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Sphalerons in the Standard Model

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1. Overview

Context of this talk: **high-energy physics**.

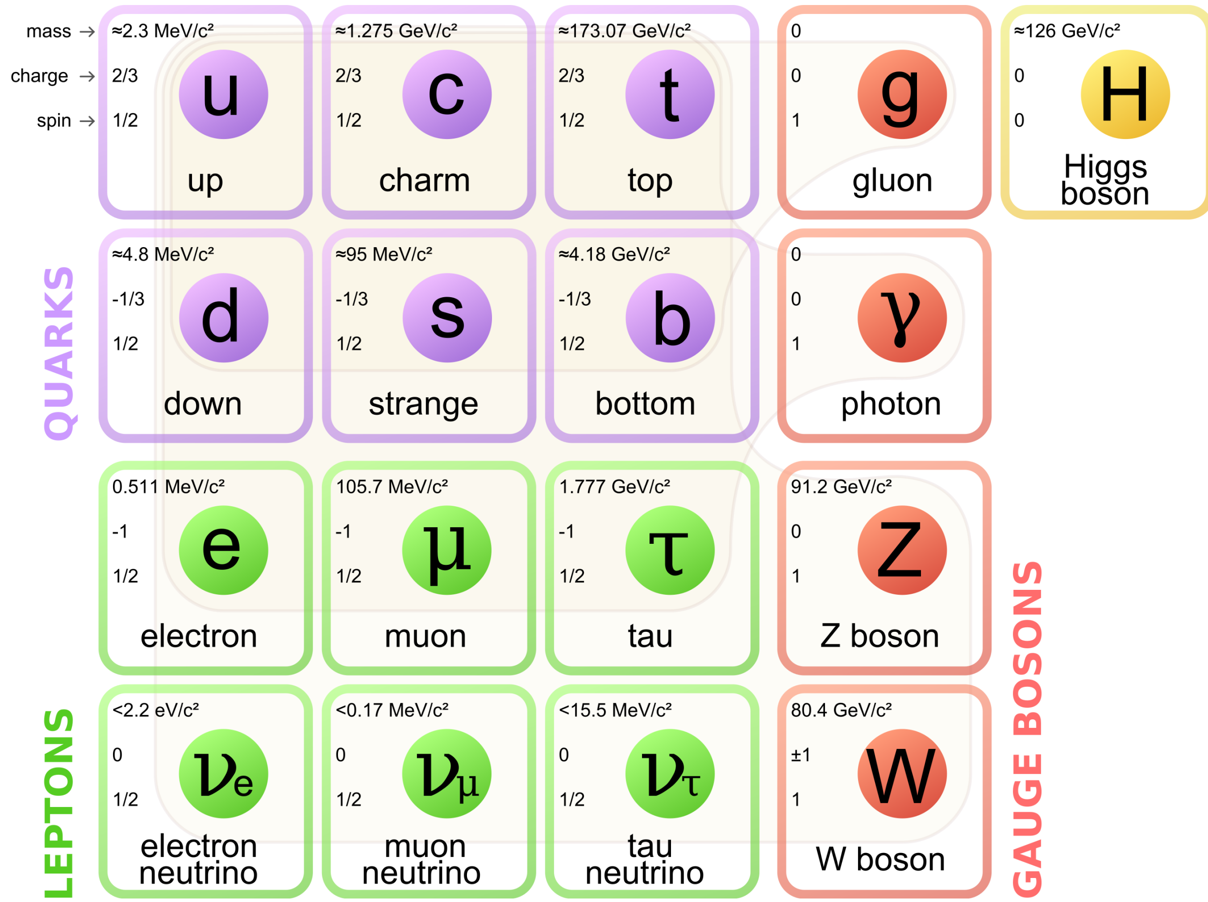
Now, **all** of our current knowledge of high-energy physics is contained in the so-called **Standard Model** (SM).

Incomplete list of SM founding fathers:

. . . , Yang and Mills, 1954; Glashow, 1961;
Englert and Brout, 1964; Higgs, 1964; Guralnik, Hagen, and Kibble, 1964;
Fadde'ev and Popov, 1967;
Weinberg, 1967; Salam, 1968; Glashow, Iliopoulos, and Maiani, 1970;
't Hooft, 1971; 't Hooft and Veltman, 1972;
Weinberg, 1973; Fritzsch, Gell-Mann, and Leutwyler, 1973;
Gross and Wilczek, 1973; Politzer, 1973; . . .

