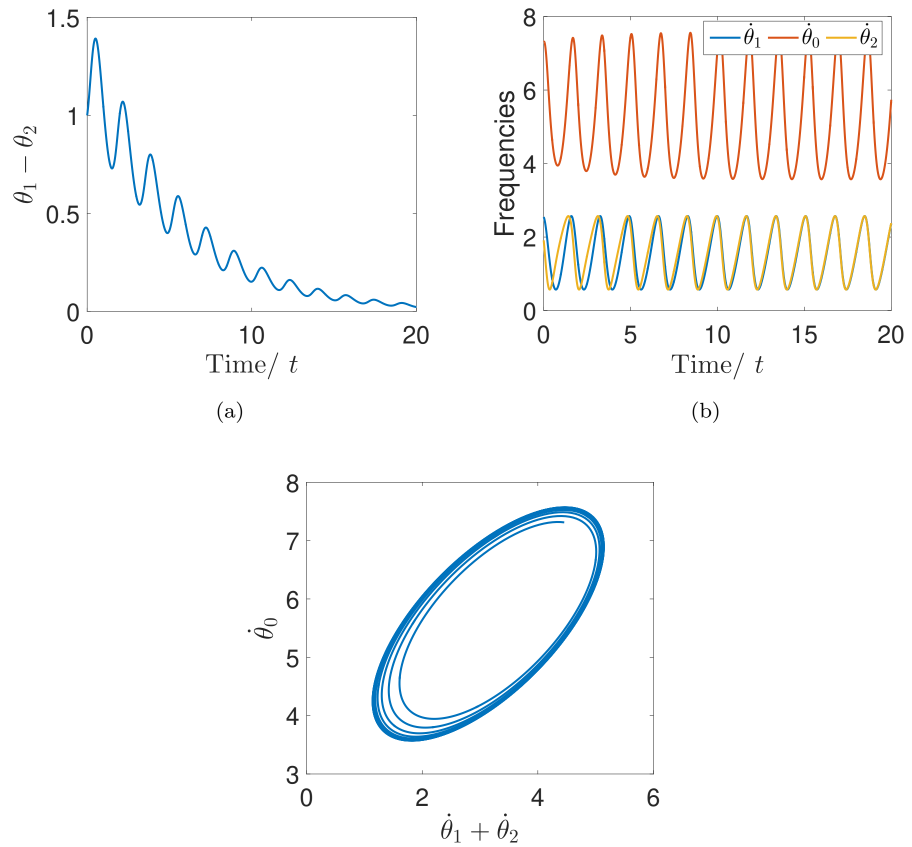
7.4.3 Numerical Examples

In this subsection, we perform some simulations to demonstrate our results in Subsections 7.4.1 and 7.4.2.

Let the parameters in the model (7.18) be: the natural frequency is $\omega = 0.5\pi$; the coupling strength $A = 1$; the phase shift $\alpha = \arctan\left(\sqrt{3}\right)$. From Theorem 7.3, one knows that phase locked remote synchronization is not stable. Then, we introduce a natural frequency detuning to the central oscillator by letting $u = 4$. The simulation results are shown in Fig. 7.6.

We observe from Fig. 7.6(a) that the phase difference between the oscillators 1 and 2 eventually converges to 0 despite some fluctuations, implying the exponential stability of the remote synchronization. Interestingly, from Fig. 7.6(b) we see that the central oscillators always has a different frequency from the peripheral ones, even when remote synchronization occurs. This means that remote synchronization can take place without requiring frequency synchronization throughout the network. Moreover, we find the frequencies $\dot\theta_0$ and $\dot\theta_1 + \dot\theta_2$ converges to the limit cycle given by (7.41) asymptotically, consistent with our findings in the previous section.

It is worth mentioning that the natural frequency detuning is not even very large ($u = 4$), but still able to stabilize the remote synchronization given a considerable phase shift. We believe our result in Theorem 7.4 is still conservative. It is interesting to seek for less conservative results in the future.



(a) (b)

(c)

Figure 7.6: Simulation results with a natural frequency detuning $u = 4$: (a) Phase difference between peripheral oscillators 1 and 2; (b) the frequencies of all the three oscillators; (c) the convergence to the limit cycle.

