Upper and Lower Bounds for Controllable Subspaces of Networks of Diffusively Coupled Agents

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Abstract—**This technical note studies the controllability of diffusively coupled networks where some agents, called leaders, are under the influence of external control inputs. First, we consider networks where agents have general linear dynamics. Then, we turn our attention to infer network controllability from its underlying graph topology. To do this, we consider networks with agents having single-integrator dynamics. For such networks, we provide lower and upper bounds for the controllable subspaces in terms of the distance partitions and the maximal almost equitable partitions, respectively. We also provide an algorithm for computing the maximal almost equitable partition for a given graph and a set of leaders.**

Index Terms—**Almost equitable partition, controllability, distance partition, linear diffusive coupling, multi-agent networks.**

I. Introduction

Recently, significant work has been done to study distributed and cooperative control of multi-agent networks [2], [3]. It is of particular interest to study the case when the agents are coupled together through linear diffusive couplings since rich collective behaviors, such as synchronization [4] and clustering [5], [6], may arise as a result of local interactions among agents without centralized coordination or control. To reduce the complexity of controller design, one is especially interested in knowing how to influence the behavior of the overall system by just controlling a small fraction of the agents [7], [8]. We call such agents that are under the forcing of external control inputs the *leaders* and correspondingly the rest of the agents *followers*. Hence, to study whether any desired collective behavior can be achieved in finite time by controlling the leaders is equivalent to the study of the controllability of the overall systems consisting of all the leaders and followers. For example, the controllability problem has been related to the problem of controlling a formation of mobile robots by manipulating the trajectories of the leaders such that all the robots can move from any initial positions to any desired final positions within finite time [9].

The controllability of diffusively coupled multi-agent networks was first studied in [10] and later in [11]–[16]. However, most existing results deal with networks where agents have single-integrator dynamics, except for agents with double-integrator dynamics [17] and agents with higher-order-integrator dynamics [18]. In this technical note, we first study diffusively coupled networks where agents have identical (general) linear dynamics. We reveal in Theorem 1 the dependence of controllability of such networks on agent dynamics and network topologies. Existing results on agents with higher order integrator dynamics [17], [18] can be considered as special cases of Theorem 1.

To infer network controllability from its topology, we then focus on diffusively coupled networks with single-integrator agents. For the controllable subspace of such a network, we provide:

i) a lower bound on its dimension in terms of the distance partitions (in Theorem 2);
ii) an upper bound in terms of the maximal almost equitable partitions (in Theorem 4).

The contribution of these results to the state-of-the-art is two-fold. Firstly, the distance partitions in Theorem 2 yield easily computable lower bounds. Secondly, the upper bounds we provide are valid in the case of multi-leader scenarios unlike the existing upper bounds in the literature [11], [15] which *only* deal with single-leader cases. Also, we provide an algorithm for obtaining the maximal almost equitable partition for given leaders.

The technical note is organized as follows. In Section II, we first introduce the class of multi-agent networks that is of interest in this technical note and later provide necessary and sufficient conditions for controllability of such networks. Section III is devoted to the study of multi-agent networks with single-integrator agent dynamics in order to reveal the effect of the underlying topology on the controllability. In this section, we provide a lower bound for the controllable subspace of such a network in terms of distance partitions and an upper bound in terms of the so-called almost equitable partitions. Section IV presents an algorithm in order to compute the almost equitable partition which bounds the controllable subspace from above. Finally, the technical note ends with the conclusions in Section V.

II. Multi-Agent Networks and Their Controllability

A. Diffusively Coupled Multi-Agent Networks

Consider a multi-agent system consisting of n agents labeled by the set $\mathcal{V} = \{1, 2, \dots, n\}$ where n is a positive integer. We assign the roles of leaders and followers to the agents and define $\mathcal{V}_{\mathrm{L}} = \{v_1, v_2 \dots, v_m\}$ where m is a positive integer with $m \leq n$ and $\mathcal{V}_{\mathrm{F}} = \mathcal{V} \setminus \mathcal{V}_{\mathrm{L}}$ to denote the sets of indices of the leaders and followers, respectively.

To each follower $i \in \mathcal{V}_{\mathrm{F}}$, we associate a linear dynamical system

$$\dot{x}_i = Ax_i + Cz_i$$

and to each leader $i \in \mathcal{V}_{\mathrm{L}}$ and $i = v_\ell$ a linear dynamical system

$$\dot{x}_i = Ax_i + Cz_i + Bu_\ell$$

where $x_i \in \mathbb{R}^p$ denotes the state of agent i, $u_\ell \in \mathbb{R}^q$ the external input to agent $i = v_\ell$, $z_i \in \mathbb{R}^s$ the coupling variable for the agent i, and all involved matrices are of appropriate dimensions.

Two distinct agents i and j are said to be *neighbors* if their states are known by each other. Throughout this technical note, we assume that the neighboring relationships are fixed. Such neighboring relationships can be described by a simple undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the edge set such that $(i, j) \in \mathcal{E}$ if agents i and j are neighbors. The coupling variable z_i for each agent $i \in \mathcal{V}$ is determined by the so-called diffusive coupling rule based on the neighboring relations as follows:

$$z_i = K \sum_{(i,j) \in \mathcal{E}} (x_j - x_i)$$

where $K \in \mathbb{R}^{s \times p}$ is the matrix describing the coupling strengths.