1. Overview

SM:



Elementary particles of the SM [`https://commons.wikimedia.org/wiki`].

1. Overview

But there is more to the SM than particles and Feynman diagrams.

In the $SU(3)$ Yang–Mills theory of the QCD sector of the SM:

the **instanton I** [Belavin, Polyakov, Schwartz, and Tyupkin, 1975].

In the $SU(2) \times U(1)$ Yang–Mills–Higgs theory of the electroweak SM:

the **sphaleron S** [Klinkhamer and Manton, 1984].

1. Overview

Terminology:

an "instanton" is a localized, **finite-action** solution of the classical field equations for imaginary time τ ($\tau^2 \leq 0$);

a "soliton" is a static, **stable**, **finite-energy** solution of the classical field equations for real time t ($t^2 \geq 0$);

a "sphaleron" is a static, **unstable**, **finite-energy** solution of the classical field equations for real time t.

1. Overview

Generally speaking, instantons (and solitons) are relevant to the **equilibrium** properties of the theory, whereas sphalerons are relevant to the **dynamics**.

Specifically, the two types of nonperturbative solutions of the SM are relevant to the following physical effects:

instantons for the gluon condensate and the η' mass,

sphalerons for the origin of the cosmic matter–antimatter asymmetry.

1. Overview

OUTLINE:

1. Overview
2. SU(2) x U(1) sphaleron S
3. Spectral flow and anomalies
4. SU(3) sphaleron $\widehat{S}$
5. Conclusion
6. References

2.0 S – General remarks

How to discover nonperturbative solutions, such as the instanton I or the sphaleron S?

Well, just follow this recipe:

1. make an appropriate *Ansatz* for the fields;
2. solve the resulting reduced field equations.

Of course, the subtlety in getting the “appropriate” *Ansatz* of step 1.

Here, topological insights have played a role.

2.1 SU(2) x U(1) sphaleron S

The electroweak Standard Model (EWSM), with $\sin^2\theta_w \approx 0.23$ and $m_H \approx 125$ GeV, has, most likely, no topological solitons but does have two sphalerons, S [1] and S* [2]. The extended $SU(3)$ theory also has a third sphaleron, $\widehat{\text{S}}$ [3].

The solution S is the best known [1, 4] and its energy is numerically equal to

$$E_S \sim 10 \text{ TeV} \,,$$

and parametrically equal to

$$E_S \sim v/g \sim M_W/\alpha \,,$$

with the Higgs vacuum expectation value v, the $SU(2)$ coupling constant g, the mass $M_W = \frac{1}{2}\, g\, v$ of the charged vector bosons $W^\pm$, and the fine-structure constant $\alpha = e^2/(4\pi) = g^2\,\sin^2\theta_w/(4\pi)$.

2.1 SU(2) x U(1) sphaleron S

In simple terms, the sphaleron solution S of the EWSM

- is a slightly elongated blob of field energy with size of order $1/M_W \sim 10^{-2}$ fm and energy density of order $(1/\alpha)\, M_W^4$;
- has “tangled” fields (hence, the existence of fermion zero modes; see the discussion on spectral flow below) ;
- corresponds to an unstable configuration of fields, which, after a small perturbation, decays to the vacuum by emission of many particles (number of order $1/\alpha \sim 100$) .

But how does S fit in configuration space?

2.1 SU(2) x U(1) sphaleron S

One particular slice of configuration space (more details later):

