Semi-Classical approach in Quantum Field Theories coupled to Strong Sources

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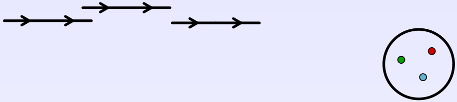
High Energy Scattering in QCD

- What happens when two protons/nuclei collide at high energy?
- Can it be calculated from first principles ?
  (i.e. using Quantum-Chromodynamics)

A proton contains more than three quarks...



- ▷ at low energy, mostly three valence quarks

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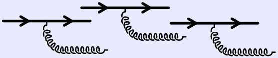

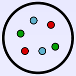

▷ when energy increases, additional gluons are present

Note : these gluons come from $\mathrm{q} \to \mathrm{q} + \mathrm{g}$ quantum fluctuations, and appear long-lived in the observer's frame due to Lorentz time dilation

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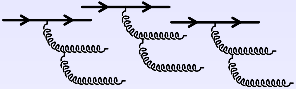
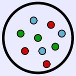
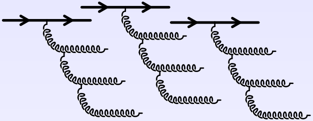
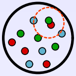




▷ as long as the density of constituents remains small, the evolution is linear: the number of partons produced at a given step is proportional to the number of partons at the previous step

A proton contains more than three quarks...





▷ eventually, the partons start overlapping in phase-space

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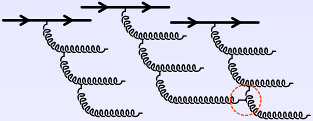
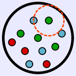


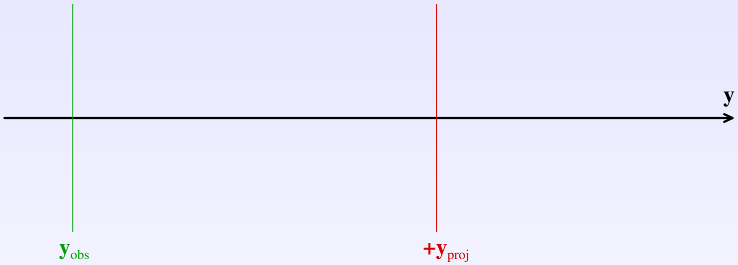


▷ parton recombination becomes favorable

▷ after this point, the evolution is non-linear:

the number of partons created at a given step depends non-linearly on the number of partons present previously

Degrees of freedom at various rapidities ($y \sim \ln(p_z)$)



McLerran-Venugopalan model :

- Fast partons : frozen dynamics, negligible $p_\perp$ $\Rightarrow$ classical current
- Slow partons : evolve with time $\Rightarrow$ gauge fields

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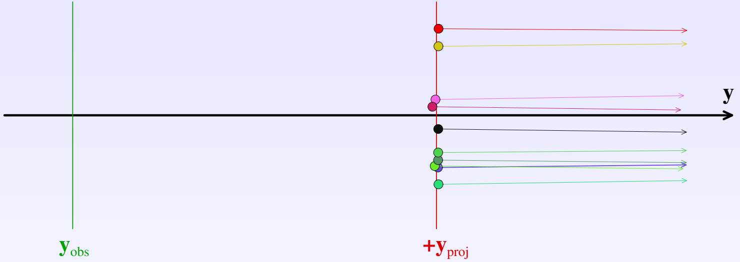

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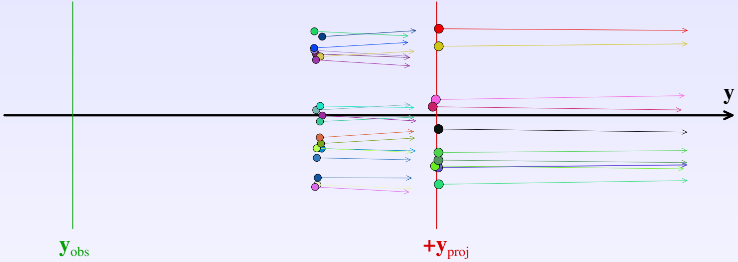

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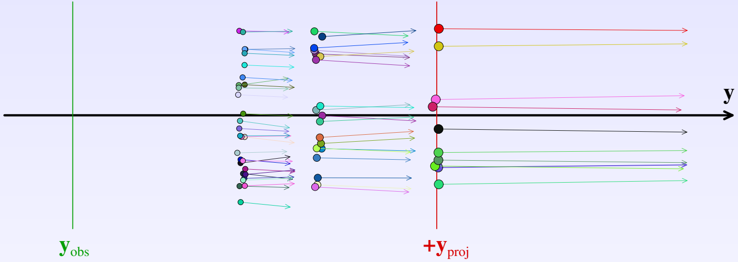

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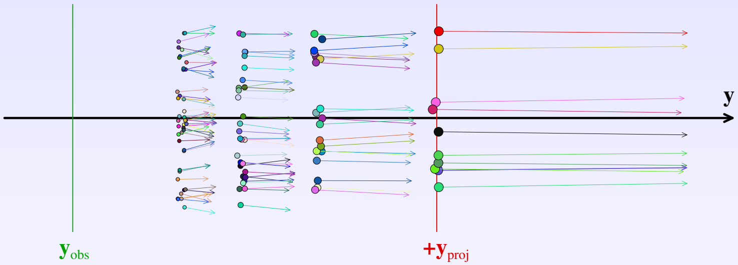

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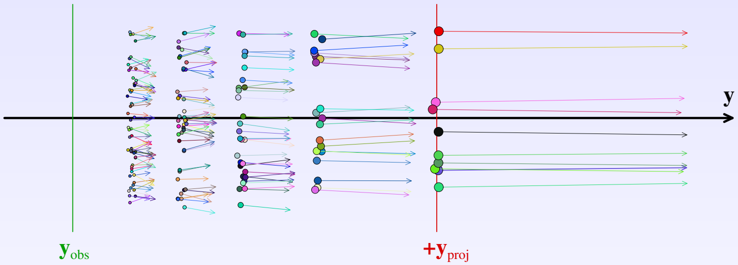

