Computation of the $H_\infty$-norm: an efficient method

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Abstract
An efficient algorithm to compute the $H_\infty$-norm of a real rational transfer matrix is presented. The new algorithm is then compared with the existing method using an example and the number of floating point operations necessary for the new method turns out to be drastically lower.

Keywords: H-infinity norm, rational transfer matrix, numerical efficiency, behaviors, dissipativity.

1 Introduction

We present an algorithm to compute the $H_\infty$-norm of a real rational transfer matrix. Currently, most algorithms to calculate the $H_\infty$-norm are search algorithms. In [3] a fast algorithm based on the relation between the singular values of the transfer matrix and the eigenvalues of a related Hamiltonian matrix is studied. In [1] a bisection algorithm is presented for repeatedly computing upper and lower bounds for the $H_\infty$-norm. An algorithm for searching the norm is also an issue in [2]. Matlab© uses an iterative algorithm to compute the $H_\infty$-norm. The algorithm discussed in this paper circumvents the kind of iteration inherent in such techniques. A similar idea is followed in [4]. Our algorithm differs slightly from the one in [4] and is also more general. We provide an example, where the algorithm in [4] fails. For this example, moreover, the floating point operations required by our algorithm is less than one-tenth of that required by the algorithm used in Matlab©.

The paper is structured as follows. Section 2 contains a brief review of the relation between $H_\infty$-norm and dissipativity, and the consequences for the computation of the $H_\infty$-norm. In sections 3 and 4 we discuss the main issues related to our algorithm. Section 5 contains a step-by-step method that can be utilized to compute the $H_\infty$-norm of a transfer matrix. In section 6 we consider an example to draw some comparisons, and finally in section 7 we briefly look into the implementation of the algorithm using the routines existing in packages. The remainder of this section is devoted to a brief review of the notation.

$\mathbb{R}$ is the field of real numbers and $\mathbb{C}$ is the complex plane. We often use $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, and $\mathbb{C}^\ast := \mathbb{C} \setminus \{0\}$. $\mathbb{R}[\xi]$ is the ring of polynomials in one indeterminate $\xi$ with real coefficients and $\mathbb{R}^{p \times m}[\xi]$ is the set of matrices with $p$ rows and $m$ columns in which each entry is an element.

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from this ring. When only the number of columns needs to be specified we use a • to avoid indicating the number of rows, i.e. \( \mathbb{R}^{\star \times m} \) is the set of matrices with \( m \) columns.

We shall often deal with real polynomials in two indeterminates \( \gamma \) and \( \omega \). This ring is denoted by \( \mathbb{R}[\gamma, \omega] \). Given \( g \in \mathbb{R}[\gamma, \omega] \) we sometimes shall consider the roots \( \omega_i \) of the polynomial in one indeterminate obtained from \( g \) by fixing the value of \( \gamma \). To emphasize this we write \( g_\gamma(\omega) \in \mathbb{R}[\gamma, \omega] \) instead, and speak of the roots of the polynomial \( g_\gamma \).

We shall sometimes need to refer to the degree of this polynomial with respect to one of these indeterminates; we then use the corresponding indeterminate in the subscript of ‘degree’. For example, \( \deg_\omega \) is the degree in \( \omega \) of a given polynomial in indeterminates \( \gamma \) and \( \omega \). Sometimes, the subscript shall also be used to emphasize the indeterminate in a one variable polynomial.

2 Dissipative systems and the \( \mathcal{H}_\infty \)-norm

Let \( G \in \mathbb{R}^{p \times m}(s) \) be a proper real rational transfer matrix and suppose it has no poles in the closed right half complex plane. Its \( \mathcal{H}_\infty \)-norm is defined as the supremum over the closed right half complex plane of the maximum singular value of \( G(s) \)

\[
\|G\|_{\mathcal{H}_\infty} := \sup_{\text{Re}(s) \geq 0} \sigma_{\max}(G(s)).
\]

It is well-known that, for the case at hand, it is enough to consider the \( \mathcal{L}_\infty \)-norm, i.e. \( \|G\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(\omega)) \). Let \( G(s) \) be given by a right-coprime factorization \( M_2(s)M_1^{-1}(s) \) with \( M_1, M_2 \in \mathbb{R}^{\star \times \star}[s] \).