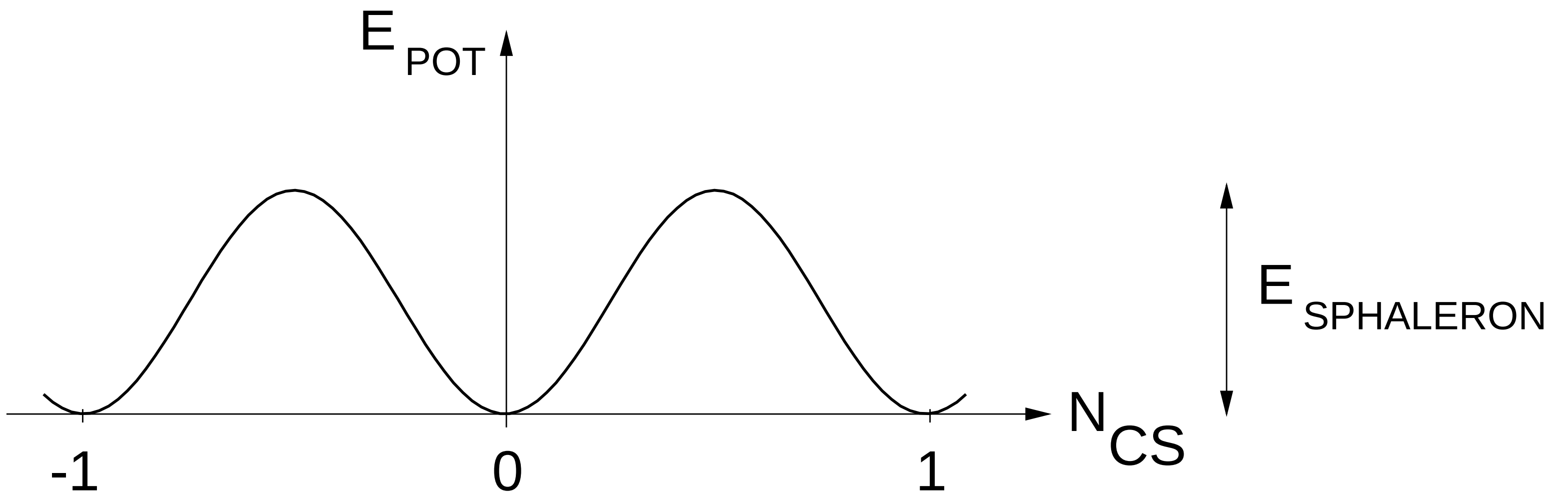


Figure 1: Potential energy over a slice of configuration space.

Small oscillations near the vacuum with $N_{\text{CS}} = 0$ (or any other integer) correspond to the SM elementary particles on the chart of p. 3.

The nontrivial structure of Fig. 1 is directly relevant to the main physics application of the sphaleron S, namely electroweak baryon number violation, to which we turn now.

2.2 Electroweak baryon number violation

Conditions for cosmological baryogenesis [Sakharov, 1967]:

1. C and CP violation	Yes	(SM)
2. Thermal nonequilibrium	Yes	(FRW)
3. Baryon number (B) violation	?	

Strictly speaking, only one established theory is expected to have B violation:

the electroweak Standard Model (EWSM).

[Side remark: the *ultimate* fate of black holes is uncertain and, hence, it is not known if black-hole physics violates baryon number conservation or not.]

2.2 Electroweak baryon number violation

The relevant physical processes of the EWSM at

$$T \ll M_W \approx 10^2 \text{ GeV} \,,$$

have a rate (tunneling **through** the barrier of Fig. 1) which is negligible [7],

$$\Gamma^{\text{(tunneling)}} \propto \exp[\,-2\,\mathcal{S}_{\text{BPST}}\,/\,\hbar\,] = \exp[\,-4\,\pi\,\sin^2\theta_w\,/\,\alpha\,] \approx e^{-400} \approx 0\,,$$

with an exponent given by twice the action of the BPST instanton.

For $T \gtrsim 10^2$ GeV, the rate (thermal excitation **over** the barrier of Fig. 1) contains a Boltzmann factor [1],

$$\Gamma^{\text{(thermal)}} \propto \exp[\,-E_S\,/\,(k\,T)\,]\,,$$

in terms of the barrier height, the sphaleron energy E_S.

Note the respective factors of $\hbar$ and k in the two rates Γ: different physics!

2.2 Electroweak baryon number violation

Clearly, we should study electroweak baryon number violation (EWBNV) for the conditions of the early universe,

$$T \gtrsim 10^2 \text{ GeV} .$$

This is a difficult problem, but entirely well-posed. Obviously, we must really deal with the fermions [7–13].