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By defining $x = \mathrm{col}(x_1, x_2, \ldots, x_n)$ and $u = \mathrm{col}(u_1, u_2, \ldots, u_m)$, we can write the dynamics of the above multi-agent system into a compact form

$$\dot{x} = \hat{L}x + \hat{M}u \tag{1}$$

where $\hat{L} = I_n \otimes A - L \otimes CK$ with L being the Laplacian matrix of $\mathbb{G}$, $\hat{M} = M \otimes B$ with $M \in \mathbb{R}^{n\times m}$ defined by

$$M_{i\ell} = \begin{cases} 1, & \text{if } i = v_\ell \\ 0, & \text{otherwise.} \end{cases}$$

Here "$\otimes$" denotes the Kronecker product (see e.g., [19]). Note that the matrix product CK is sometimes called the *inner coupling matrix* in the study of the synchronization behavior in complex networks [4]. In the next subsection, we study the controllability of system (1).

B. Controllability of Diffusively Coupled Multi-Agent Systems

In the following theorem, we provide necessary and sufficient conditions for the controllability of the multi-agent system (1).

Theorem 1: The pair $(\hat{L}, \hat{M})$ is controllable if and only if the pair (L, M) is controllable and for each eigenvalue λ of L the pair $(A - \lambda CK, B)$ is controllable.

Proof: (Necessity) We only prove the necessity of the controllability of the pair (L, M), and the necessity of the controllability of the pair $(A - \lambda CK, B)$ can be proved in a similar manner. Suppose (L, M) is uncontrollable. Then there exists some nonzero $x \in \mathbb{R}^n$ such that $x^T L = \lambda x^T$ and $x^T M = 0$. Let $(\theta, y) \in \mathbb{C} \times \mathbb{C}^p$ be a left-eigenpair of the matrix $A - \lambda CK$. Note that $(\theta, x \otimes y)$ is a left-eigenpair of $\hat{L}$ and $(x \otimes y)^H \hat{M} = (x^T M) \otimes (y^H B) = 0$ where z^H denotes the conguate transpose of the vector z. This implies that the pair $(\hat{L}, \hat{M})$ is uncontrollable.

(Sufficiency) On the contrary, suppose that $(\hat{L}, \hat{M})$ is uncontrollable. Since L is symmetric, one can always find an orthonormal matrix U such that $U^T LU = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_i's are eigenvalues of L. Now consider the following two matrices $\tilde{L}$ and $\tilde{M}$ that are obtained from $\hat{L}$ and $\hat{M}$ respectively according to

$$\begin{aligned} \tilde{L} &\triangleq (U^T \otimes I_p)\hat{L}(U \otimes I_p) \\ &= \mathrm{blockdiag}\,(A - \lambda_1 CK, \ldots, A - \lambda_n CK) \\ \tilde{M} &\triangleq (U^T \otimes I_p)\hat{M} = U^T M \otimes B. \end{aligned}$$

Since $(\hat{L}, \hat{M})$ is uncontrollable and $U^T \otimes I_p$ is nonsingular, the pair $(\tilde{L}, \tilde{M})$ is also uncontrollable. In view of the block diagonal structure of $\tilde{L}$, we know that there must exist an index s with $1 \le s \le n$ such that the corresponding matrix pair $(A - \lambda_s CK, (U^T M)_s \otimes B)$ is uncontrollable, where for a matrix M we use $(M)_s$ to denote its sth row. This, however, implies that (L, M) is uncontrollable in case $(U^T M)_s = 0$ or $(A - \lambda_s CK, B)$ is uncontrollable in case $(U^T M)_s \neq 0$. Hence, we arrive at a contradiction. ■

The main results of [17], [18] on the controllability of networks of agents with higher-order-integrator dynamics can be recovered from Theorem 1 as special cases.

Theorem 1 also reduces the computational cost for checking controllability because the dimensions of the controllability matrices of the pairs (L, M) and $(A - \lambda CK, B)$ are much lower than that of the pair $(\hat{L}, \hat{M})$ when the number of agents n is large or the dimension p of agent dynamics is high.

One can roughly interpret the two conditions stated in Theorem 1 to be the effects of network topologies and agent dynamics on controllability. In the next section, we reveal the relationship between network controllability and the underlying graph topologies.

III. Controllability of Single Integrators and Graph Partitions

When the agents are governed by single-integrator dynamics, system (1) becomes

$$\dot{x} = -Lx + Mu. \tag{2}$$

Then the controllability of the network is completely determined by the underlying topology given by the pair of matrices L and M. In what follows, we want to provide lower and upper bounds for the controllable subspace $\mathcal{K} = \mathrm{im}M + L\mathrm{im}M + \cdots + L^{n-1}\mathrm{im}M$ of system (2). Towards this end, we quickly review graph partitions.