We need some basic results from the behavioral theory to systems and control (see [10]). Given \( M_1 \) and \( M_2 \) as above, we define the behavior \( \mathfrak{B}(M_1, M_2) \) as follows

\[
\mathfrak{B}(M_1, M_2) := \{(u, y) \in C^\infty(\mathbb{R}, \mathbb{R}^{m+p}) \mid \exists \ell \in C^\infty(\mathbb{R}, \mathbb{R}^n) \text{ such that } u = M_1(\frac{d}{dt})\ell \text{ and } y = M_2(\frac{d}{dt})\ell \}.
\]

Define \( M \in \mathbb{R}^{(n+p) \times m}[s] \) by \( M := \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \). Thus \( \mathfrak{B}(M_1, M_2) \) is the image of the operator \( M(\frac{d}{dt}) \) when this operator acts on \( C^\infty(\mathbb{R}, \mathbb{R}^n) \). A detailed exposition on behavioral theory can be found in [8]. \( G \) having no poles on the imaginary axis is equivalent to \( M_1(s) \) having no zeros on \( i\mathbb{R} \). Define \( \Sigma_\gamma \in \mathbb{R}^{(n+p) \times (n+p)} \) by

\[
\Sigma_\gamma := \begin{pmatrix} \gamma I_n & 0 \\ 0 & -I_p \end{pmatrix}
\]

where \( I_n \) and \( I_p \) are identity matrices of sizes compatible with \( u \) and \( y \), respectively.

The following theorem yields alternative ways to describe the \( \mathcal{H}_\infty \)-norm:

**Theorem 1** With the above notation the following 3 statements are equivalent:

1. \( \|G\|_{\mathcal{H}_\infty} \leq \sqrt{T} \),
2. \( M^T(-i\omega)\Sigma_\gamma M(i\omega) \geq 0 \) for all \( \omega \in \mathbb{R} \), and
3. For all \( \omega \in \mathbb{R} \) and \( v \in \mathbb{C}^n \), \( \frac{\|M_2(i\omega)v\|^2}{\|M_1(i\omega)v\|^2} \leq \gamma \).

Proof: In [11] it is proven that the first two statements are equivalent to \( \Sigma_\gamma \)-dissipativeness of \( \mathfrak{B}(M_1,M_2) \). The equivalence of the last two statements is straightforward:

\[
\frac{\|M_2(i\omega)v\|^2}{\|M_1(i\omega)v\|^2} \leq \gamma \text{ for all } v \in \mathbb{C}^n \quad \iff \quad \|M_2(i\omega)v\|^2 \leq \gamma \|M_1(i\omega)v\|^2 \text{ for all } v \in \mathbb{C}^n \quad \iff \quad M^T(-i\omega)\Sigma_\gamma M(i\omega) \geq 0
\]

Associated with the second statement of the above theorem, we define the matrix \( H_\gamma(\omega) \in \mathbb{C}^{n \times n}[\gamma, \omega] \)

\[
H_\gamma(\omega) := \gamma M_1^T(-i\omega)M_1(i\omega) - M_2^T(-i\omega)M_2(i\omega) = M^T(-i\omega)\Sigma_\gamma M(i\omega)
\]

and the function \( \Gamma : \mathbb{R} \times \mathbb{C}^n \to \mathbb{R}^+ \)

\[
\Gamma(\omega, v) := \frac{\|M_2(i\omega)v\|^2}{\|M_1(i\omega)v\|^2}
\]

associated with the third statement. Since \( G \) is proper, the definition of \( \Gamma \) can be extended to \( \mathbb{R} \times \mathbb{C}^n \) by taking the limit for \( \omega \to \infty \).

According to theorem 1, to find the \( H_\infty \)-norm of \( G \), it is sufficient to compute the minimum \( \gamma \) such that \( H_\gamma(\omega) \geq 0 \) for all \( \omega \in \mathbb{R} \), or \( \Gamma(\omega, v) \leq \gamma \) for all \( \omega \in \mathbb{R} \) and \( v \in \mathbb{C}^n \). We denote this minimum by \( \bar{\gamma} \). Once we have found this minimum it is easy to compute the \( H_\infty \)-norm of the system, which is nothing but \( \sqrt{\bar{\gamma}} \). Thus the problem of computation of the \( H_\infty \)-norm is reformulated into a property of polynomial matrices.

3 Determination of \( \bar{\gamma} \)

As noted, we need to find the minimal \( \gamma \) such that \( H_\gamma(\omega) \) is non-negative for all \( \omega \in \mathbb{R} \), or \( \Gamma(\omega, v) \leq \gamma \) for all \( \omega \in \mathbb{R} \), \( v \in \mathbb{C}^n \). Clearly \( \Gamma(\omega, cv) = \Gamma(\omega, v) \) for all \( c \neq 0 \). So we can restrict \( v \) to \( S^{n-1} \), the unit sphere in \( \mathbb{C}^n \). Then \( \Gamma \) is a continuous function on the compact set \( \mathbb{R} \times S^{n-1} \), and hence it attains its maximum \( \bar{\gamma} \) at some point \( (\bar{\omega}, \bar{v}) \).

