Quantum speed limit vs. classical displacement energy

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What is this lecture about?

Symplectic topology (Conley, Zehnder, Gromov, Eliashberg, Floer, Hofer,..., 1980 – ...), discovered surprising rigidity phenomena involving symplectic manifolds, their subsets, and their diffeomorphisms.

Question: What are quantum footprints of symplectic rigidity?

Difficulty: Quantum-classical correspondence is not sharp (H.J. Groenewold, 1946)

Today's story: Quantum counterpart of symplectic displacement energy, a fundamental symplectic invariant (Hofer, 1990)

Fidelity

H- finite dimensional complex Hilbert space
$\mathcal{L}(H)$- Hermitian operators, observables
$\mathcal{S}(H)$ - quantum states, $\rho \in \mathcal{L}(H)$, $\rho \geq 0$, trace$(\rho) = 1$.

Fidelity: $\theta, \sigma \in \mathcal{S}(H)$
$\Phi(\theta, \sigma) = \|\sqrt{\theta}\sqrt{\sigma}\|_{tr}$.
Measures overlap between quantum states.

Example: For pure states $\xi, \eta \in H$, $|\xi| = |\eta| = 1$,
$\Phi(\xi, \eta) = |\langle \xi, \eta \rangle|$.

Dislocation

$F_t \in \mathcal{L}(H)$ - quantum Hamiltonian.
Schrödinger equation
$\dot{U}_t = -\frac{i}{\hbar} F_t U_t$,
$U_t : H \to H$ unitary evolution, $U_0 = \mathbb{1}$, $U_1 = U$.

Quantum Hamiltonian F_t *a*-dislocates a state $\theta \in \mathcal{S}$ if
$\Phi(\theta, U\theta U^{-1}) \leq a$, $a \in [0, 1)$.

Appears e.g. in quantum computation. Margolus-Levitin (1998) address the question about *"the maximum number of distinct [i.e., non-overlapping] states that the system can pass through, per unit of time. For a classical computer, this would correspond to the maximum number of operations per second."*

Quantum speed limit

The total energy of the quantum evolution is given by $\hbar^{-1}\ell_q(F)$, $\ell_q(F) := \int_0^1 \|F_t\|_{op} dt$.

Quantum speed limit: universal bound on the energy required to a-dislocate a quantum state:

$$\Phi(a, U\theta U^{-1}) \leq a \ \Rightarrow \ \ell_q(F) \geq \arccos(a)\hbar$$

Mandelstam-Tamm, 1945 "time-energy uncertainty", Uhlmann (1992) Margolus-Levitin, 1998, Anderson-Heidari, 2014

Quantum speed limit

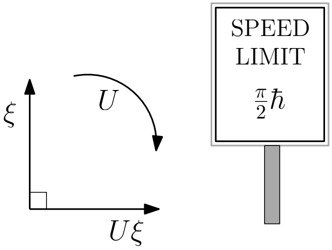


Figure: “Displacing” a pure quantum state

We explore semiclassical dislocation of semiclassical states.

Mathematical model of classical mechanics

(M^{2n}, ω)–symplectic manifold (e.g, 2-sphere)
ω– **symplectic form**. Locally $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$.
M-phase space of mechanical system.

Energy determines evolution: $f : M \times [0,1] \to \mathbb{R}$ – Hamiltonian function (energy). Hamiltonian system:

$$\begin{cases} \dot{q} = \frac{\partial f}{\partial p} \\ \dot{p} = -\frac{\partial f}{\partial q} \end{cases}$$

Family of **Hamiltonian diffeomorphisms**

$$\varphi_t : M \to M, \;\; (p(0), q(0)) \mapsto (p(t), q(t))$$

Ham-group of Hamiltonian diffeomorphisms
Key feature: $\varphi_t^* \omega = \omega$.

$X \subset M$ **displaceable** if $\exists\ \varphi \in Ham\colon\ \varphi X \cap X = \emptyset$ (Hofer, 1990)

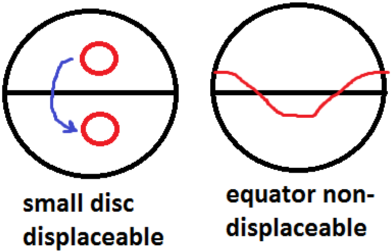


Figure: (Non)-displaceability on the 2-sphere

Displacement energy

Let f_t, $t \in [0, 1]$ be classical Hamiltonian generating Hamiltonian diffeomorphism φ. Total energy

$\ell_c(f) = \int_0^1 ||f_t|| dt$, where $||g|| := \max |g|$-uniform norm.

Dispacement energy of a displaceable subset $X \subset M$
$e(X) := \inf \ell_c(f)$ over all displacing Hamiltonians f.

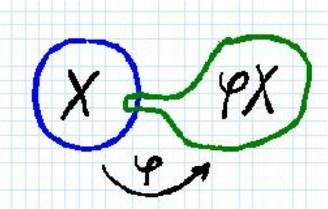
Fundamental invariant in modern symplectic topology. Yields biinvariant metric geometry of *Ham*.

Rigidity: $e(X) > 0$ for all open X
$e(B) \sim r^2$ for ball of radius r^2.
Hofer (1990), Viterbo, P., Lalonde-McDuff

Counterpoint: If $\mathrm{Vol}(X) < \frac{1}{2} \cdot \mathrm{Vol}(M)$, for all $\epsilon > 0, \delta > 0$ there exists f_t such that

$\mathrm{Vol}(\varphi X \cap X) < \epsilon$, $\ell_c(f) < \delta$.

Based on Katok's lemma, 1970, Ostrover-Wagner, 2005.