A. Graph Partitions

Let $\mathbb{G}$ be an undirected graph with the vertex set $\mathcal{V}$. A subset $\mathcal{C}$ of $\mathcal{V}$ is called a *cell*. A collection of cells $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ is called a *partition* if the cells are mutually disjoint and $\bigcup_i \mathcal{C}_i = \mathcal{V}$. We use $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ to denote the partition. The *characteristic matrix* $P(\pi) \in \mathbb{R}^{n\times k}$ of the partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ is defined by

$$P_{ij}(\pi) \triangleq \begin{cases} 1 & \text{if } i \in \mathcal{C}_j \\ 0 & \text{otherwise.} \end{cases}$$

Next, we introduce particular partitions and employ them in order to obtain bounds for the controllable subspace of system (2).

B. Lower Bounds by Distance Partitions

The *distance* between two vertices $i, j \in \mathcal{V}$ is the length of the shortest path from i to j in $\mathbb{G}$ and will be denoted by $\mathrm{dist}(i, j)$. For convenience, we say $\mathrm{dist}(i, i) = 0$ for any $i \in \mathcal{V}$. The *diameter* of $\mathbb{G}$ is defined by $\mathrm{diam}(\mathbb{G}) \triangleq \max_{i,j\in\mathcal{V}} \mathrm{dist}(i, j)$. Obviously, when $\mathbb{G}$ is connected [9] and $n > 1$, it holds that $0 < \mathrm{diam}(\mathbb{G}) < n$. Let $\mathbb{G}$ be a connected graph and $v \in \mathcal{V}$. The *distance partition relative to* v consists of the cells $\mathcal{C}_i = \{w \in \mathcal{V} | \mathrm{dist}(w, v) = i\}$ for $0 \le i \le \mathrm{diam}(\mathbb{G})$. We denote the distance partition relative to v by $\pi_{\mathrm{D}}(v)$. The following lemma is a direct consequence of the definition of $\pi_{\mathrm{D}}(v)$.

Lemma 1: For any $z \in \mathcal{C}_i$ and $w \in \mathcal{C}_j$, it holds that $|i - j| \le \mathrm{dist}(z, w)$.

The following theorem provides a lower bound for the dimension of the controllable subspace $\mathcal{K}$ in terms of the distance partition relative to the leaders.

Theorem 2: If $\mathbb{G}$ is connected, then $\dim \mathcal{K} \ge \max_{v_i\in\mathcal{V}_{\mathrm{L}}} \mathrm{card}(\pi_{\mathrm{D}}(v_i))$.

Proof: We first prove that if $\mathcal{V}_{\mathrm{L}} = \{v\}$ then $\dim \mathcal{K} \ge \mathrm{card}(\pi_{\mathrm{D}}(v))$. Without loss of generality, we take $v = 1$, $\pi_{\mathrm{D}}(1) = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_r\}$ with $r \le \mathrm{diam}\mathbb{G}$

$$\begin{aligned} &\mathcal{C}_0 = \{1\}, \text{and} \\ &\mathcal{C}_q = \{i_q + 1, i_q + 2, \ldots, i_{q+1}\} \end{aligned}$$

where $1 \le q \le r$ and $1 = i_1 < i_2 < \cdots < i_{r+1} = n$. In view of Lemma 1, we know that no vertex in $\mathcal{C}_i$ has a neighbor in $\mathcal{C}_j$ if $|i - j| > 1$. This means that L is of the form

$$L = \begin{bmatrix} \deg(1) & \mathbf{1}^T & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mathbf{1} & L_{11} & L_{12} & 0 & \cdots & 0 & 0 & 0 \\ 0 & L_{21} & L_{22} & L_{23} & \cdots & 0 & 0 & 0 \\ 0 & 0 & L_{32} & L_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & L_{r-2,r-2} & L_{r-2,r-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & L_{r-1,r-2} & L_{r-1,r-1} & L_{r-1,r} \\ 0 & 0 & 0 & 0 & \cdots & 0 & L_{r,r-1} & L_{r,r} \end{bmatrix} \tag{3}$$

where $\mathbf{1}$ is the all-one column vector of dimension $\mathrm{card}(\mathcal{C}_1)$ and L_{kl} are $\mathrm{card}(\mathcal{C}_k) \times \mathrm{card}(\mathcal{C}_l)$ matrices for all $1 \leq k, l \leq r$. Since $v = 1$, $M = e_1 = [1 \quad 0 \quad \cdots \quad 0]^T$. Let $E = [e_1 \quad Le_1 \quad \cdots \quad L^r e_1]$. Then, we get

$$E = \begin{bmatrix} 1 & \deg(1) & * & * & \cdots & * & * \\ 0 & \mathbf{1} & * & * & \cdots & * & * \\ 0 & 0 & L_{21}\mathbf{1} & * & \cdots & * & * \\ 0 & 0 & 0 & L_{32}L_{21}\mathbf{1} & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & L_{r-1,r-2}\cdots L_{21}\mathbf{1} & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & L_{r,r-1}\cdots L_{21}\mathbf{1} \end{bmatrix},$$

where '*' denotes the corresponding elements of less interest. Since $\mathbb{G}$ is connected, each diagonal block must be nonzero. Then $\mathrm{rank}E = \mathrm{card}(\pi_{\mathrm{D}}(1)) = r + 1$. Therefore, by further using the fact that $r \leq \mathrm{diam}\mathbb{G} \leq n - 1$, we have

$$\begin{aligned} \mathrm{card}(\pi_{\mathrm{D}}(v)) &= \mathrm{rank}E \\ &\leq \mathrm{rank}\,[e_1 \quad Le_1 \quad \cdots \quad L^{n-1}e_1] \\ &= \dim \mathcal{K}. \end{aligned}$$