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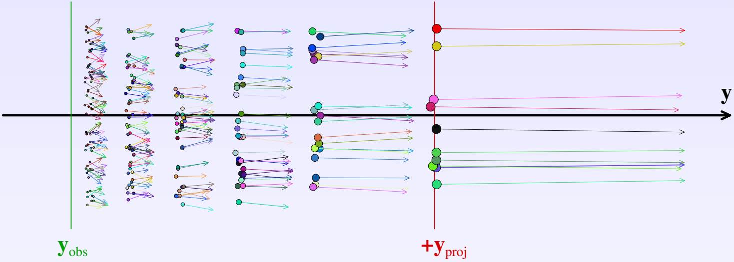

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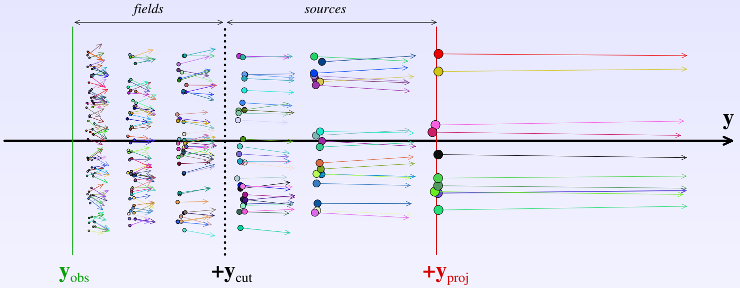

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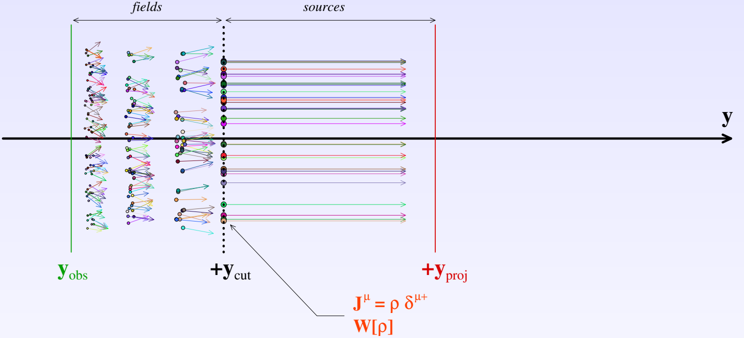

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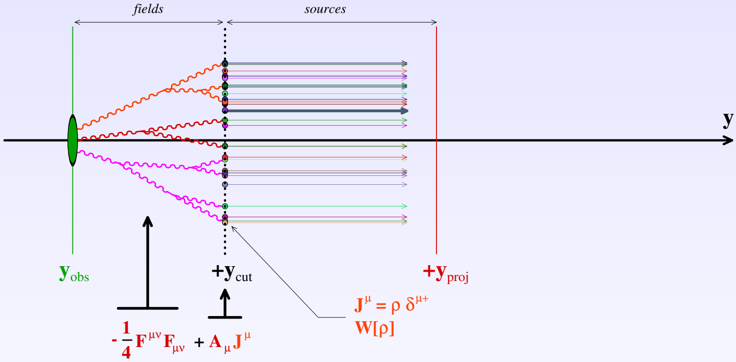

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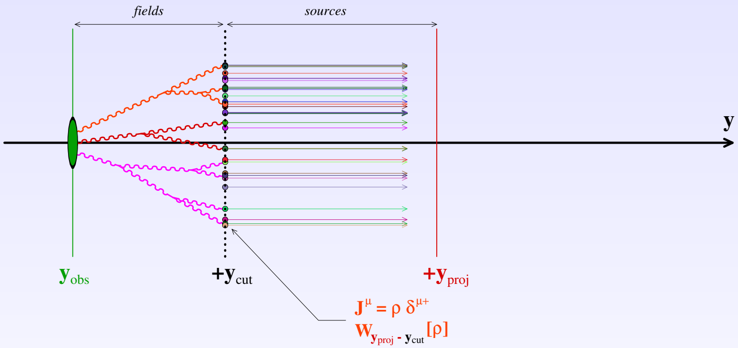
Cancellation of the cutoff dependence



- The cutoff $y_{\rm cut}$ is arbitrary and should not affect the result
- The probability distribution $W[\rho]$ changes with the cutoff

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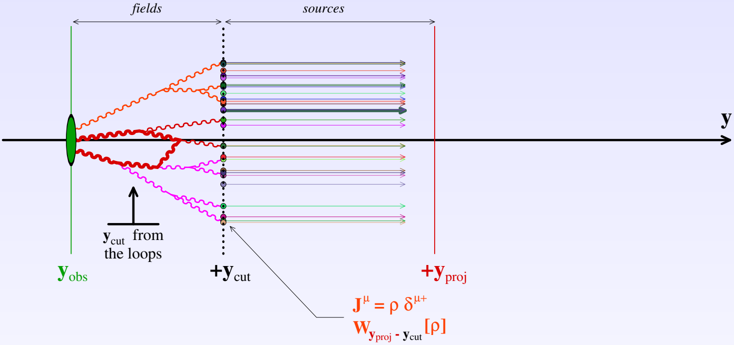


- The cutoff y_{cut} is arbitrary and should not affect the result
- The probability distribution $W[\rho]$ changes with the cutoff
- Loop corrections are also cutoff-dependent and cancel the cutoff dependence coming from $W[\rho]$

Effective description

$$\mathcal{S} \equiv \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J_\mu A^\mu \right)$$

Quantum Field Theory with (Strong) Sources

Scalar field toy model

All the peculiarities of quantum field theories coupled to an external source can be studied on a simpler example:

Scalar field coupled to an external source

$$\mathcal{S} \equiv \int d^4x \left(-\frac{1}{2}\phi(\square + m^2)\phi - \underbrace{\frac{g^2}{4!}\phi^4}_{U(\phi)} + J\phi \right)$$

- Assume the system starts at $t = -\infty$ in the vacuum state
- For interesting things to happen, the source should be time dependent

Results in the non-interacting case $(g = 0)$

Particle production amplitudes:

$$\left\langle \mathbf{p}_1 \cdots \mathbf{p}_{n\,\text{out}} \middle| 0_{\text{in}} \right\rangle = \widetilde{J}(p_1) \cdots \widetilde{J}(p_n)\ \exp\left(-\frac{1}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^2 2E_\mathbf{p}} \left|\widetilde{J}(p)\right|^2 \right)$$

- Multiplicity : Poisson distribution

$$P(n) = \frac{\overline{N}^n\ e^{-\overline{N}}}{n!} \qquad \overline{N} = \int \frac{d^3\mathbf{p}}{(2\pi)^2 2E_\mathbf{p}} \left|\widetilde{J}(p)\right|^2$$