The following sections are, however, rather technical and will be skipped for the moment:

2.3 EWBNV – Classic result

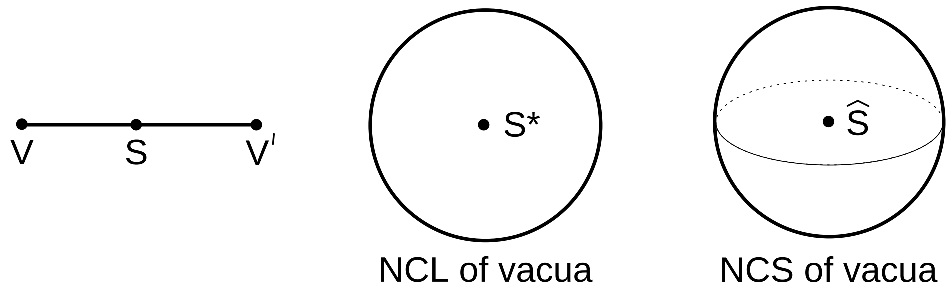
2.4 EWBNV – Open question

2.5 EWBNV – Partial answer

3. Spectral flow and anomalies

Three sphalerons (S, S^*, and $\widehat{S}$) are relevant to the SM, each related to having a **nontrivial vacuum structure**.

Different parts of configuration space look like a line segment, a disk, and and a ball, with vacuum fields on their boundaries:



where NCL stands for noncontractible loop and NCS for noncontractible sphere.

Note that the V–S–V′ line segment above corresponds to a slice of configuration space. Identifying V and V′ gives a circle, unwrapping it gives the real line, and then evaluating the corresponding field energies gives the sine-square structure of Fig. 1.

3. Spectral flow and anomalies

A useful **diagnostic** over configuration space can be obtained from the eigenvalue equation of the time-dependent Dirac Hamiltonian:

$$H(\vec{x},t)\,\Psi(\vec{x},t) = E(t)\,\Psi(\vec{x},t)\ ,$$

where H is a <u>functional</u> of the background gauge field $\vec{A}(\vec{x},t)$.

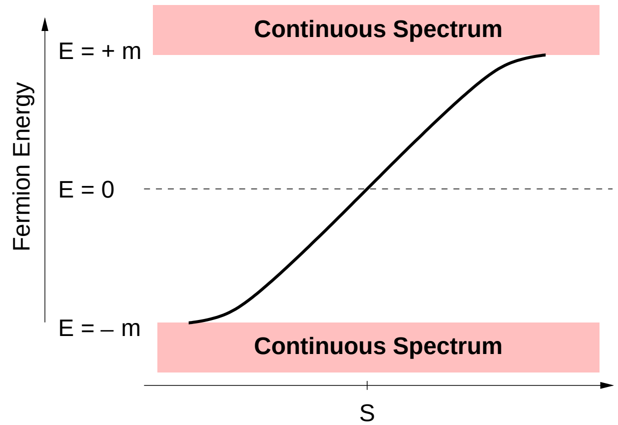
Then, fermion number violation via the sphaleron S is related to the **spectral flow** $\mathcal{F}$. See, e.g., Refs. [8, 13].

<u>Definition</u>:

$\mathcal{F}[\,t_f, t_i\,]$ is the number of eigenvalues of the Dirac Hamiltonian that cross zero from below minus the number of eigenvalues that cross zero from above, for the time interval $[\,t_i, t_f\,]$ with $t_i < t_f$.

3. Spectral flow and anomalies

All three sphalerons are related to a generalized form of **spectral flow** (with fermion masses from the Higgs field). The picture for S is well known (cone-like for S^* and $\widehat{S}$):



In turn, these sphalerons are associated with **anomalies**:

S with the chiral $U(1)$ anomaly [Adler–Bell–Jackiw, 1969],

S^* with the chiral nonperturbative $SU(2)$ anomaly [Witten, 1982],

$\widehat{S}$ with the chiral non-Abelian anomaly [Bardeen, 1969].

3. Spectral flow and anomalies

The sphalerons are then relevant to the following physical processes:

S to B+L violation for the matter-antimatter asymmetry in the early universe,

S* to multiparticle production in high-energy scattering with $\sqrt{s} \geq E_{S^*}$,

$\widehat{\text{S}}$ to nonperturbative dynamics of QCD.

The physics application of S is well known, even though far from being understood completely (as discussed before, but in the skipped parts...).

For the rest of the talk, let us focus on $\widehat{\text{S}}$, which has an interesting mathematical structure but a less clear physics application.