Now we consider the case when $\mathcal{V}_{\mathrm{L}} = \{v_1, v_2, \ldots, v_m\}$. Clearly, we have $\dim \mathcal{K} \geq \mathrm{card}(\pi_{\mathrm{D}}(v_i))$ for any $v_i \in \mathcal{V}_{\mathrm{L}}$. Therefore, $\dim \mathcal{K} \geq \max_{v_i \in \mathcal{V}_{\mathrm{L}}} \mathrm{card}(\pi_{\mathrm{D}}(v_i))$. ■

Next, we introduce almost equitable partitions in order to provide an upper bound for the controllable subspace.

C. Upper Bounds by Almost Equitable Partitions

For a graph $\mathbb{G}$, a partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ is said to be an *almost equitable partition* if for any distinct cells $\mathcal{C}_i$ and $\mathcal{C}_j$, every vertex in $\mathcal{C}_i$ has the same number of neighbors in $\mathcal{C}_j$ [20]. We denote the set of all the almost equitable partitions of $\mathbb{G}$ by Π_{AEP}. Almost equitable partitions have the following key property that is related to the Laplacian matrices of the corresponding graphs. In the sequel, we say that a subspace $\mathcal{X} \subseteq \mathbb{R}^n$ is *T-invariant* if $T\mathcal{X} \subseteq \mathcal{X}$ where $T : \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 2 [20, Prop. 1]: A partition π is almost equitable if and only if $\mathrm{im}P(\pi)$ is L-invariant.

To come up with an upper bound for the controllable subspace, we need to compare different partitions. We say that a partition π_1 is *finer* than another partition π_2 if each cell of π_1 is a subset of some cell of π_2 and we write $\pi_1 \leq \pi_2$. It is a direct consequence of the definition that

$$\pi_1 \leq \pi_2 \quad \Longleftrightarrow \quad \mathrm{im}P(\pi_2) \subseteq \mathrm{im}P(\pi_1). \tag{4}$$

Let $\pi_{\mathrm{L}} = \{\{v_1\}, \{v_2\}, \ldots, \{v_m\}, \mathcal{V} \setminus \mathcal{V}_{\mathrm{L}}\}$. Note that $\mathrm{im}M \subseteq \mathrm{im}P(\pi_{\mathrm{L}})$. Define

$$\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}}) = \{\pi \mid \pi \in \Pi_{\mathrm{AEP}} \text{ and } \pi \leq \pi_{\mathrm{L}}\}. \tag{5}$$

The following theorem shows that each partition belonging to $\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ provides an upper bound for the controllable subspace $\mathcal{K}$.

Theorem 3: For any $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$, $\mathcal{K} \subseteq \mathrm{im}P(\pi)$.

Proof: It follows from Lemma 3 that $\mathrm{im}P(\pi)$ is L-invariant for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$. As noted before, $\mathrm{im}M \subseteq \mathrm{im}P(\pi_{\mathrm{L}})$. In view of (4), this means that $\mathrm{im}M \subseteq \mathrm{im}P(\pi)$ for each π with $\pi \leq \pi_{\mathrm{L}}$. In particular, we have $\mathrm{im}M \subseteq \mathrm{im}P(\pi)$ for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$. Therefore, the subspace $\mathrm{im}P(\pi)$ is L-invariant and contains $\mathrm{im}M$. Since the controllable subspace $\mathcal{K}$ is the smallest subspace with these properties, we get $\mathcal{K} \subseteq \mathrm{im}P(\pi)$ for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$. ■

Remark 1: Theorem 3 applies when there are multiple leaders. As such, it extends the similar result of [12, Prop. 2], (see also [21]) which deals with the single-leader case.

A natural question to ask is how to sharpen the upper bounds provided by Theorem 3. Obviously, the tightest bound which can be obtained by this theorem is given by

$$\mathcal{K} \subseteq \bigcap_{\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})} \mathrm{im}P(\pi).$$

However, this bound is not very practical as it requires the computation of all the almost equitable partitions which are finer than the partition π_{L}. The relation (4) suggests that one can provide an upper bound in terms of a partition which is maximal in a certain sense. More precisely, if one can show that there exists a partition $\pi^* \in \Pi_{\mathrm{AEP}}$ such that $\pi \leq \pi^*$ for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$, then one can conclude that

$$\mathcal{K} \subseteq \mathrm{im}P(\pi^*).$$

In [12] (see also [21]), such a bound is provided for the single-leader case without formally proving the existence of such a partition. In what follows, we investigate the structure of the set Π_{AEP} in detail and show that such a maximal partition exists and is unique. Furthermore, we will present an algorithm in order to compute this maximal partition. To do so, we need to introduce some notations.