- Notes:
  - The final multiplicity grows without bounds at large J
  - The typical field amplitude $\langle \phi(x) \rangle$ is proportional to J
  - Exclusive quantities (e.g. $P(n)$) contain a small factor $\exp(-\overline{N})$
  - Inclusive quantities (e.g. the moments of $P(n)$) are not suppressed
  - No correlations between the produced particles

A note on vacuum graphs

- In ordinary perturbation theory, one disregards disconnected "vacuum" graphs, because their sum is a phase
- In the presence of a time dependent source, particles are produced. Consider the following expression of unitarity:

$$1 = \underbrace{\left|\left\langle 0_{\rm out}\middle|0_{\rm in}\right\rangle\right|^2}_{<1} + \sum_{\alpha\neq\emptyset} \underbrace{\left|\left\langle \alpha_{\rm out}\middle|0_{\rm in}\right\rangle\right|^2}_{\neq 0 \text{ if } J\neq 0}$$

- Therefore:

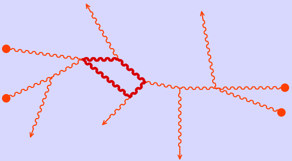
$$\text{sum of vacuum graphs} = \left\langle 0_{\rm out}\middle|0_{\rm in}\right\rangle \neq e^{i\,(\text{real phase})}$$

(they actually give the factor $\exp(-\overline{N})$ in the previous example)

Power counting

- When interactions are present, the expansion in g can be organized as a diagrammatic series

Order of magnitude of a connected graph

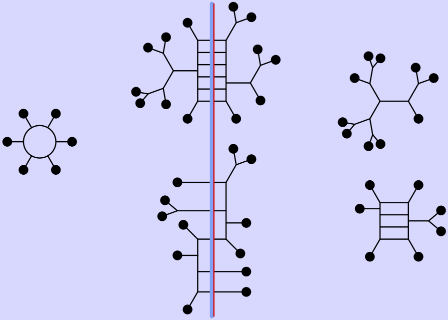


$$\sim \quad \underbrace{g^{n_E-2}}_{\text{ext. lines}} \; \underbrace{(\hbar g^2)^{n_L}}_{\text{loops}} \; \underbrace{(gJ)^{n_J}}_{\text{sources}}$$

- Sources $J \gtrsim g^{-1}$ are strong (one cannot expand in powers of J)
- Tree diagrams give the classical contribution ($\hbar \to 0$)

Exclusive quantities and vacuum diagrams

Example of graph in the probability of producing 11 particles



- Quantities where the final state is fully specified are very hard to calculate. Their diagrammatic expansion contains disconnected "vacuum" graphs (the $\exp(-\overline{\mathrm{N}})$ in the non-interacting case)

Inclusive quantities

- Average particle multiplicity $\sim 1/g^2 \gg 1$
- Probability of a given final state $\sim \exp(-1/g^2) \ll 1$

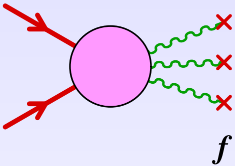
  $\Longrightarrow$ not very useful

- Inclusive observables :
  average of some quantity over all possible final states

$$\left\langle \mathcal{O} \right\rangle \equiv \sum_{\substack{\text{all final}\\ \text{states }\mathbf{f}}} \mathcal{P}(AA \to \mathbf{f})\ \mathcal{O}[\mathbf{f}]$$

Schwinger-Keldysh formalism : technique to perform the sum over final states without computing the individual transition probabilities $\mathcal{P}(AA \to \mathbf{f})$

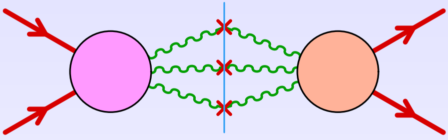
Schwinger-Keldysh formalism



Time-ordered perturbation theory :

$$G_{++}(p) = \frac{i}{p^2 + i\epsilon}$$

Schwinger-Keldysh formalism



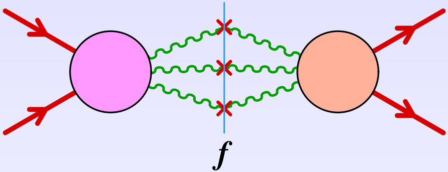
Time-ordered perturbation theory :

$$G_{++}(p) = \frac{i}{p^2 + i\epsilon}$$

Anti time-ordered perturbation theory :

$$G_{--}(p) = \frac{-i}{p^2 - i\epsilon}$$

Schwinger-Keldysh formalism



Time-ordered perturbation theory :

$$G_{++}(p) = \frac{i}{p^2 + i\epsilon}$$

Anti time-ordered perturbation theory :

$$G_{--}(p) = \frac{-i}{p^2 - i\epsilon}$$

Schwinger-Keldysh formalism :

- Across the cut : $G_{+-}(p) \equiv 2\pi\,\theta(-p^0)\,\delta(p^2)$
- Final state sum : sum over all the assignments of the labels $+$ and $-$ to vertices and sources

Inclusive observables at Leading Order

- The Leading Order is given by a sum of tree diagrams
- The sum over the $\pm$ labels turns all propagators into retarded

$$G_{++} - G_{+-} = G_{R}$$

- Expressible in terms of solutions of the classical equations of motion :

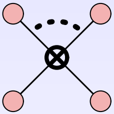
$$(\square + m^2)\varphi + \underbrace{\frac{g^3}{6}\varphi^3}_{U'(\varphi)} = J$$

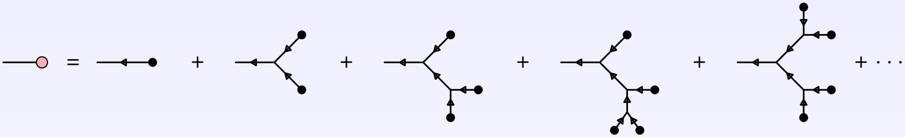
- Boundary conditions : $\lim_{x^0 \to -\infty} \varphi(x), \partial_0\varphi(x) = 0$

  (WARNING : not true for exclusive observables !)