Let \( n \) be the degree of \( \det M_1(s) \), and let \( p_\gamma(\omega) \in \mathbb{R}[\gamma, \omega] \) be the determinant of \( H_\gamma(\omega) \). We write \( p_\gamma(\omega) \) as a polynomial in \( \omega \):

\[
p_\gamma(\omega) = a_0(\gamma) + \omega^2 a_2(\gamma) + \cdots + \omega^{2n-2} a_{2n-2}(\gamma) + \omega^{2n} a_{2n}(\gamma).
\]

Note that \( p_\gamma \) is an even polynomial in \( \omega \), since \( H_\gamma(-\omega) = H_\gamma^T(\omega) \). Further, for each \( \gamma > \bar{\gamma} \), \( p_\gamma(\omega) > 0 \) and \( p_\gamma(\omega) \geq 0 \) for all \( \omega \in \mathbb{R} \). Following is one of the main results of this paper.

Theorem 2 \( \|G\|_{H_\infty} = \sqrt{\bar{\gamma}} \). If \( \bar{\omega} = \infty \) then \( \bar{\gamma} \) is the largest root of \( a_{2n}(\gamma) \). If \( a_{2n}(\bar{\gamma}) \neq 0 \) then \( p_\gamma \) has a double root at \( \bar{\omega} \).
Proof: For the proof we refer to appendix A.

If \( a_{2n}(\bar{\gamma}) \neq 0 \), and hence \( \bar{\omega} < \infty \), then \( p_\gamma \) is nonnegative, but not identically zero. All its real roots, including the one at \( \bar{\omega} \), are of even multiplicity. It has at least a double root at \( \bar{\omega} \), so its derivative also has a root at \( \bar{\omega} \). Therefore, \( p_\gamma \) and \( q_\gamma \) are not coprime, with \( q_\gamma(\omega) \in \mathbb{R}[\gamma, \omega] \) defined as \( q_\gamma(\omega) := \frac{\partial p_\gamma}{\partial \omega}(\omega) \). The idea of our method is to compute all \( \gamma \)'s that cause \( p_\gamma \) and \( q_\gamma \) to lose coprimeness. In view of the above, \( \bar{\gamma} \) is in this set; in fact, it is the largest element in the set for which \( p_\gamma \) has a real root. (Note that for some \( \gamma \)'s, \( p_\gamma \) and \( q_\gamma \) could also have one or more common roots, but all in \( \mathbb{C} - \mathbb{R} \). These \( \gamma \)'s are not the interesting ones.)

Analogous to equation (3) where we wrote \( p_\gamma(\omega) \) as a polynomial in \( \omega \), we write \( q_\gamma(\omega) \) as follows

\[
q_\gamma(\omega) = \frac{\partial p_\gamma}{\partial \omega}(\gamma, \omega) = \omega b_1(\gamma) + \omega^3 b_3(\gamma) + \cdots + \omega^{2n-1} b_{2n-1}(\gamma)
\]

with \( b_i \in \mathbb{R}[\gamma] \). (Since \( p_\gamma \) is even, \( q_\gamma \) is odd and hence \( b_{2k} = 0 \) for all \( k \in \mathbb{Z}_+ \).) Note that \( b_{2k-1} = 2ka_{2k} \).

The way we proceed further is as follows, on the one hand find \( \gamma_\infty \), the largest real root of \( a_{2n}(\gamma) \). On the other hand determine the set of \( \gamma \)'s for which \( p_\gamma \) and \( q_\gamma \) are not coprime. Find \( \gamma_f \), the largest \( \gamma \) in this set for which \( p_\gamma \) has a real root. Then \( \bar{\gamma} = \max(\gamma_\infty, \gamma_f) \). Note that \( \gamma_\infty \geq 0 \).

4 Losing coprimeness

We use the standard result that under the condition that their leading terms are both nonzero, two polynomials are coprime if and only if their Sylvester resultant is nonsingular (see [9, chapter 12]). The Sylvester resultant \( S \in \mathbb{R}^{(4n-1) \times (4n-1)}[\gamma] \) of \( p \) and \( q \) is defined as follows

\[
S = \begin{pmatrix}
a_0 & a_1 & \cdots & a_{2n-2} & a_{2n-1} & a_{2n} & 0 & \cdots & 0 \\
0 & a_0 & \cdots & a_{2n-3} & a_{2n-2} & a_{2n-1} & a_{2n} & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & a_0 & a_1 & a_2 & \cdots & a_{2n-1} & a_{2n} \\
b_0 & b_1 & \cdots & b_{2n-2} & b_{2n-1} & 0 & 0 & \cdots & 0 \\
0 & b_0 & \cdots & b_{2n-3} & b_{2n-2} & b_{2n-1} & 0 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{2n-1} & 0 & \cdots \\
0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{2n-2} & b_{2n-1} & \cdots \\
\end{pmatrix}
\] \hspace{1cm} (4)

The determinant of this matrix is a polynomial in \( \gamma \): \( \det S \in \mathbb{R}[\gamma] \). There is a possibility that \( \det S \) is identically zero (i.e. the zero polynomial). We shall deal with this case after we analyze the situation when \( \det S \neq 0 \).