Let Π denote the set of all the partitions of $\mathbb{G}$. With the partial order "$\leq$", the set Π becomes a complete lattice (see e.g., [22]) which means that every subset of Π has both its greatest lower bound and least upper bound within Π. We use $\vee\Pi'$ to denote the least upper bound of a subset Π'. By definition, the least upper bound $\vee\Pi'$ has the following property:

$$\pi \leq \tilde{\pi} \quad \text{for all } \pi \in \Pi' \quad \Longrightarrow \quad \vee\Pi' \leq \tilde{\pi}. \tag{6}$$

The complete lattice structure of Π readily implies that the set $\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ admits a unique least upper bound $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}}) = \vee\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ such that $\pi \leq \pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$. However, the least upper bound of a subset of Π does not need to belong to the subset in general. As such, one needs to further show that $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ belongs to $\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ in order to conclude that $\mathcal{K} \subseteq \mathrm{im}P(\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}}))$. To do so, we first state the following auxiliary lemma.

Lemma 3 [1, Lemma 1]: For any subset Π' of Π, it holds that $\bigcap_{\pi \in \Pi'} \mathrm{im}P(\pi) = \mathrm{im}P(\bigvee \Pi')$.

Based on this lemma, we are in a position to prove that the maximal partition $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ belongs to the set $\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$.

Lemma 4: It holds that $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}}) \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$.

Proof: It follows from (5) and (6) that $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}}) \leq \pi_{\mathrm{L}}$. Therefore, it remains to show that $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ is an almost equitable partition. To see this, note that

$$\mathrm{im}P(\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}})) = \mathrm{im}P(\bigvee \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})) = \bigcap_{\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})} \mathrm{im}P(\pi) \tag{7}$$

due to Lemma 3. In view of Lemma 3, $\mathrm{im}P(\pi)$ is L-invariant for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$. Since the intersection of L-invariant subspaces must be L-invariant too, it follows from (7) that $\mathrm{im}P(\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}}))$ is L-invariant. So one can conclude from Lemma 3 that $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ is an almost equitable partition. ■

Combining Theorem 3 with Lemma 4, we can state the following tightened bound for the controllable subspace.

Theorem 4: It holds that $\mathcal{K} \subseteq \mathrm{im}P(\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}}))$.

Remark 2: The bounds presented in Theorem 2 and Theorem 4 are tight for general graphs in the sense that one can construct graphs such that those bounds hold with equality. Consider the system (2) associated with the graph depicted on the left of Fig. 1. If agents 1 and 4

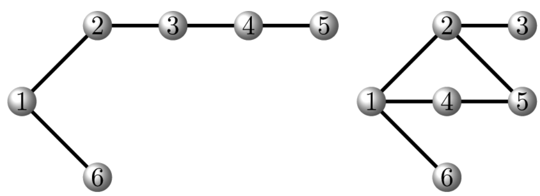

Fig. 1. Examples illustrating the tightness of the lower/upper bounds.

are chosen as leaders, then the lower bound holds with equality which is strictly less than the upper bound. If agents 1 and 3 are chosen as leaders, then the upper bound holds with equality which is strictly greater than the lower bound. For the system associated with the graph shown on the right of Fig. 1, if we choose agent 1 to be the single leader, neither of the two bounds is achieved. Moreover, one can check that there is no partition for which the image of its characteristic matrix is equal to the controllable subspace.

The bounds in Theorems 2 and 4 coincide for some specific graphs, for instance distance regular graphs with a single-leader as shown in [1].

The lower bound in Theorem 2 is easy to check since distance partitions can be obtained rather straightforwardly. However, the computation for the upper bound presented in Theorem 4 is not so straightforward since there are no algorithms to obtain almost equitable partitions with the constraint that multiple cells (corresponding to the leaders in our setting) have been strictly specified. In the next section, we develop an algorithm through which the least upper bound $\pi^*_{\rm AEP}(\pi_{\rm L})$ of the set $\Pi_{\rm AEP}(\pi_{\rm L})$ can be computed starting from a given partition $\pi_{\rm L}$.

IV. Algorithm to Compute $\pi^*_{\rm AEP}(\pi_{\rm L})$

To present the algorithm, we need to define a few concepts first. Let $\mathbb{R}^{n\times\bullet}$ denote all matrices with n rows. Let $\psi : \mathbb{R}^{n\times\bullet} \to \Pi$ be the mapping such that for any matrix $X \in \mathbb{R}^{n\times\bullet}$, it holds that i and j are in the same cell of $\psi(X)$ if and only if the ith and jth rows of the matrix X are the same. Note that

$$\pi = \psi(P(\pi)) \tag{8}$$

for any partition $\pi \in \Pi$.