Inclusive observables at Leading Order

$$\left\langle \mathcal{O}_{\rm LO} \right\rangle =$$

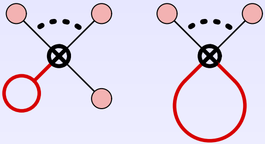




- The propagators are retarded
- The valence of the vertices depends on the interaction term

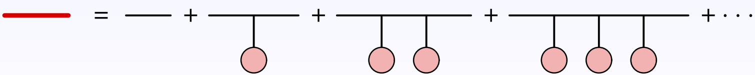
Inclusive observables at Next to Leading Order

$$\langle \mathcal{O}_{\text{NLO}} \rangle =$$



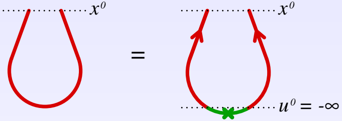
- Left : 1-loop correction to the classical field φ
- Right : 1-loop formed by connecting two classical fields

- The lines in the loops are dressed by the classical field φ :



Equal-time dressed propagators

- (Dressed) equal-time time-ordered propagators can be obtained by stitching two (dressed) retarded propagators:



$$G_{++}(x,y) = \int d^3\mathbf{u}\, d^3\mathbf{v}\;\; G_R(x,u)\, \overleftrightarrow{\partial}_{u^0}\, \Gamma_2(u,v)\, \overleftrightarrow{\partial}_{v^0}\, G_A(v,y)$$

- The "stitch" at $t = -\infty$ is given by

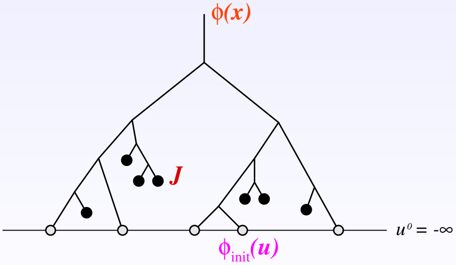
$\Gamma_2(u,v) =$ 

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}}\; e^{i p\cdot(u-v)}$$

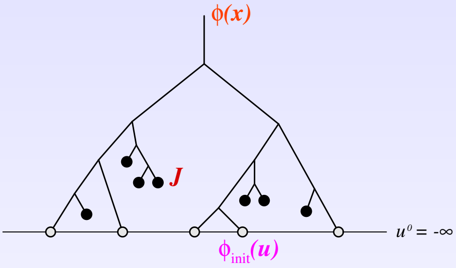
Classical field as a functional of its initial condition

Green's formula

$$\varphi(x) \;=\; \int d^4y\, G_R^0(x,y)\,\Big[J(y) - U'(\varphi(y))\Big] + \int_{u^0=-\infty} d^3\mathbf{u}\, G_R^0(x,u)\;\overleftrightarrow{\partial}_{u^0}\;\varphi_{\rm init}(u)$$



Dressed retarded propagator



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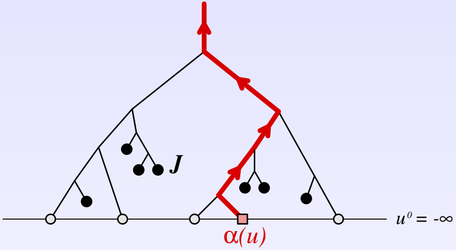


$$G_R(x,u) \sim \frac{\delta\varphi(x)}{\delta\varphi_{\rm init}(u)}$$

- More precisely:

$$\int d^3\mathbf{u}\;\Big[\alpha(u)\mathbb{T}_{\mathbf{u}}\Big]\;\varphi(x) = \int d^3\mathbf{u}\;\;G_R(x,u)\;\;\overleftrightarrow{\partial}_{u^0}\;\;\alpha(u)$$

$$\downarrow$$

$\mathbb{T}_{\mathbf{u}} \equiv$ generator of shifts of the initial condition at point $\mathbf{u}$

Inclusive observables at Next to Leading Order

Link between LO and NLO

$$\left\langle \mathcal{O}_{\text{NLO}} \right\rangle = \left[\frac{\hbar}{2} \int d^3\mathbf{u}\, d^3\mathbf{v}\;\; \underbrace{\Gamma_2(\mathbf{u},\mathbf{v})\; \mathbb{T}_{\mathbf{u}} \mathbb{T}_{\mathbf{v}}}_{\text{Laplace operator on phase-space}} \right] \mathcal{O}_{\text{LO}}(\varphi_{\text{init}}) \Big|_{\varphi_{\text{init}} \equiv 0}$$

$$\Gamma_2(\mathbf{u},\mathbf{v}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\; e^{ip\cdot(u-v)}$$

- $\mathbb{T}_{\mathbf{u},\mathbf{v}}$ only act on the initial condition
- At NLO, the time evolution remains classical, and the $\hbar$ correction comes entirely from the initial state
- Quantum corrections to the time evolution arise at order $\hbar^2$

Classical instabilities
Resummation

Mode functions

- Define:

$$\alpha_{\mathbf{k}}(x) \equiv \int d^3\mathbf{u}\,\left[e^{i\mathbf{k}\cdot\mathbf{u}}\mathbb{T}_{\mathbf{u}}\right]\,\varphi(x)$$

$$\Big[\square_x + m^2 + \underbrace{\frac{g^2}{2}\varphi^2(x)}_{U''(\varphi)}\Big]\;\alpha_{\mathbf{k}}(x) = 0$$

$$\alpha_{\mathbf{k}}(x) \underset{x^0\to-\infty}{\longrightarrow} e^{ik\cdot x}$$

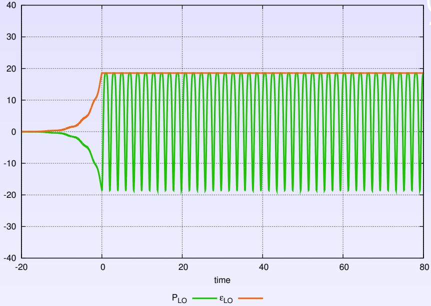
The $\{\alpha_{\mathbf{k}}\}$ are a basis of the space of linearized perturbations around the classical field $\varphi(x)$

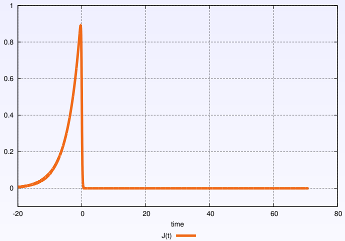
Instabilities

- When the classical equation of motion has unstable solutions, some of the $\alpha_{\mathbf{k}}(x)$ have an unbounded growth when $x^0 \to +\infty$
- This behaviour is common in field theory:
  - Scalar field with a ϕ^4 interaction : parametric resonance
  - Yang-Mills theory : Weibel instability
- Consequence : $\big\langle \mathcal{O}_{\rm NLO} \big\rangle$ growths (exponentially) with time, and eventually becomes larger than $\big\langle \mathcal{O}_{\rm LO} \big\rangle$
  $\Longrightarrow$ breakdown of the perturbative expansion