Suppose \( \det S \neq 0 \). This implies that for almost all \( \gamma \), \( p_\gamma \) and \( q_\gamma \) are coprime. They are not coprime precisely when \( \gamma \) is a root of \( \det S \), but not a root of \( a_{2n} \). (In the case that \( \gamma \) is a root of \( a_{2n} \), both leading coefficients would be zero.) We find the largest root \( \gamma_f \) of the polynomial \( \det S \) for which \( p_\gamma \) has a real root.
A relevant remark here is that $a_{2n}^2$ divides $\det S$. This can be seen as follows. Since $b_{2n-1} = 2na_{2n}$, the last column of $S$ is divisible by $a_{2n}$. Further, because $p_\gamma$ is even, $a_{2n-1} = 0$ and hence $b_{2n-2} = 0$ also. Hence the second last column of $S$ is also divisible by $a_{2n}$. This proves that $a_{2n}^2$ divides $\det S$. Thus $\gamma_\infty$ is already a root of $\det S$. So although to find $\gamma_f$ we have to disregard the zero’s of $a_{2n}$, in fact we are looking for the largest real $\gamma$ such that $\det S(\gamma) = 0$, and either $a_{2n}(\gamma) = 0$, or $p_\gamma$ has a real root.

Note that, similarly, $a_0$ divides $\det S$, because the last column of $S$ is divisible by $a_0$ (since $b_0 = 0$). When $\gamma$ equals a root of $a_0$ then 0 is a common root of $p_\gamma$ and $q_\gamma$. Thus the maximal real root of $a_0$ is also a lower bound for $\bar{\gamma}$. These lower bounds will be useful for a quick computation of $\bar{\gamma}$. The computation of the roots of $\det S$ can be simplified by dividing out $a_{2n}^2 a_0$, and computing the roots of a lower order polynomial.

We now come to the case that $\det S = 0$. In this case for all $\gamma$, $p_\gamma$ and $q_\gamma$ are not coprime. This happens when $p$ and $q$, now considered as polynomials in two variables, are not coprime (see appendix B).

We calculate the factors of $p$ and $q$ in $\mathbb{R}[\gamma, \omega]$ and call the greatest common factor $r \in \mathbb{R}[\gamma, \omega]$. From theorem 7 we conclude that for $\bar{p} := p/r$ and $\bar{q} := q/r$, the essential properties of $\bar{p}$ & $\bar{q}$ are similar to those of $p$ & $q$: $\bar{p}_\gamma$ has a double root at $\bar{\omega}$, and $\bar{q}_\gamma(\bar{\omega}) = 0$, and hence we can proceed with $\bar{p}$ and $\bar{q}$ instead.

Notice that the leading term of $\bar{q}$ is still an integer multiple of that of $\bar{p}$: $a_{2n} = \bar{a}_\ell r_\ell^\ell$ and $b_\ell r_\ell^\ell = 2na_{2n}$, so $\bar{b}_\ell = 2n\bar{a}_\ell$, and we can still divide $\det S$ by the same factor.

If we construct the Sylvester resultant $S$ as in equation (4) using $\bar{p}$ and $\bar{q}$, then we know that $\det S \neq 0$, since we divided out the greatest common factor. We go through the real roots of $\det S$ greater than $\gamma_\infty$, looking for the maximum value that results in $\bar{p}_\gamma$ and $\bar{q}_\gamma$ having a real common root. If there exists such a $\gamma$ then we take that for $\bar{\gamma}$. Otherwise, define $\bar{\gamma} := \gamma_\infty$.

It is possible to estimate the number of roots of $\det S$. As mentioned before, $p_\gamma$ is an even polynomial in $\omega$. The degree $\omega$ of $p_\gamma$ is twice the degree of $\det M_1$. When $M_2 M_1^{-1}$ is a right coprime factorization, then $\det M_1$ has degree equal to the McMillan degree of the system. The degree of $a_\ell$ is at most $m$, the number of inputs (and the size of $M_1$ and $H$). Hence the degree of $\det S$ is at most $m(4n - 1)$. This gives us an estimate of the order of $\det S$ before dividing $\det S$ by $a_{2n}^2 a_0$.