Now we present an algorithm that computes $\pi^*_{\rm AEP}(\pi_{\rm L})$ starting from partition $\pi_{\rm L}$.

Theorem 5: Let $\pi_0 = \pi_{\rm L}$. Define the sequences

$$\pi_{k+1} = \psi([\,P(\pi_k) \quad LP(\pi_k)\,]) \tag{9}$$

where $k \geq 0$. Then, there exists an integer q with $0 \leq q \leq n-m$ such that $\pi_q = \pi^*_{\rm AEP}(\pi_{\rm L}) = \pi_{q+\ell}$ for all $\ell \geq 0$.

To prove this theorem, we will use the following auxiliary lemma.

Lemma 5: Let $X, Y \in \mathbb{R}^{n\times\bullet}$. The following statements hold.

1. $\mathrm{im}X \subseteq \mathrm{im}P(\psi(X))$.
2. $\mathrm{im}X \subseteq \mathrm{im}Y$ implies that $\mathrm{im}P(\psi(X)) \subseteq \mathrm{im}P(\psi(Y))$.

Proof:

1) : It follows from the definition of ψ that for each matrix $X \in \mathbb{R}^{n\times\bullet}$, there exists a matrix Z_X such that $X = P(\psi(X))Z_X$. Consequently, $\mathrm{im}X \subseteq \mathrm{im}P(\psi(X))$.

2) : In view of (4), it suffices to prove that $\psi(Y) \leq \psi(X)$. To do so, let i and j be such that the ith and jth rows of the matrix Y are the same. Since $\mathrm{im}X \subseteq \mathrm{im}Y$, there exists a matrix Z such that $X = YZ$. Then, the ith and jth rows of the matrix X must be the same. Therefore, it follows from the definition of ψ that any cell of $\psi(Y)$ is a subset of a cell of $\psi(X)$. In other words, $\psi(Y) \leq \psi(X)$. ■

Now we are ready to prove Theorem 5.

Proof of Theorem 5: Note that

$$\mathrm{im}P(\pi_k) \subseteq \mathrm{im}\,[\,P(\pi_k) \quad LP(\pi_k)\,].$$

Then, it follows from Lemma 5.2, (4) and (8) that $\pi_{k+1} = \psi([\,P(\pi_k) \quad LP(\pi_k)\,]) \leq \psi(P(\pi_k)) = \pi_k$. Therefore, we obtain

$$\pi_{k+1} \leq \pi_k \tag{10}$$

for all $k \geq 0$. Now, we claim that the implication

$$\pi_{r+1} = \pi_r \text{ for some } r \Rightarrow \pi_r = \pi_{r+\ell} \text{ for all } \ell \geq 0 \tag{11}$$

holds. To show this, note that $\pi_r = \psi([\,P(\pi_r) \quad LP(\pi_r)\,])$ if $\pi_{r+1} = \pi_r$. Then, it follows from Lemma 5.1 and (8) that $\mathrm{im}\,[\,P(\pi_r) \quad LP(\pi_r)\,] \subseteq \mathrm{im}P(\pi_r)$. This means that

$$\mathrm{im}\, LP(\pi_r) \subseteq \mathrm{im}\, P(\pi_r). \tag{12}$$

Since $\pi_{r+1} = \pi_r$, (12) implies that $\pi_{r+\ell} = \pi_r$ for all $\ell \geq 0$. Since $\mathrm{card}(\pi_{\rm L}) = m+1$ when $m < n$ and $\mathrm{card}(\pi_{\rm L}) = m$ when $m = n$, we get $\mathrm{card}(\pi_0) \geq m$. Then, (10) and the implication (11) imply that there exists an integer q with $0 \leq q \leq n-m+1$ such that $\pi_q = \pi_{q+\ell}$ for all $\ell \geq 0$. What remains to prove is that

$$\pi_q = \pi^*_{\rm AEP}(\pi_0). \tag{13}$$

From (12), we know that $P(\pi_q)$ is L-invariant. Then, π_q is an almost equitable partition due to Lemma 3. We also know from (10) that $\pi_q \leq \pi_0$. Therefore, $\pi_q \in \Pi_{\rm AEP}(\pi_0)$. This implies that

$$\pi_q \leq \pi^*_{\rm AEP}(\pi_0). \tag{14}$$

Now, we claim that

$$\pi^*_{\rm AEP}(\pi_0) \leq \pi_k \tag{15}$$

for each $k \geq 0$. We prove this claim by induction on k. When $k = 0$, (15) follows from the definition of $\pi^*_{\rm AEP}(\pi_0)$ that $\pi^*_{\rm AEP}(\pi_0) \leq \pi_0$. Now, assume that $\pi^*_{\rm AEP}(\pi_0) \leq \pi_k$ holds for some $k \geq 0$. It follows from (4) that $\mathrm{im}P(\pi_k) \subseteq \mathrm{im}P(\pi^*_{\rm AEP}(\pi_0))$ and from Lemma 3 that $L\mathrm{im}P(\pi_k) \subseteq \mathrm{im}P(\pi^*_{\rm AEP}(\pi_0))$. Therefore, $\mathrm{im}\,[\,P(\pi_k) \quad LP(\pi_k)\,] \subseteq \mathrm{im}P(\pi^*_{\rm AEP}(\pi_0))$. Then, we obtain from Lemma 5. 2 and (8) that $\mathrm{im}P(\pi_{k+1}) \subseteq \mathrm{im}P(\pi^*_{\rm AEP}(\pi_0))$. Hence, (4) yields that $\pi^*_{\rm AEP}(\pi_0) \leq \pi_{k+1}$. Consequently, (15) is proven. In particular, we can conclude that