Example in a ϕ^4 scalar theory

- Scalar field coupled to an external source $J(t)$
- Source active for $t < 0$, then the fields evolve under their self-interactions
- Observables: energy density (red) and pressure (green)





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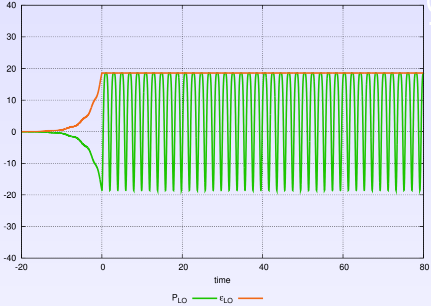
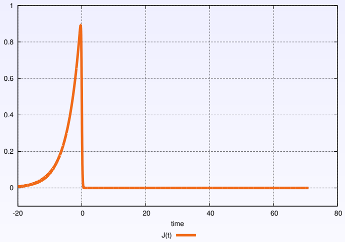
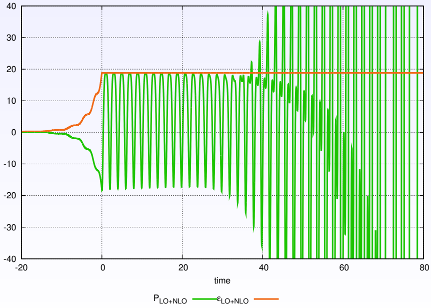


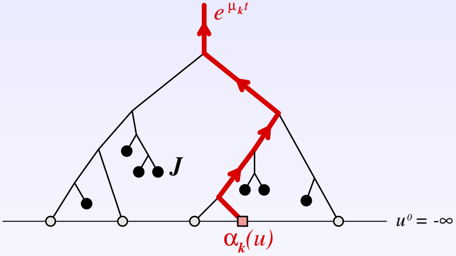




Improved power counting

- For an unstable mode:

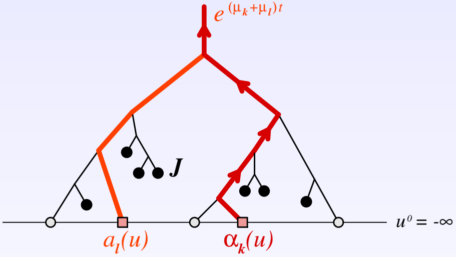
$$\alpha_{\mathbf{k}}(x) \underset{x^0 \to +\infty}{\sim} e^{\mu_{\mathbf{k}} x^0} \qquad (\mu_{\mathbf{k}} = \text{Lyapunov exponent})$$



Improved power counting

- For an unstable mode:

$$\alpha_{\mathbf{k}}(x) \underset{x^0 \to +\infty}{\sim} e^{\mu_{\mathbf{k}} x^0} \qquad (\mu_{\mathbf{k}} = \text{Lyapunov exponent})$$



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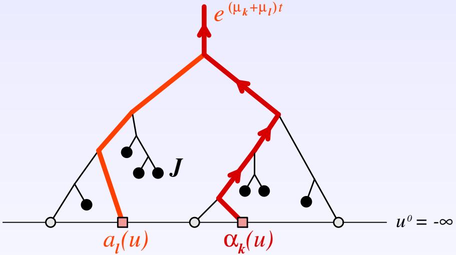
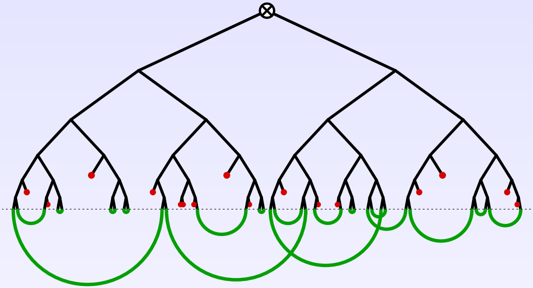


- 1 loop : up to $g^2 \hbar \, e^{2\mu_{\mathbf{k}} t}$
- n loops : up to $\left(g^2 \hbar \, e^{2\mu_{\mathbf{k}} t}\right)^n$

Leading terms



Resummation

$$\left\langle \mathcal{O}_{\text{resummed}} \right\rangle \equiv \exp\left[\frac{\hbar}{2}\int d^3\mathbf{u}\, d^3\mathbf{v}\;\; \Gamma_2(\mathbf{u},\mathbf{v})\; \mathbb{T}_{\mathbf{u}}\mathbb{T}_{\mathbf{v}}\right]\; \left\langle \mathcal{O}_{\text{LO}} \right\rangle$$

By construction:

$$\left\langle \mathcal{O}_{\text{resummed}} \right\rangle = \left\langle \mathcal{O}_{\text{LO}} \right\rangle + \left\langle \mathcal{O}_{\text{NLO}} \right\rangle + \text{subset of all higher orders}$$

Leading terms : Classical Statistical Approximation

$$\exp\left[\frac{\hbar}{2}\int d^3u\,d^3v\;\underbrace{\Gamma_2(u,v)\;\mathbb{T}_u\mathbb{T}_v}_{\text{Laplacian}}\right]\;\mathcal{O}_{\text{LO}}(\varphi_{\text{init}})$$
$$=\int[Da(u)]\;\exp\left[-\frac{1}{2\hbar}\int_{u,v}a(u)\Gamma_2^{-1}(u,v)a(v)\right]\;\mathcal{O}_{\text{LO}}(\varphi_{\text{init}}+a)$$

- In this resummation, the observable is obtained as an average over classical fields with fluctuating initial conditions
- The variance of the fluctuations ($\hbar\,\Gamma_2$) is prescribed by the NLO
- The exponentiation of the 1-loop result promotes the classical vacuum $\varphi_{\text{init}} \equiv 0$ into the coherent quantum state $\left|0_{\text{init}}\right\rangle$

Generalizations

Coherent initial state

$$\varphi_{\text{init}} \neq 0 \quad , \qquad \frac{\hbar}{2}\,\Gamma_2(u,v) = \frac{\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\; e^{ip\cdot(u-v)}$$