4.0 S-hat – Preliminary remarks

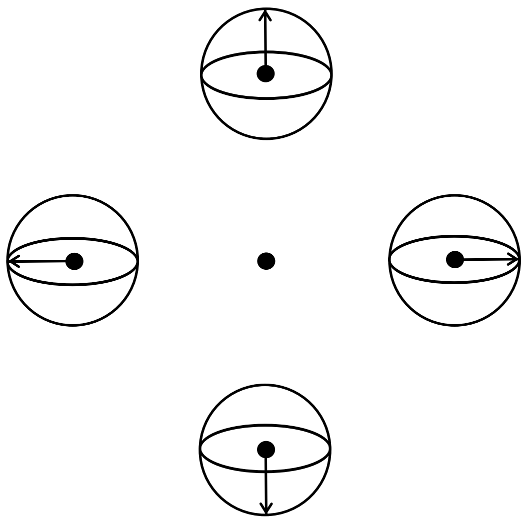
Before discussing the $SU(3)$ sphaleron $\widehat{\text{S}}$, recall three basic facts of S.

First, the $SU(2)$ sphaleron S can be embedded in $SU(3)$ YMH theory [strictly speaking, the embedded solution is the $SU(2) \times U(1)$ sphaleron].

Second, the $SU(2)$ gauge and Higgs fields of S are determined by two radial functions $f(r)$ and $h(r)$.

Third, the $SU(2)$ sphaleron S has a so-called **hedgehog structure**, i.e., a topologically nontrivial map $S_3^{\text{(space)}} \to SU_2^{\text{(internal)}} = S_3^{\text{(internal)}}$.

Here a sketch of $S_2^{\text{(space)}} \to S_2^{\text{(internal)}}$:



4.1 SU(3) sphaleron S-hat

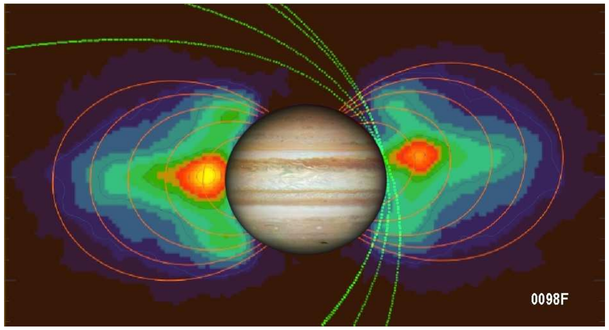
Now turn to $\widehat{\text{S}}$, which is very different.

First, $\widehat{\text{S}}$ exists in $SU(3)$ YMH but not in $SU(2)$ YMH theory.

Second, the self-consistent *Ansatz* of $\widehat{\text{S}}$ requires eight axial functions for the gauge field and three axial functions for the fundamental Higgs field.

Third, $\widehat{\text{S}}$ does not have a hedgehog structure but a **Jupiter-like structure**:

for a given half-plane through the symmetry–axis with azimuthal angle ϕ, the parallel components A_r and A_θ involve only one particular $su(2)$ subalgebra of $su(3)$, whereas the orthogonal component A_ϕ excites precisely the other five generators of $su(3)$.



4.1 SU(3) sphaleron S-hat

As to the reduced field equations, they are very difficult to solve, even numerically.

Still it is possible to obtain an upper bound on the energy [3]:

$$E_{\widehat{\mathsf{S}}}\Big|_{\lambda/g^2=0} < 1.72 \times E_{SU(2)-S}\,, \tag{2}$$

with $E_{SU(2)-S} \equiv 1.52 \times 4\pi v/g$ and λ the quartic Higgs coupling constant.

After several years of work, the numerical solution of the reduced field equations has been obtained recently [K & Nagel, 2016] and the numerical value for the energy is:

$$E_{\widehat{\mathsf{S}}}\Big|_{\lambda/g^2=0} = (1.160 \pm 0.005) \times E_{SU(2)-S}\,. \tag{3}$$