$$\pi^*_{\rm AEP}(\pi_0) \leq \pi_q.$$

Together with (14), this implies that (13) holds. ■

To illustrate the algorithm by means of examples, we consider the diffusively coupled network (2) corresponding to the graph depicted in Fig. 2. Note that the Laplacian matrix is given by

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 6 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 4 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 4 \end{bmatrix}$$

for this graph. We employ the algorithm for three different leader sets. Figs. 3, 4, and Fig. 5 depict, respectively, the partitions obtained by the recursion (9) for the leader sets $\mathcal{V}_{\rm L} = \{1\}$, $\mathcal{V}_{\rm L} = \{4\}$, and $\mathcal{V}_{\rm L} =$

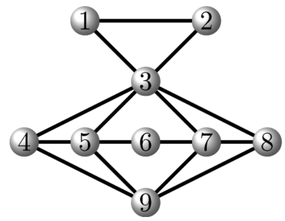

Fig. 2. Example for the algorithm.

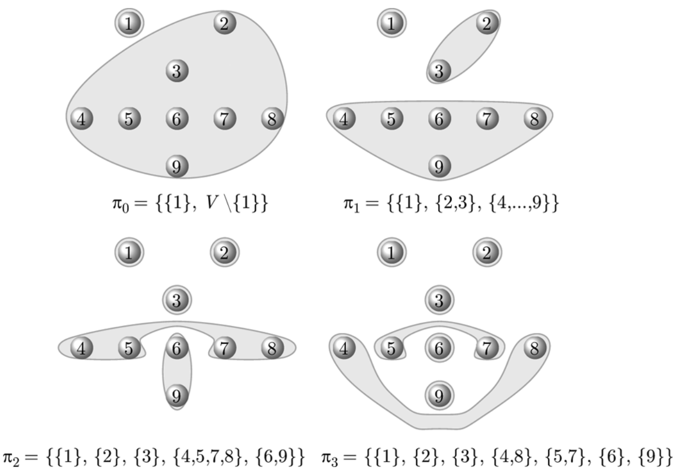

Fig. 3. Execution of the algorithm for $\mathcal{V}_{\rm L} = \{1\}$.

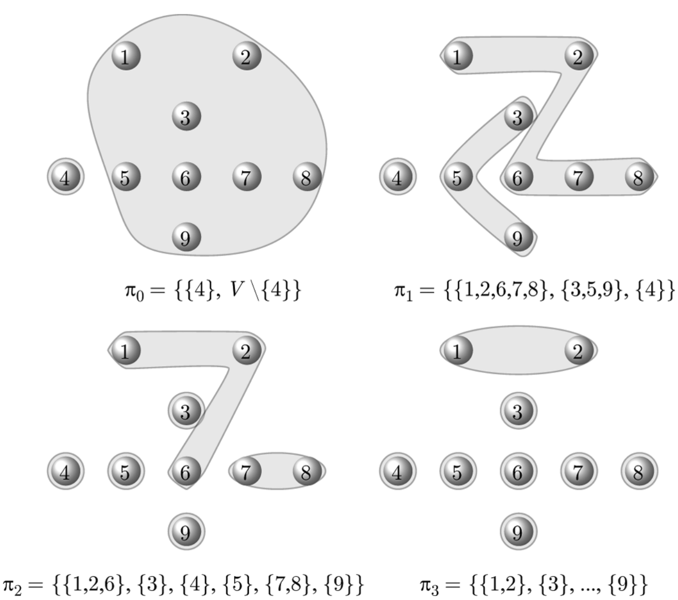

Fig. 4. Execution of the algorithm for $\mathcal{V}_{\rm L} = \{4\}$.

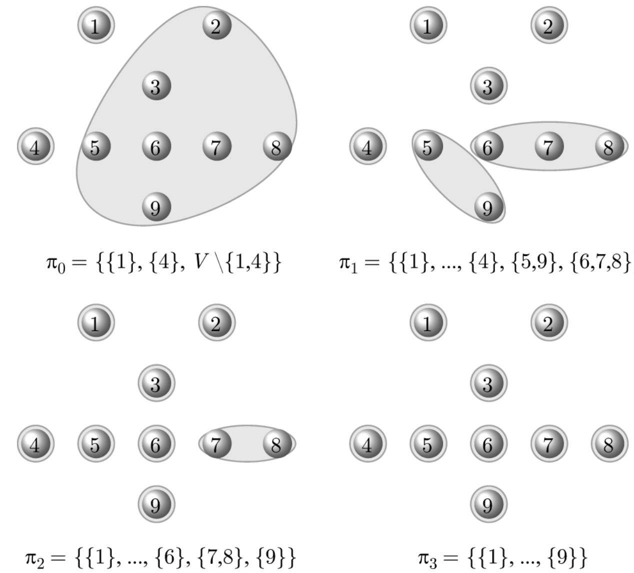

Fig. 5. Execution of the algorithm for $\mathcal{V}_{\rm L} = \{1, 4\}$.