Initial state filled with a distribution of particles

$$\varphi_{\text{init}} = 0 \quad , \qquad \frac{\hbar}{2}\,\Gamma_2(u,v) = \hbar\int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\; e^{ip\cdot(u-v)}\,\left[\frac{1}{2} + f_0(\mathbf{p})\right]$$

$$\frac{1}{2} \quad \Longleftrightarrow \quad \text{zero point fluctuations}$$

$$f_0(\mathbf{p}) \quad \Longleftrightarrow \quad \text{initial particle distribution}$$

More on the Classical Statistical Approximation

Quantum Mechanics

CSA in Quantum Mechanics

- Consider the Liouville–von Neumann equation :

$$i\hbar\,\frac{\partial\widehat{\rho}_\tau}{\partial\tau}=\big[\widehat{H},\widehat{\rho}_\tau\big]$$

- Introduce the Wigner transforms :

$$\begin{aligned}W_\tau(\boldsymbol{x},\boldsymbol{p}) &\equiv \int d\boldsymbol{s}\; e^{i\boldsymbol{p}\cdot\boldsymbol{s}}\,\big\langle \boldsymbol{x}+\frac{\boldsymbol{s}}{2}\big|\widehat{\rho}_\tau\big|\boldsymbol{x}-\frac{\boldsymbol{s}}{2}\big\rangle\\ \mathcal{H}(\boldsymbol{x},\boldsymbol{p}) &\equiv \int d\boldsymbol{s}\; e^{i\boldsymbol{p}\cdot\boldsymbol{s}}\,\big\langle \boldsymbol{x}+\frac{\boldsymbol{s}}{2}\big|\widehat{H}\big|\boldsymbol{x}-\frac{\boldsymbol{s}}{2}\big\rangle\end{aligned}$$

- LvN equation is equivalent to Moyal-Groenewold equation

$$\begin{aligned}\frac{\partial W_\tau}{\partial\tau} &= \mathcal{H}(\boldsymbol{x},\boldsymbol{p})\,\frac{2}{i\hbar}\,\sin\Big(\frac{i\hbar}{2}\big(\overleftarrow{\partial}_{\boldsymbol{p}}\overrightarrow{\partial}_{\boldsymbol{x}}-\overleftarrow{\partial}_{\boldsymbol{x}}\overrightarrow{\partial}_{\boldsymbol{p}}\big)\Big)\,W_\tau(\boldsymbol{x},\boldsymbol{p})\\ &= \underbrace{\{\mathcal{H},W_\tau\}}_{\text{Poisson bracket}}+\mathcal{O}(\hbar^2)\end{aligned}$$

CSA in Quantum Mechanics

- Approximating the right hand side by the Poisson bracket
  $\Longleftrightarrow$ classical time evolution
  $\Longrightarrow$ $\mathcal{O}(\hbar^2)$ error

- In addition : $\hbar$ dependence in the initial state
  Uncertainty principle, $\Delta x \cdot \Delta p \geq \hbar$
  $\Longrightarrow$ the Wigner distribution $W_{\tau=0}(x, p)$ must have a width $\gtrsim \hbar$

- All the $\mathcal{O}(\hbar)$ effects can be accounted for by a Gaussian initial distribution $W_{\tau=0}(x, p)$

Path Integral

CSA from the Schwinger-Keldysh path integral

$$\langle \mathcal{O} \rangle = \int [D\phi_+ D\phi_-] \; \mathcal{O}[\phi] \; e^{i(S[\phi_+]-S[\phi_-])}$$

- ϕ_+ = amplitude $\quad\phi_-$ = conjugate amplitude
- $\phi_+ - \phi_-$ = quantum interference

- Introduce : $\phi_1 \equiv \phi_+ - \phi_-$, $\phi_2 \equiv \frac{1}{2}(\phi_+ + \phi_-)$

$$\underbrace{S[\phi_+] - S[\phi_-]}_{\text{odd in } \phi_1} = \phi_1 \cdot \frac{\delta S[\phi_2]}{\delta \phi_2} + \text{terms cubic in } \phi_1$$

- Strong field regime : $\phi_\pm$ large, but $\phi_+ - \phi_-$ small
  Neglect the terms cubic in ϕ_1
  $D\phi_1 \;\; \rightarrow\;$ classical Euler-Lagrange equation for ϕ_2
- The only remaining fluctuations are in the initial condition for ϕ_2

Perturbation Theory

CSA in perturbation theory

- Start from Schwinger-Keldysh perturbation theory:

$$G_{++}(p) = \frac{i}{p^2 + i\epsilon} + 2\pi\, f_0(p)\delta(p^2) \qquad G_{--}(p) = \Big[G^*_{++}(p)\Big]^*$$
$$G_{-+}(p) = 2\pi\,(\theta(p^0) + f_0(p))\delta(p^2) \qquad G_{+-}(p) = G_{-+}(-p)$$

$$\Gamma_{++++} = -ig^2 \qquad \Gamma_{----} = +ig^2$$

- Rotate from the basis $\phi_\pm$ to the basis $\phi_{1,2}$
- New perturbative rules :

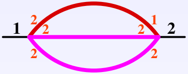
$$G_{21}(p) = \frac{i}{p^2 + ip^0\epsilon} \qquad G_{12}(p) = \frac{i}{p^2 - ip^0\epsilon}$$
$$G_{22}(p) = 2\pi\,(\tfrac{1}{2} + f_0(p))\delta(p^2) \qquad G_{11}(p) = 0$$

$$\Gamma_{1222} = -ig^2 \qquad \Gamma_{1112} = -\tfrac{i}{4}g^2$$

- CSA : neglect the 1112 vertex

Non renormalizability of the CSA

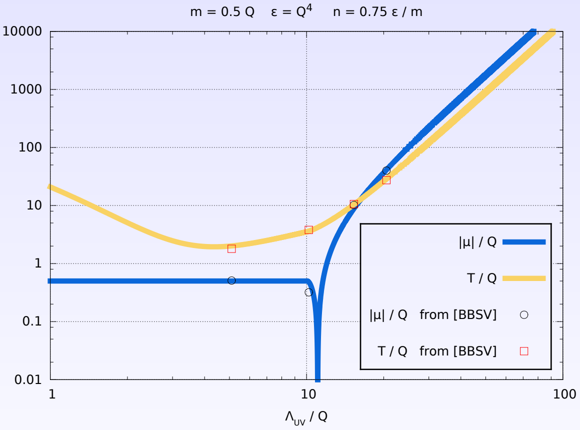
- CSA $\neq$ underlying theory at 2-loops and beyond
- Vacuum fluctuations make the CSA **non-renormalizable**.
  Example of problematic graph :