### 5 The algorithm

After having explained the main content of the procedure, we now come to a concise algorithm that takes in the transfer matrix as input and gives out the $\mathcal{H}_\infty$-norm as output.

**Data:** $G \in \mathbb{R}^{p \times n}(s)$, a real rational transfer matrix, such that $G$ is proper and has no poles in the closed right half plane.

**Output:** $\|G\|_{\mathcal{H}_\infty}$.

**Algorithm:**

1. Obtain a right coprime factorization of $G$ into $G(s) = M_2(s)M_1^{-1}(s)$ with $M_1, M_2 \in \mathbb{R}^{\bullet \times n}[s]$.
2. Compute $p_\gamma(\omega) := \det[\gamma M_1^T(-i\omega)M_1(i\omega) - M_2^T(-i\omega)M_2(i\omega)]$. 

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3. Let \( a_\ell \in \mathbb{R}[\gamma] \) denote the coefficient of the term with highest degree in \( \omega \). Denote the largest of the roots of \( a_\ell \) by \( \gamma_\infty \). (We shall see in lemma 5 that all roots of \( a_\ell \) are non-negative.)

4. Determine \( q_\gamma(\omega) := \frac{\partial}{\partial \omega} p_\gamma(\omega) \).

5. Find the greatest common factor of \( p_\gamma(\omega) \) and \( q_\gamma(\omega) \). Divide both by this factor to obtain \( \bar{p}_\gamma(\omega) \) and \( \bar{q}_\gamma(\omega) \), respectively.

6. Write \( \bar{p}_\gamma \) and \( \bar{q}_\gamma \) as follows
   \[
   \bar{p}_\gamma(\omega) = a_0(\gamma) + \omega^2 a_2(\gamma) + \cdots + \omega^{2n} a_{2n}(\gamma) \\
   \bar{q}_\gamma(\omega) = \omega b_1(\gamma) + \omega^3 b_3(\gamma) + \cdots + \omega^{2n-1} b_{2n-1}(\gamma) .
   \]
   with \( a_1, b_i \in \mathbb{R}[\gamma] \).

7. Compute the Sylvester resultant \( S \) as in equation (4).

8. Compute the roots of \( \det S \in \mathbb{R}[\gamma] \) and consider the ordered set \( T := \{ \tau_1 > \tau_2 > \ldots > \tau_m = \gamma_\infty \} \) of the real roots of \( \det S \) that are not less than \( \gamma_\infty \).

9. For \( k = 1 \) to \( m - 1 \), check if a common root of \( \bar{p}_\tau \) and \( \bar{q}_\tau \) is real (or alternatively, if \( \bar{p}_\tau \) has a real root). If yes, then \( \bar{\gamma} := \tau_k \).

10. Otherwise, \( \bar{\gamma} := \tau_m \).

11. \( \| G \|_{2\infty} := \sqrt{\bar{\gamma}} \).

6 An example

We demonstrate the algorithm using the following not-too-trivial and not-too-complicated example. Consider the transfer matrix
   \[
   G(s) = \frac{1}{5} \begin{pmatrix}
   \frac{3(s+1)}{s^2+s+1} & \frac{4(s+1)}{s^4+s^3+s^2+s+1} \\
   \frac{4(s+1)}{s^4+s^3+s^2+s+1} & -\frac{3(s+1)}{s^4+s^3+s^2+s+1}
   \end{pmatrix}.
   \]

\( p_\gamma(\omega) \) and \( q_\gamma(\omega) \) for this example are as follows
   \[
   p_\gamma(\omega) = 25 \left( \gamma (16 - 7 \omega^2 + \omega^4) - 1 - \omega^2 \right)^2 , \\
   q_\gamma(\omega) = 50 (\gamma (16 - 7 \omega^2 + \omega^4) - 1 - \omega^2)(-14\omega - 2\omega + 4\omega^3) \gamma .
   \]

In this case, \( \det S \) evaluated as in equation (4) turns out to be the zero polynomial. (This is the reason why the algorithm employed in [4] fails for this example.) We then divide out the greatest common divisor of \( p_\gamma \) and \( q_\gamma \) to obtain
   \[
   \bar{p}_\gamma = 16\gamma - 1 - (7\gamma + 1)\omega^2 + \gamma \omega^4 , \\
   \bar{q}_\gamma = -(28\gamma + 4)\omega + 8\gamma \omega^3 .
   \]