$\{1, 4\}$. In all these cases, the last partition correspond to the maximal almost equitable partition $\pi^*_{\rm AEP}(\pi_{\rm L})$.

V. Concluding Remarks

We have studied controllability of networks of agents with linear dynamics. After investigating the effect of network topologies on controllability, we have focused on network with agents having single-integrator dynamics. For this case, we have presented a lower bound for controllable subspace in terms of the distance partitions and an upper bound in terms of the maximal almost equitable partitions. To compute the upper bound, we have provided an algorithm that finds the maximal almost equitable partition for given leaders.

As future research directions, we are interested in studying controllability of multi-agent networks when their associated graphs are directed and/or time-varying. Also, we envision that the use of ideas and notions of geometric control theory in the context of multi-agent systems would lead to graph topological interpretation of many other fundamental control theoretic problems.

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Output Synchronization of Networked Passive Systems With Event-Driven Communication

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***Abstract*—In this note, we study the output synchronization problem of networked passive systems with event-driven communication, in which the information exchange among the coupled agents are event-based rather than pre-scheduled periodically. We propose a setup for the interconnected agents to achieve output synchronization with event-driven communication in the presence of constant communication delays. The results presented here are important extensions of applying event-driven communication to control of multi-agent systems, especially when it is difficult to derive a common upper bound on the admissible network induced delays, or when the network induced delays between coupled agents are larger than the inter-event time implicitly determined by the event-triggering condition.**

***Index Terms*—Communication delay, control of multi-agent systems, event-driven communication, graph theory, output synchronization, passivity.**

I. Introduction

Several researchers have recently proposed event-based control as a promising approach to reduce the control related communication and computation load in many control applications. In a typical event-based implementation, the control signals are kept constant until the violation of an "event-triggering condition" on certain signals triggers the re-computation of the control actions. Compared with time-driven control, where constant sampling period is adopted to guarantee control system's stability in the worst-case scenario, the possibility of reducing the number of re-computations, and thus of transmissions, while guaranteeing desired levels of performance makes event-based control very appealing in Networked Control Systems (NCSs). A comparison of time-driven and event-driven control for stochastic systems favoring the latter can be found in [1]; a deterministic event-triggered control strategy is introduced in [2]; similar results on deterministic self-triggered feedback control have been reported in [3], [4]; output-based event-triggering control with guaranteed L_∞-gain for linear time-invariant systems has been studied in [5]; an event-triggered real-time scheduling approach for stabilization of passive and output feedback passive (OFP) systems has been proposed in [6], and extensions to more general dissipative systems with time-varying network induced delays have been reported in [7], [8] and [9]; event-triggering stabilization for distributed networked control systems has been studied in [10]; in [11], a self-triggered coordination strategy for optimal deployment of mobile robotics is proposed.

There has also been a growing interest in coordination and cooperative control of multi-agent systems supported by significant developments in the fields of communication and computation technologies during the past decade. Several results concerning distributed cooperative control strategies for multi-agent systems have appeared in the literature involving agreement or consensus algorithms [12]–[15], formation control and group coordination [16], [17], distributed estimation [18], to name a few. In addition to the design of distributed control strategies, there are issues regarding the implementation of the control algorithms, which have not received enough attention so far. Important aspects in the implementation of distributed algorithms for cooperative control of multi-agent systems include communication data transmission and control actuation update strategies. A lot of work in the literature assume that the control law processed in the distributed controller and the data transmissions scheduled between agents are implemented in a conservative way, where a tight upper bound is selected as the maximal allowable inter-execution time to guarantee stability of the multi-agent systems in all possible operational scenarios. This traditional time-driven control approach may lead to inefficient implementation of the distributed control algorithms in terms of processor usage or communication bandwidth. All of those issues mentioned above bring event-based control as a promising alternative approach because of its potential in reducing communication load and implementation cost for the purpose of control.

Most of the work on event-triggered control focuses on sensor-actuator NCSs, and there has not been enough emphasis on applying event-triggered control to the cooperative control of multi-agent systems, although some recent work on event-triggered consensus problems have been reported in [19] and [20]. However, a severe limitation of the proposed control strategy in [19] is the fact that each agent has to monitor the states of its neighboring agents continuously in order to evaluate the triggering condition. In [20], the authors further propose a distributed self-triggering cooperative control strategy for the consensus problem based on their previous work [19]. Although the authors have proved that the inter-event time implicitly determined by the triggering condition is positive, it is still difficult to derive a common lower bound on the inter-event time because the updates of the neighboring agents may make an agent satisfy its triggering condition imme-

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