Im 

$$= -\frac{g^4}{1024\pi^3}\left(\Lambda_{\text{UV}}^2 - \frac{2}{3}p^2\right)$$

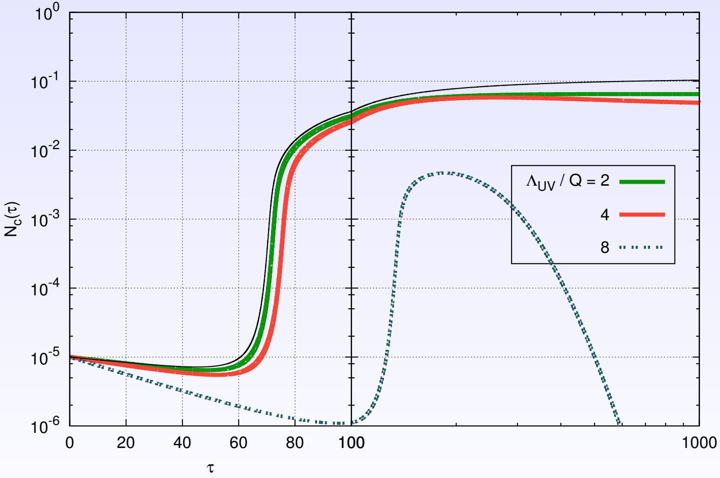
$\Longrightarrow$ divergence in an operator not present in the Lagrangian

Cutoff dependence at late time



- Weak cutoff dependence in the range : $\Lambda_{\rm UV} \sim (3-6) \times$ (physical scales)
- But : no continuum limit

Occupation in the zero mode for various UV cutoffs



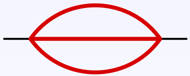
Kinetic Theory

From QFT to Kinetic Theory

Dyson-Schwinger equations $\longrightarrow$ Boltzmann equation : $p^\mu \partial_\mu f = C_{\mathbf{p}}[f]$

- Schwinger-Keldysh expression of the collision term:

$$C_{\mathbf{p}}[f] = \frac{i}{2}\Big[f(p)\Sigma_{-+}(p) - (1+f(p))\Sigma_{+-}(p)\Big]$$



$$\Longrightarrow \quad C_{\mathbf{p}}[f] = \frac{g^4}{4E_{\mathbf{p}}}\int_{\mathbf{k}}\int_{\mathbf{p}'}\int_{\mathbf{k}'}(2\pi)^4\delta(P+K-P'-K')$$
$$\times\Big[f(\mathbf{p}')f(\mathbf{k}')\big(1+f(\mathbf{p})\big)\big(1+f(\mathbf{k})\big)$$
$$-f(\mathbf{p})f(\mathbf{k})\big(1+f(\mathbf{p}')\big)\big(1+f(\mathbf{k}')\big)\Big]$$

Collision term in the semi-classical approximation

- Expression in the $\phi_{1,2}$ basis:

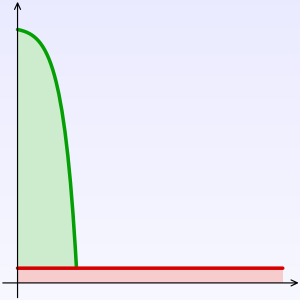
$$C_{\mathbf{p}}[f] = \frac{i}{2}\Big[\Sigma_{11}(p) + \big(\tfrac{1}{2} + f(p)\big)\Big(\Sigma_{21}(p) - \Sigma_{12}(p)\Big)\Big]$$

- Neglecting the 1112 vertex, the collision term becomes:

$$\begin{aligned} C_{\mathbf{p}}[f] \;=\; & \frac{g^4}{4E_{\mathbf{p}}}\int_{\mathbf{k}}\int_{\mathbf{p}'}\int_{\mathbf{k}'}(2\pi)^4\delta(P+K-P'-K') \\ & \times\Big[\big(\tfrac{1}{2}+f(\mathbf{p}')\big)\big(\tfrac{1}{2}+f(\mathbf{k}')\big)\big(1+f(\mathbf{p})+f(\mathbf{k})\big) \\ & -\big(\tfrac{1}{2}+f(\mathbf{p})\big)\big(\tfrac{1}{2}+f(\mathbf{k})\big)\big(1+f(\mathbf{p}')+f(\mathbf{k}')\big)\Big] \end{aligned}$$

- $\frac{1}{2}, 1$: originate from the zero point occupancy
- Terms in f^3 and f^2 correct, but spurious f^1 terms
- Obeys H-theorem, Fixed point: $f(p) = \frac{T}{E_{\mathbf{p}}-\mu} - \frac{1}{2}$
- But : T, μ depend on the ultraviolet cutoff

Should we keep or drop the vacuum $1/2$?



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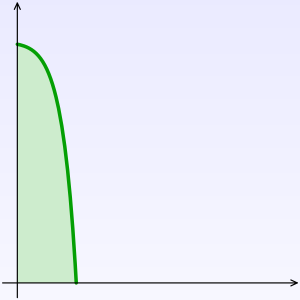

?

Should we keep or drop the vacuum $1/2$?

- The $1/2$'s are responsible for UV problems, but...
- They ensure that the collision term is correct at orders f^3 and f^2 (without them, one has a pure classical wave approximation)
- They are important in certain kinematic situations

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- No f^2 terms without the $1/2$'s :

$$\partial_t f_4 \sim g^4 \int_{123} \cdots \left[f_1 f_2 (f_3 + f_4) - f_3 f_4 (f_1 + f_2)\right] + \cdots \left[\cancel{f_1 f_2 - f_3 f_4}\right]$$

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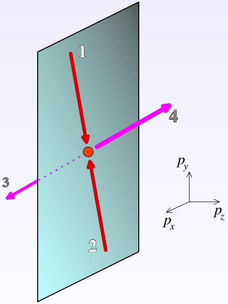