The roots of the determinant of the Sylvester resultant obtained using \( \bar{p}_\gamma \) and \( \bar{q}_\gamma \) turn out to be \( 9+\sqrt{49} \), \( \frac{1}{16} \), and \( 0 \). We now check at which among these values that are positive, \( \bar{p}_\gamma \) and \( \bar{q}_\gamma \) have a real common root. This happens at \( 9+\sqrt{49} \) and \( \frac{1}{16} \). We thus have \( \bar{\gamma} = 9+\sqrt{49} \) and \( \| G \|_{2\infty} = \sqrt{\bar{\gamma}} = 1.1194629 \). Using our method, the number of floating point operations (flops) utilized in Matlab\textsuperscript{®} for computing the \( H_\infty \)-norm is 4508, while the built-in routine within Matlab\textsuperscript{®} utilizes 179700 flops\textsuperscript{\textsuperscript{1}}. (The routine in Matlab\textsuperscript{®} is based on an iterative algorithm from [3] (see [7]).)

\textsuperscript{1}The command ‘flops’ is no longer in present Matlab versions. Also, the comparison of the two methods was made for the same precision of \( 10^{-6} \).
7 Conclusive remarks

We have presented and elaborated on an efficient method to compute the $\mathcal{H}_\infty$-norm of a transfer matrix. Implementation of this algorithm requires the computation of the Sylvester resultant of two multivariate polynomials with respect to a given variable. This computation is a standard routine in packages on symbolic mathematics (for example, Maple® and Mathematica®). These routines can be employed in the above algorithm. If the resultant is the zero polynomial we need to compute the common factor as explained in section 4. This can be done using Maple®.

We have finally considered a simple example where the numerical efficiency of our method is about forty times better than the current method.

References


A Proof of Theorem 2

In this appendix we prove theorem 2. We first study some properties of \( H \) as defined in equation (1)

\[
H_\gamma(\omega) := \gamma M_1^T(-i\omega)M_1(i\omega) - M_2^T(-i\omega)M_2(i\omega)
\]

with \( M_1 \) nonsingular in the closed right half plane. We also discuss the related functions \( p_\gamma(\omega) := \det H_\gamma(\omega) \) and \( \ell(\gamma, \omega, v) := v^*H_\gamma(\omega)v \) (5)

(where \( v^* \) denotes the complex conjugate transpose of \( v \in \mathbb{C}^{m*} \)) and \( \Gamma \) defined as in equation (2) by

\[
\Gamma(\omega, v) := v^*M_2^T(-i\omega)M_2(i\omega)v
\]

\[
\ell(\gamma, \omega, v) := v^*H_\gamma(\omega)v
\]

For a fixed \( \omega \in \mathbb{R} \) and \( v \in \mathbb{C}^{m*} \), \( \ell(\gamma, \omega, v) \) is strictly increasing in \( \gamma \), and if \( \gamma \gg 0 \) then \( \ell(\gamma, \omega, v) > 0 \). In fact, it is easy to see that \( \ell(\gamma, \omega, v) > 0 \) if and only if \( \gamma > \Gamma(\omega, v) \). The properness of \( G \) guarantees that \( \Gamma(\omega, v) \) remains finite when \( \omega \to \infty \). As remarked in section 3, \( \Gamma \) is a continuous function on the compact set \( \mathbb{R} \times \mathbb{S}^{m-1} \). It attains its maximum \( \bar{\gamma} \) for some value \( \bar{\omega} \) and vector \( \bar{v} \). Note that this is exactly the value \( \bar{\gamma} \) that we are looking for.

Lemma 3 Let \( \bar{\gamma}, \bar{\omega} \) and \( \bar{v} \) be as above. Then

1. \( \ell(\gamma, \omega, v) > 0 \) for all \( \gamma > \bar{\gamma} \),
2. \( \ell(\bar{\gamma}, \omega, v) \geq 0 \)
3. if \( \bar{\omega} \neq \infty \), then \( \ell(\bar{\gamma}, \bar{\omega}, \bar{v}) = 0 \) and \( \frac{\partial \ell}{\partial \omega}(\bar{\gamma}, \bar{\omega}, \bar{v}) = 0 \).
4. either \( \ell(\bar{\gamma}, \omega, \bar{v}) > 0 \) for all \( \omega \) in a small punctured neighborhood of \( \bar{\omega} \), or \( \ell(\bar{\gamma}, \omega, \bar{v}) = 0 \) for all \( \omega \).

Proof: The first two statements follow directly from the definition of \( \Gamma \) and the fact that \( \ell \) is increasing in \( \gamma \).