7.5 Concluding Remarks

Motivated by synchronization observed in distant cortical regions in the human brain, especially neuronal synchrony of unconnected areas through relaying, we have studied remote synchronization of Kuramoto oscillators coupled by star networks. We have shown that the symmetry of outgoing connections from the central oscillator plays a critical role in facilitating phase synchronization between peripheral oscillators. By carefully adjusting the strengths of these couplings, interesting patterns of stable remote synchronization, such as cluster synchronization, can be achieved. We have also studied the case when there is a phase shift. Sufficient conditions have been obtained to ensure the stability of remote synchronization. To analytically study some empirical findings in the literature [97, 98], we have further considered an even simpler network motif. We have proven that the introduction of natural frequency detuning to the central oscillator can enhance remote synchronization. The new criteria obtained in Chapter 6 are used to construct the proof.

Simulations have been performed to validate our results. Some results suggest that the sufficient condition for remote synchronization are still conservative. We are interested in obtaining less conservative results in the future. Moreover, it is also interesting to study stability of remote synchronization in more complex networks.

8 Conclusion and Outlook

Conclusion

In this thesis, we have studied distributed coordination algorithms and partial synchronization in complex networks in Part I and Part II, respectively.

In Part I, we focus on distributed coordination algorithms in stochastic settings. Inspired by coordinating behaviors observed in nature, distributed coordination algorithms serve as a foundation for a number of network algorithms for various purposes such as information fusion, load balancing, placement of mobile sensors, etc. When implementing those network algorithms, the stochastic influence such as random changes of network structures and communication delays and noise cannot be ignored. Besides, randomness is sometimes also deliberately introduced to improve global performance. Such stochastic network algorithms can be modeled by stochastic systems, which are often analyzed by stochastic Lyapunov theory. However, existing Lyapunov criteria are often not directly applicable especially when the stochastic processes are not confined to some certain ones such as i.i.d. Motivated by this, we have developed some new Lyapunov criteria for discrete-time stochastic systems in Chapter 3. In contrast to the existing Lyapunov theory for discrete-time stochastic systems, a constructed Lyapunov function is not required to decrease after every time step anymore. Instead, stability can be guaranteed if it decreases after finite time steps. We then use them in Chapter 4 to study the following distributed coordination algorithms: 1) the products of random stochastic matrices, 2) asynchrony-induced agreement problems in periodic networks, and 3) distributed linear equation solving. Sharper results compared to those in the literature have been obtained.

In Part II, we have uncovered some possible underlying mechanisms that could give rise to partial synchronization in complex networks. We have investigated in Chapter 5 how partial synchronization can take place among directly connected regions. We have found that strong regional (or local) connection is a possible mechanism. If some oscillators in a network are connected tightly, they can evolve in unison, while the rest that are weakly coupled remain incoherence. The Lyapunov functions based on the incremental 2-norm and the incremental ∞-norm are used to construct the proof. In addition, we have studied how partial synchronization is possible to occur among units that have no direct connections, an interesting phenomenon termed remote synchronization. This phenomenon has also been widely detected in the human brain, where distant cortical regions without direct neural links also experience functional correlations. In order to study remote synchronization, we have developed some new criteria for partial stability of nonlinear systems in Chapter 6. These criteria enable us to study the partial stability of some nonlinear systems that are not easy to analyze using existing results. Then, we study remote synchronization in simple network structures, i.e., star networks, in Chapter 7. We have found that the symmetry of outgoing connections from the central oscillator is crucial to shaping remote synchronization, and is possible to induce several clusters for the peripheral oscillators. We have further investigated how detuning the natural frequency of the central oscillator in a star network with two peripheral nodes can strengthen remote synchronization. Finally, we use the obtained Lyapunov criteria on partial stability to prove that natural frequency detuning of the central oscillator actually makes the remote synchronization more robust against the phase shifts.

Outlook

The Lyapunov theory plays a fundamental role in the control field. We are interested in further developing control Lyapunov criteria for both discrete-time and continuous-time stochastic systems since they can be applied to many practical problems including distributed optimization.

As it has been observed in the human brain, partial synchronization is perhaps more common than global synchronization. It is certainly more interesting to study partial synchronization further since it would help us to better understand the sophisticated mechanisms behind the synchrony patterns in the brain. Particularly, we are even more interested in studying remote synchronization in more complex networks than star networks. We believe that network symmetries would still play a crucial role in rendering remote synchronization. Before considering general networks, We plan to start with some simpler ones such as line works and bipartite networks. Instead of considering an identical phase shift for all oscillators, we plan to study the case when phase shifts are heterogeneous. Phase shifts are usually used to model small time delays in the Kuramoto model. When the delays are large, the Kuramoto-Sakaguchi model is not accurate anymore. In this case, it is better to employ a time-delayed Kuramoto model and investigate the role that delays play in remote synchronization.