4.1 SU(3) sphaleron S-hat

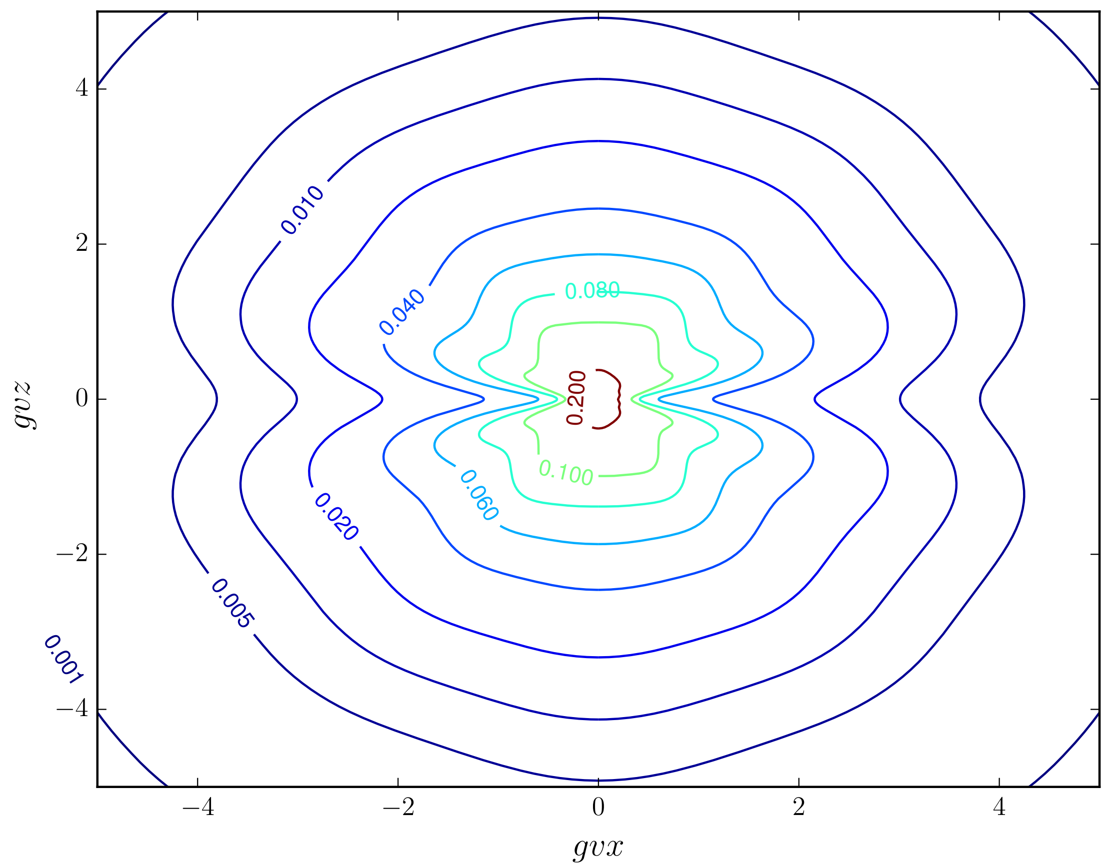


Figure 2: $\widehat{\mathrm{S}}$ energy-density contours (in units of $g^2\, v^4$) for $\lambda/g^2 = 0$.

4.1 SU(3) sphaleron S-hat

Mathematically, it is remarkable that the energy of $\widehat{\mathrm{S}}$ with eight gauge fields is close to that of S with only four gauge fields. Most likely, this is due to the highly-ordered (Jupiter-like) structure mentioned earlier.

Physically, it is important that the $\widehat{\mathrm{S}}$ barrier is low, as it implies that related processes are little suppressed at high energies/temperatures.

For the QCD version of $\widehat{\mathrm{S}}$, the energy scale would be set by quantum effects, $\Lambda \sim 100$ MeV.

5. Conclusion

The mathematical physics of the sphaleron solutions is relatively straightforward. Really difficult are the physics applications.

Let us mention three **outstanding puzzles** related to the three sphalerons S, S*, and $\widehat{\text{S}}$:

First, how does the B+L violation proceed microscopically at high energies or high temperatures (the scale being set by $E_S \sim 10$ TeV) and what is the proper selection rule?

Second, does EWSM multiparticle production in high-energy scattering with parton center-of-mass energy $\sqrt{s} \sim E_{S^*} \sim 20$ TeV reach the unitarity limit?

Third, do the $\widehat{\text{S}}$ gauge fields produce new physical effects in QCD?