If \( \ell(\bar{\gamma}, \omega, \bar{v}) > 0 \), then clearly \( \bar{\gamma} > \Gamma(\bar{\omega}, \bar{v}) \), contradicting the fact that \( \Gamma \) attains its maximum \( \bar{\gamma} \) at \( (\bar{\omega}, \bar{v}) \). This proves that \( \ell(\bar{\gamma}, \bar{\omega}, \bar{v}) = 0 \). Now suppose \( \frac{\partial \ell}{\partial \omega}(\bar{\gamma}, \bar{\omega}, \bar{v}) \neq 0 \), then there exists an \( \omega \) close to \( \bar{\omega} \) such that \( \ell(\bar{\gamma}, \omega, \bar{v}) < 0 \), contradicting statement 2 of this lemma.

Since \( \ell \) is analytic in \( \omega \) the last statement follows. \( \square \)

Next we analyze \( p_\gamma \). Since \( H_\gamma(\omega) \) is positive (non-negative) for \( \gamma > (\geq) \bar{\gamma} \), \( p_\gamma \) has to be positive (nonnegative) for all \( \gamma > (\geq) \bar{\gamma} \). Further, if \( \bar{\omega} \neq \infty \) then \( p_\gamma(\bar{\omega}) = 0 \), since \( H(\bar{\gamma}, \bar{\omega}) \) is non-negative, and \( v^*H(\bar{\gamma}, \bar{\omega})v = 0 \). In addition, \( p_\gamma \) has at least a double root in \( \bar{\omega} \) (it could even be identically zero). This proves the following lemma.

Lemma 4 If \( \bar{\omega} \neq \infty \) then \( p_\gamma \) has a double root at \( \bar{\omega} \).
We now explore some properties of the roots of $a_{2n}$ the leading coefficient of $p_\gamma$. These turn out to be related to the singular values of a certain constant matrix. Consider $D \in \mathbb{R}^{p \times m}$ defined as

$$D := \lim_{s \to \infty} (M_2(s)M_1^{-1}(s)).$$

($D$ exists because $G$ is assumed to be proper.) Consider the singular values of $D$: $\sigma_i$, $i = 1, \cdots, \min(m, p)$. We claim that $a_{2n}(\gamma)$ has degree unequal to $m$ and that $\sigma_i^2$ are roots of $a_{2n}(\gamma)$ counted with multiplicity; and if $m > p$ (i.e. if $D$ is wide), then the remaining $m - p$ roots of $a_{2n}(\gamma)$ are zero. Under the assumption that $\det(M_1(s))$ is monic, this is equivalent to claiming $a_{2n}(\gamma) = \det(\gamma I_m - D^T D)$. This is shown in the following lemma.

**Lemma 5** Let $p_\gamma(\omega) \in \mathbb{R}[\gamma, \omega]$ be the determinant of $H_\gamma(\omega)$ as defined above in equation (1). Suppose $a_{2n}(\gamma)$ is the leading coefficient of $p_\gamma(\omega)$. Then $a_{2n}(\gamma) = \det(I_m - D^T D)$.

**Proof:** Notice that $(M_1^T)^{-1}(-i\omega)H_\gamma(\omega)M_1^{-1}(i\omega) = \gamma I_m - G^T(-i\omega)G(i\omega)$. Further, since $G$ is proper, we can write this as follows

$$\gamma I_m - G^T(-i\omega)G(i\omega) = \gamma I_m - D^T D + \frac{1}{\omega} P(\omega)$$

with $P(\omega) \in \mathbb{C}^{n \times m}(\omega)$ a proper complex rational matrix. This implies that

$$\det(\gamma I_m - G^T(-i\omega)G(i\omega)) = \det(\gamma I_m - D^T D) + \frac{1}{\omega} h_\gamma(\omega),$$

with $h_\gamma(\omega)$ such that it is polynomial in $\gamma$ and proper rational in $\omega$.

Further,

$$\det[(M_1^T)^{-1}(-i\omega)H_\gamma(\omega)M_1^{-1}(i\omega)] = \frac{\det(M_1^T(-i\omega))(\det H_\gamma(\omega))(\det M_1(i\omega))^{-1}}{\det(M_1^T(-i\omega)M_1(i\omega))}.$$  

This implies that

$$p_\gamma(\omega) = (\det(M_1^T(-i\omega)M_1(i\omega))) \det(\gamma I_m - D^T D) + \text{terms with degree}_\omega \text{ strictly less than } 2n.$$  

Since we assumed that $M_1(s)$ is such that $\det M_1(s)$ is monic, we obtain that $a_{2n}(\gamma)$ is indeed equal to $\det(\gamma I_m - D^T D)$. This completes the proof. \[\square\]

**Lemma 6** Let $p_\gamma(\omega) = a_{2n}(\gamma)\omega^{2n} + \cdots + a_0(\gamma)$, and let $\bar{\omega} = \infty$, then $\bar{\gamma} = \max \{ \gamma \in \mathbb{R} \mid a_{2n}(\gamma) = 0 \}$.