The analysis will be quite challenging. The partial stability theory can be very helpful for the analysis. It is quite interesting to further develop partial stability theory and applied it to the study of remote synchronization. We also plan to test our theoretical findings via experiments of the brain by cooperating with neuroscientists.

We believe theoretical study using appropriate mathematical models of the brain will be very important to explain and predict brain behaviors, and also contribute to the treatment of various brain diseases in the future.

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Summary

Coordinating behaviors in populations of interacting units have been widely observed in many natural systems. Many attempts have been made to understand these behaviors, which have also inspired a lot of applications. This thesis is devoted to: 1) the study of distributed coordination algorithms, which is one of those applications, and 2) the investigation on the underlying mechanisms of a particular type of coordinating behaviors, i.e., partial synchronization.

Part I of this thesis focuses on the study of distributed coordination algorithms. When implementing distributed coordination algorithms, the computational processes are inevitably influenced by some random factors, which can be random changes in network structures or stochastic communication delays. Besides, some randomness may also be introduced deliberately to improve global performance. Taking the randomness into account, distributed coordination algorithms can be modeled by stochastic systems. However, traditional methods cannot be directly used for stochastic stability analysis of these systems in many circumstances. There is a great need for developing new results to study the stability of stochastic systems. This is exactly the aim of Chapter 3. In Chapter 3, we develop a new Lyapunov criterion, termed the finite-step Lyapunov criterion, for discrete-time stochastic systems. From the existing Lyapunov criterion, a constructed Lyapunov function needs to decrease at every time step to ensure stability. In sharp contrast, we relax this requirement by allowing it to decrease after some finite time steps. This relaxation provides a larger range of choices when constructing a Lyapunov function. In Chapter 4, we then show how the obtained Lyapunov criteria can be utilized to solve the problems we encountered in dealing with several distributed coordination algorithms including 1) convergence of products of random sequences of stochastic matrices, 2) asynchronous-updating-induced agreement of agents coupled by periodic networks, and 3) algebraic equation solving via distributed averaging algorithms in a randomly changing network.

Global synchronization across the entire brain is always a sign of certain brain diseases, while partial synchronization usually takes place in the healthy brain. This motivates us to study partial synchronization in Part II, trying to uncover the

underlying mechanisms that could give rise to this special type of coordinating behavior. Towards this end, we employ the Kuramoto model and its variation, i.e., Kuramoto-Sakaguchi model, to describe the dynamics of oscillators. Two classes of partial synchronization are studied in this part: 1) synchronization among a set of oscillators that have direct connections; 2) synchronization among a set of oscillators that have no direct link, termed *remote synchronization*. We have studied the first class in Chapter 5. Inspired by the organization of cortical neurons in the brain, a two-level network structure is considered. The oscillators are all-to-all connected, forming local communities at the lower level; at the higher level, the communities are interconnected by a sparse network. We show that strong coupling strengths among the set of directly connected oscillators can lead to partial synchronization. Remote synchronization in star networks is investigated in Chapters 6 and 7. To prove the stability of remote synchronization, one often needs to show the partial stability of a nonlinear system. However, existing criteria for partial stability are not directly applicable in our case, which motivates us to develop some new criteria for partial stability analysis of nonlinear systems in Chapter 6. We first prove that a constructed Lyapunov function does not need to have a negative-definite time derivative. If it decreases after a finite time, asymptotic (or exponential) stability can be ensured. We then show that the exponential stability of a class of slow-fast systems can be studied by analyzing the averaged systems obtained by periodic averaging. In Chapter 7, we first consider directed star networks and show the important role that the symmetries of the connections going out from the central oscillator play in rendering remote synchronization among peripheral oscillators. Finally, we focus on an undirected star network of two peripheral oscillators. Using the new criteria developed in Chapter 6, we prove that the natural frequency detuning of the central oscillator can actually enhance remote synchronization.

Samenvatting

Coördinerend gedrag in populaties van interactieve units wordt veel geobserveerd in natuurlijke systemen. Veel pogingen zijn gemaakt om dit gedrag begrijpen, dat ook dient als een inspiratie voor veel toepassingen. Deze thesis wijdt zich toe aan: 1) de studie van gedistribueerde coördinatie algoritmen, dat een van die toepassingen is, en 2) het onderzoeken van de onderliggende mechanismen van een bepaalde type coördinatie, i.e., partiele synchronisatie.