6. References

[1] F.R. Klinkhamer & N.S. Manton, PRD 30, 2212 (1984).
[2] F.R. Klinkhamer, NPB 407, 88 (1993), arXiv:hep-ph/9306208.
[3] F.R. Klinkhamer & C. Rupp, NPB 709, 171 (2005), arXiv:hep-th/0410195.
[4] J. Kunz, B. Kleihaus, and Y. Brihaye, PRD 46, 3587 (1992).
[5] G. Aad *et al.* [ATLAS Collab.], PLB 716, 1 (2012), arXiv:1207.7214.
[6] S. Chatrchyan *et al.* [CMS Collab.], PLB 716, 30 (2012), arXiv:1207.7235.
[7] G. 't Hooft, PRL 37, 8 (1976).
[8] N.H. Christ, PRD 21, 1591 (1980).
[9] T.M. Gould & S.D.H. Hsu, NPB 446, 35 (1995), arXiv:hep-ph/9410407.
[10] V.V. Khoze, NPB 445, 270 (1995), arXiv:hep-ph/9502342.
[11] F.R. Klinkhamer & Y.J. Lee, PRD 64, 065024 (2001), arXiv:hep-th/0104096.
[12] F.R. Klinkhamer, in: *XIV-th Rencontre de Blois*, arXiv:hep-ph/0209227.
[13] F.R. Klinkhamer & C. Rupp, JMP 44, 3619 (2003), arXiv:hep-th/0304167.
[14] S.L. Adler, PR 177, 2426 (1969).
[15] J.S. Bell & R. Jackiw, Nuovo Cimento A 60, 47 (1969).
[16] W.A. Bardeen, PR 184,1848(1969).
[17] E. Witten, PLB 117, 324 (1982).

2.3 EWBNV – Classic result (skipped)

Consider $SU(2)$ Yang–Mills–Higgs theory with vanishing Yukawa couplings. Actually, forget about the Higgs, which may be reasonable above the EW phase transition.

Triangle anomaly in the AAA-diagram, provided the VVV-diagram is anomaly-free [14, 15].

The gauge vertices of the EWSM are V–A and must be nonanomalous (gauge invariance is needed for unitarity). Then, the $B + L$ current becomes anomalous [7]:

$$\begin{aligned} \Delta(B-L) &= 0\,, \\ \underbrace{\Delta(B+L)}_{\text{change of fermion number}} &= \underbrace{2\,N_{\text{fam}}}_{\text{integer}} \times \underbrace{\Delta N_{\text{CS}}}_{\text{gauge field characteristic}}\,. \end{aligned}$$

2.3 EWBNV – Classic result (skipped)

In the $A_0 = 0$ gauge, one has the Chern–Simons number

$$N_{\mathsf{CS}}(t) = N_{\mathsf{CS}}[\vec{A}(\vec{x}, t)]$$

and

$$\Delta N_{\mathsf{CS}} \equiv N_{\mathsf{CS}}(t_{\mathsf{out}}) - N_{\mathsf{CS}}(t_{\mathsf{in}}) \ .$$

For the record (using differential forms and the Yang–Mills field strength 2-form $F \equiv dA + A^2$), we have

$$N_{\mathsf{CS}}[A] \equiv \frac{1}{8\pi^2} \int_{M_3} \left(A dA + \frac{2}{3} A^3 \right) = \frac{1}{8\pi^2} \int_{M_3} \left(AF - \frac{1}{3} A^3 \right) \ .$$

2.3 EWBNV – Classic result (skipped)

't Hooft [7] calculated the **tunneling** amplitude using the BPST instanton. This BPST instanton, which is a finite action solution over Euclidean spacetime (imaginary-time theory), gives

$$\Delta N_{\mathsf{CS}} = Q[A_{\text{finite action}}] \in \mathbb{Z}\ ,$$

where the topological charge Q is the **winding number** of the map

$$S^3\big|_{|x|=\infty} \to SU(2) \sim S^3\ .$$

This holds only for transitions from near-vacuum to near-vacuum, i.e., at very low temperatures or energies. As mentioned above, the rate is then effectively zero, but, at least, $\Delta(B+L)$ is an **integer**, namely $2\,N_{\mathsf{fam}} \times \Delta N_{\mathsf{CS}}$.

2.4 EWBNV – Open question (skipped)

For **real-time** processes at nonzero energies or temperarures, the topological charge Q is, in general, **noninteger**.

Hence, the question [12]

$$\Delta(B+L) \propto \text{which gauge field characteristic ?}$$

In the following, we consider pure $SU(2)$ Yang–Mills theory with a single isodoublet of left-handed fermions.