**Proof:** Let $\bar{\omega} = \infty$, then $\bar{\gamma} = \lim_{\omega \to \infty} \Gamma(\omega, \bar{\omega})$.

Consider $z(\omega, v) := M_1(i\omega)v/\|M_1(i\omega)v\|$ with $v \in \mathbb{C}^n$; then $\Gamma(\omega, v) = \|G(i\omega)z(\omega, v)\|^2$. Since $M_1(i\omega)$ is invertible for all $\omega \in \mathbb{R}$, $z$ can take any value on the unit sphere $S$, and hence for a fixed $\omega$ the maximum of $\Gamma(\omega, v)$ over all $v$ equals the square of the induced 2-norm of $G(i\omega)$. Taking the limit for $\omega \to \infty$ yields the same for $\bar{\omega}$. Clearly $\bar{\gamma}$ is this maximum, and therefore
\[ \gamma \] is equal to the square of the largest singular value of \( G(\infty) \), which by lemma 5, is the largest root of \( a_n \).

The implication of the above lemma is well-known: the maximum singular value of \( D \) is a lower bound for \( \|G\|_{\infty} \) (see [1, 2, 3], for example). Using the above lemmas, we easily prove theorem 2.

**Theorem 2** If \( \bar{\omega} = \infty \) then \( \bar{\gamma} \) is the largest root of \( a_{2n}(\gamma) \), otherwise \( p_{\gamma} \) has a double root at \( \bar{\omega} \).

**Proof:** The first statement is proved in lemma 6, and the second in lemma 4.

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**B  Some properties of polynomials in two indeterminates**

The ring \( \mathbb{R}[\gamma, \omega] \) is a unique factorization domain. A subset \( P \) of polynomials determines an algebraic variety \( V(P) = \{ z \in \mathbb{C}^2 \mid p(z) = 0 \text{ for all } p \in P \} \). The irreducible components of \( V(P) \) correspond uniquely to sets of irreducible factors of the elements of \( P \) (see [5]).

Now suppose \( P = \{p, q\} \). Then the irreducible components of \( V(P) \) are either one or zero dimensional. If they are one dimensional then \( p \) and \( q \) must have a common factor. So, if \( p_{\gamma} \) and \( q_{\gamma} \) are not coprime for all \( \gamma \), meaning that \( V(P) \) has a one dimensional component, then \( p \) and \( q \) have a common factor.

We now concentrate on the situation in section 4, where \( q = \frac{\partial p}{\partial \omega} \) and \( r \) is the greatest common factor of \( p \) and \( q \).

**Theorem 7** Let \( p_{\gamma} \) be positive for all \( \gamma > \bar{\gamma} \), and let \( \bar{\omega} \) be a real root of \( p_{\bar{\gamma}} \). Then \( r_{\gamma} \) is either positive (or negative) for all \( \gamma > \bar{\gamma} \), and if \( \bar{p} = p/r \) and \( \bar{q} = q/r \), then \( \bar{p} \) is positive (or negative) for all \( \gamma > \bar{\gamma} \). Further, multiplicities of \( \bar{\omega} \) as a root of \( \bar{p}_{\bar{\gamma}} \) and \( \bar{q}_{\bar{\gamma}} \) are even and odd, respectively.

**Proof:** Suppose that for some \( \gamma \), \( r_{\gamma} \) is neither positive nor negative for all \( \omega \in \mathbb{R} \), then it has a zero somewhere, and hence \( p_{\gamma} \) has a zero, too. This proves the first and the second statements. For the final statements we assume that \( r_{\gamma}(\bar{\omega}) = 0 \), because there is nothing to prove otherwise. Since for \( \gamma > \bar{\gamma} \), \( p_{\gamma} \) and \( r_{\gamma} \) are positive (without loss of generality), \( p_{\bar{\gamma}} \) and \( r_{\bar{\gamma}} \) have \( \bar{\omega} \) as a root of even multiplicity. This immediately proves that multiplicity of \( \bar{\omega} \) as a root of \( \bar{p}_{\bar{\gamma}} \) has to also be even. \( q_{\bar{\gamma}} \), being the derivative of \( p_{\bar{\gamma}} \), has \( \bar{\omega} \) as a root of odd multiplicity. Hence, it follows that multiplicity of \( \bar{\omega} \) as a root of \( \bar{q}_{\bar{\gamma}} \) also is odd.

The above theorem thus allows us to use just \( \bar{p} \) and \( \bar{q} \) to construct the Sylvester resultant and eventually compute \( \bar{\gamma} \), since at \( \bar{\gamma} \) no additional important information is contained in \( r \).