No measure-theoretic symplectic rigidity

Quantization

Think of Berezin-Toeplitz quantization.
(M, ω)- closed Kähler manifold, quantizable: $[\omega]/(2\pi) \in H^2(M, \mathbb{Z})$
(e.g., $M = S^2$ of area 2π).

$H_\hbar$-sequence of complex Hilbert spaces, $\hbar \to 0$, $\dim H_\hbar \to \infty$.

Toeplitz operators $T_\hbar : C^\infty(M) \to \mathcal{L}(H_\hbar)$,

$$T_\hbar(f) = \int_M f(x) R_\hbar(x) P_{x,\hbar} d\mathrm{Vol}(x) \ ,$$

where $R_\hbar : M \to \mathbb{R}$ - (Rawnsley) function ,
$P_{x,\hbar} : H_\hbar \to H_\hbar$ - coherent state projectors .

For classical state τ (probability measure on M) define

$$Q_\hbar(\tau) = \int_M P_{x,\hbar} d\tau(x) \in \mathcal{S}(H_\hbar)$$

"classical" quantum state, Giraud-Braun-Braun 2008

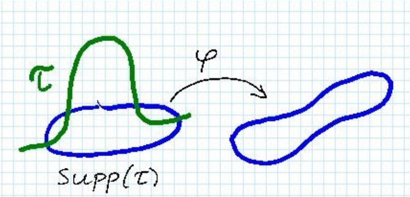
Displacement yields dislocation

f_t-classical Hamiltonian, $t \in [0,1]$, τ-classical state.
$F_t = T_\hbar(f_t)$- quantum Hamiltonian, $\theta = Q_\hbar(\tau)$ - quantum state.

Theorem (Charles-P., 2016)

If f_t displaces supp$(\tau) \Rightarrow F_t$ $O(\hbar^\infty)$-dislocates θ.

Figure: φ-time-one-map of the flow of f_t



Dislocation yields displacement

Assume τ has smooth density, $f_{t,\hbar}$ depends on $\hbar$ and bounded with 4 derivatives, $\dim M = 2n$.

Theorem (Charles-P., 2016, sketch)

If $F_{t,\hbar}$ $o(\hbar^n)$-dislocates $\theta \Rightarrow f_{t,\hbar}$ displaces $supp(\tau)$ and

$$\ell_q(F_{t,\hbar}) \geq e(supp(\tau)) \; .$$

Conclusion: Speed limit becomes more restrictive ~ 1 than the universal bound $\sim \hbar$.

Uses positivity of displacement energy e (symplectic rigidity) and sharp remainder estimates for Berezin-Toeplitz quantization (Charles-P., 2015).

Rigidity vs. flexibility on quantum side

Theorem (Charles-P., 2016)

Assume $Vol(supp(\tau)) < \frac{1}{2} \cdot Vol(M)$. Then $\forall \epsilon, \delta > 0$ there exists f_t such that F_t ϵ-dislocates θ and $\ell_q(F_t) < \delta$.

Conclusion: Competition between rigidity ($\ell_q > e$) vs. flexibility ($\ell_q < \delta$) is governed by the rate of dislocation.

	RIGIDITY	FLEXIBILITY
RATE OF DISLOCATION	$o(\hbar^n)$	ϵ

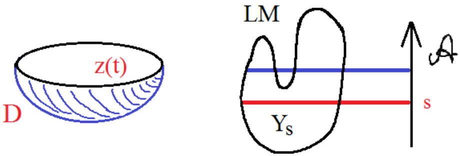
Remarks

Zooming into small scales: Theorems 1,2 extend to dislocation of semiclassical states which "occupy" a ball of radius $\hbar^{\varepsilon}$, $\varepsilon \in [0, 1/2)$ in the phase space. The speed limit on such a scale is $\sim \hbar^{2\varepsilon}$ which, again, is more restrictive than the universal quantum speed limit $\sim \hbar$.

Other mechanisms of dislocation: Implication dislocation $\Rightarrow$ displacement is specific for "classical" quantum states $Q_{\hbar}(\tau)$. There are other mechanisms of semiclassical dislocation which do not involve displacement, i.e. for Lagrangian states.

Displacement energy via Floer theory

LM- space of contractible loops $z : S^1 \to M$
Action functional: $\mathcal{A}(z) : LM \to \mathbb{R},\ z \mapsto \int_0^1 f(z(t))dt - \int_D \omega$
D-disc spanning z, $f : M \times S^1 \to \mathbb{R}$ – Hamiltonian



Gradient flow: nonlinear Cauchy-Riemann eq. (Gromov, Floer)
Critical points: 1-periodic orbits of Hamilt. flow. Responsible for topological changes of sublevels $Y_s = \{\mathcal{A} < s\} \subset LM$ as s varies. (Viterbo, Schwarz, Oh)

Critical values yield bounds for displacement energy

Quantum indeterminism $\leftrightarrow$ **symplectic quasi-states** (Entov-P.,2006), positive functionals linear on (Poisson) commutative subalgebras of $C(M)$ but not on the whole space. Gleason's theorem (1957): $\dim H \geq 3 \Rightarrow$ every quantum quasi-state linear, cf. discussion on hidden variables in quantum mechanics (also,Groenewold, 1946).
Anti-Gleason phenomenon in classical mechanics: quasi-states coming from Floer theory.

Rigidity of Poisson brackets $\leftrightarrow$ **noise production** in phase-space registration of a quantum particle (P., 2014, Charles-P., 2016)

Symplectic capacities, monotone invariants based on periodic orbits $\leftrightarrow$ **Gutzwiller type trace formula** (A. Uribe, 2016); in progress Charles, Le Floch, P., Uribe