(The fermion number $B+L$ of the EWSM follows by multiplying with $2\,N_{\mathsf{fam}}$. Recall that $B-L$ remains conserved in the EWSM.)

Furthermore, the gauge fields will be called dissipative if their energy density approaches zero uniformly as $t \to \pm\infty$.

2.5 EWBNV – Partial answer (skipped)

Spectral flow was already defined in Sec. 3. Here a sketch for the NCL through the sphaleron S:

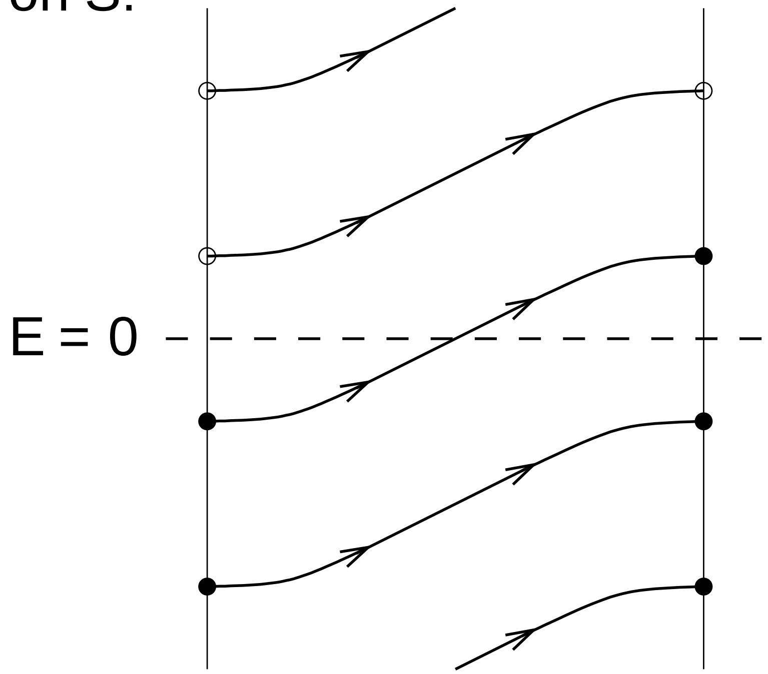


Figure 3: Spectral flow with $\mathcal{F}[t_f, t_i] = +1 - 0 = +1$. Filling the (infinite) Dirac sea at the initial time t_i results in one extra fermion at the final time t_f.

2.5 EWBNV – Partial answer (skipped)

Strongly-dissipative* SU(2) gauge fields at finite energy have [8, 9, 10]:

$$\mathcal{F} \;=\; \Delta N_{\text{CS}}[A_{\text{associated vacuum}}] \equiv \Delta N_{\text{winding}} \in \mathbb{Z}\,.$$

Now, there exist three weakly-dissipative,* spherically symmetric gauge field solutions [Lüscher & Schechter, 1977] with

1. (low energy) $\Delta N_{\text{winding}} = 0$ and $\mathcal{F} = 0\,,$
2. (moderate energy) $\Delta N_{\text{winding}} = 1$ and $\mathcal{F} = 1\,,$
3. (high energy) $\Delta N_{\text{winding}} = 1$ and $\mathcal{F} = -1\,.$

$$\Rightarrow \;\left[\mathcal{F} \neq \Delta N_{\text{winding}}\right]_{\text{spherically symmetric fields}}\,.$$

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* For the precise definition of strongly/weakly-dissipative, see [11].

2.5 EWBNV – Partial answer (skipped)

In fact, there is another gauge field characteristic [11]:

$$\Delta N_{\text{twist}} = 0 \qquad \text{for case 1 and 2}\,,$$

$$\Delta N_{\text{twist}} = -2 \qquad \text{for case 3}\,.$$

$$\Rightarrow \; \left[\, \mathcal{F} = \Delta N_{\text{winding}} + \Delta N_{\text{twist}} \,\right]_{\text{spherically symmetric fields}} \,.$$

For **weakly-dissipative or nondissipative** gauge fields, one has thus

$$\Delta(B+L) \;\; = \;\; 2\, N_{\text{fam}} \times \Big(\Delta N_{\text{CS}} \left[A_{\,\text{associated vacuum}} \right] + \underline{\text{extra terms}} \Big) .$$

But the “extra terms” are not known in general [12].

In short, the microphysics of EWBNV is not fully understood.