Deel I van de thesis focust op de studie van gedistribueerde coördinatie algoritmen. Bij het implementeren van dergelijke systemen worden de rekenkundige processen onoverkomelijk beïnvloed door willekeurige factoren die kunnen voortkomen uit spontane veranderingen in netwerk structuren of communicatie vertragingen. Daarnaast kan een bepaalde mate van willekeurigheid met intentie worden ingebracht om de globale prestaties van de algoritmen te verbeteren. Rekening houdend met deze willekeurigheid worden gedistribueerde coördinatie algoritmen gemodelleerd door stochastische systemen. Echter, in veel gevallen kunnen traditionele methoden niet direct worden toegepast voor de stochastische stabiliteit analyse. Daarom is er een grote behoefte aan de ontwikkeling van nieuwe resultaten voor de stabiliteit van stochastische systemen. Dit is exact het doel van hoofdstuk 3. In dit hoofdstuk ontwikkelen we een nieuw Lyapunov criterium, genaamd het eindige-stap Lyapunov criterium, voor stochastische systemen in discrete tijd. Voor het bestaande Lyapunov criterium moet een geconstrueerde Lyapunov functie op elke tijdstap afnemen om stabiliteit te garanderen. Hierin verzwakken we deze eis uitzonderlijk door toe te staan dat de functie afneemt na een bepaald eindig aantal tijdstappen. Deze verzwakte eis geeft een groter aantal mogelijkheden bij de constructie van een Lyapunov functie. In hoofdstuk 4 tonen we vervolgens aan hoe het nieuwe criterium kan worden gebruikt bij het oplossen van problemen in gedistribueerde coördinatie algoritmen zoals: 1) de convergentie van producten van willekeurige reeksen van stochastische matrices, 2) door asynchrone updating geïnduceerde overeenkomst van agenten gekoppeld door een periodiek netwerk, en 3) oplossingen van algebraïsche vergelijkingen via gedistribueerde middeling algoritmen in willekeurig veranderende netwerken.

Globale synchronisatie in het brein is vaak een teken van een bepaalde brein ziekte, maar partiele synchronisatie komt juist voor in een gezond brein. Dit gegeven motiveert ons om partiele synchronisatie te bestuderen in deel II van de thesis, waarin getracht wordt het onderliggende mechanisme te ontdekken die tot een dergelijk type coördinatie kan leiden. Hiervoor gebruiken we het Kuramoto model en het Kuramoto-Sakaguchi model om de dynamica van oscillatoren te beschrijven. Twee klasse van partiele synchronisatie worden bestudeerd: 1) synchronisatie tussen een set oscillatoren met directe connecties; 2) synchronisatie tussen oscillatoren zonder directe connectie, genoemd *afgezonderde synchronisatie*. De eerste klasse wordt bestudeerd in hoofdstuk 5. Geïnspireerd door de organisatie van corticale neuronen in het brein, wordt een twee-laags netwerk structuur beschouwd. The oscillatoren zijn volledig verboden, waardoor lokale gemeenschappen worden gevormd op een lager niveau; op het hogere niveau zijn de gemeenschappen verbonden met een verspreid netwerk. We tonen aan dat een hoge koppel sterkte tussen de set van direct gekoppelde oscillatoren tot partiele synchronisatie kan leiden.

Afgezonderde synchronisatie in ster netwerken wordt bestudeerd in de hoofdstukken 6 en 7. Om de stabiliteit van de partiele synchronisatie te bewijzen, moet vaak de partiele stabiliteit van een niet-lineair systeem worden aangetoond. Echter, in ons geval zijn de bestaande criteria voor partiele stabiliteit niet direct toepasbaar, hetgeen ons motiveert om nieuwe criteria voor partiele stabiliteit af te leiden in hoofdstuk 6. We bewijzen eerst dat een geconstrueerde Lyapunov functie niet noodzakelijk een strikt negatieve tijd afgeleide hoeft te hebben. Als de tijd afgeleide afneemt na een eindige hoeveelheid tijd, dan kan asymptotische (of exponentiele) stabiliteit gegarandeerd worden. Vervolgens tonen we aan dat exponentiele stabiliteit van een klasse van slow-fast systemen kan worden bestudeerd aan de hand van van een gemiddeld systeem dat kan worden verkregen door periodieke middeling. In hoofdstuk 7, bestuderen we een gericht ster netwerk en tonen we de belangrijke rol aan van de symmetrie in de uitgaande connecties van de centrale oscillator in het ontstaan van afgezonderde synchronisatie tussen de perifere oscillatoren. Tot slot focussen we op een niet gericht ster netwerk met twee perifere oscillatoren. Door gebruik te maken van de criteria ontwikkeld in hoofdstuk 6 bewijzen we dat het ontstemmen van de natuurlijke frequentie van de centrale oscillator afgezonderde synchronisatie kan bevorderen.