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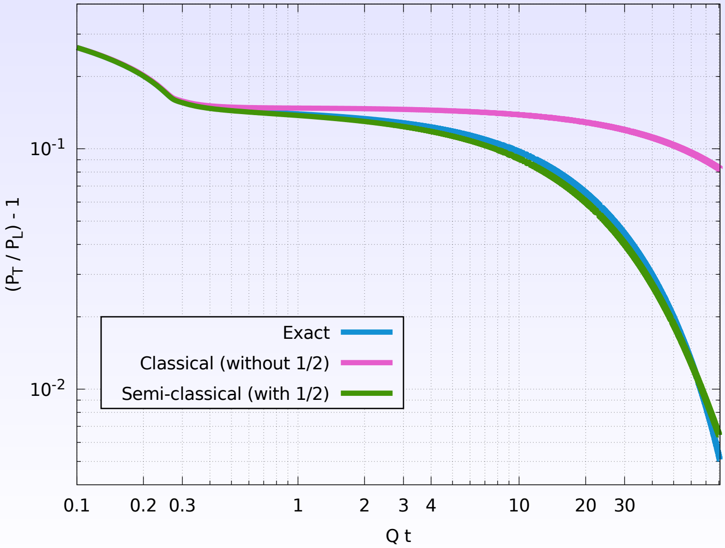
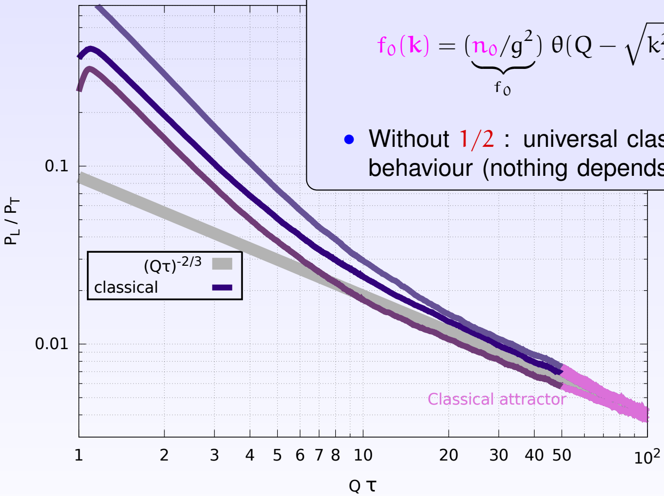
- If the distribution becomes very anisotropic, trying to produce the particle 4 at large angle results in $f_3 \approx f_4 \approx 0 \quad \Rightarrow$ nothing left
- f^3 terms $\Leftrightarrow$ stimulated emission : ineffective to produce particles in empty regions of phase-space

Isotropization in a fixed box



Isotropization in a longitudinally expanding system

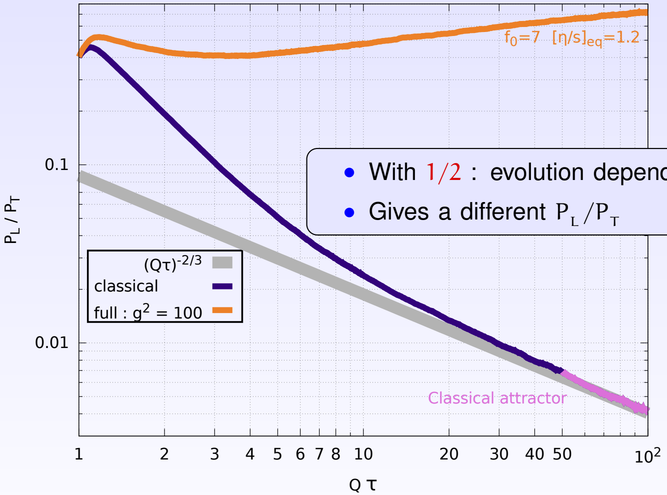


- Initial conditions:

$$f_0(\mathbf{k}) = \underbrace{(n_0/g^2)}_{f_0}\ \theta(Q - \sqrt{k_\perp^2 + \xi_0\, k_z^2})$$

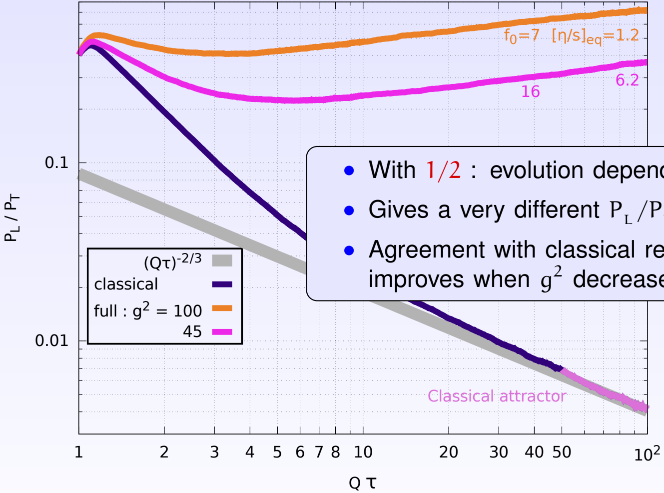
- Without $1/2$: universal classical behaviour (nothing depends on g^2)

Isotropization in a longitudinally expanding system



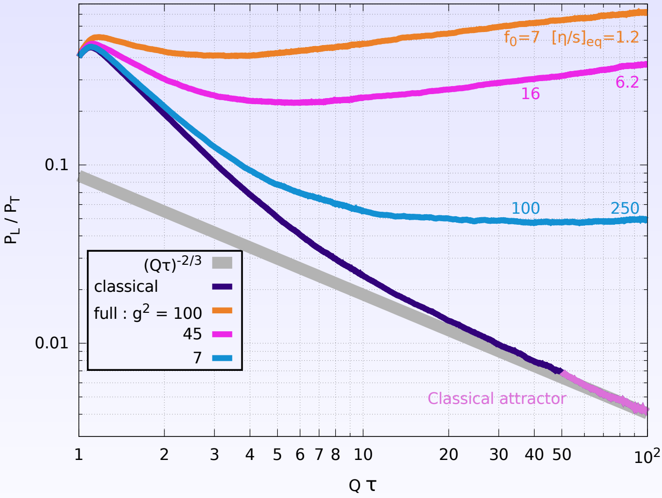
- With $1/2$: evolution depends on g^2
- Gives a different P_L/P_T

Isotropization in a longitudinally expanding system



- With $1/2$: evolution depends on g^2
- Gives a very different P_L/P_T
- Agreement with classical result improves when g^2 decreases

Isotropization in a longitudinally expanding system



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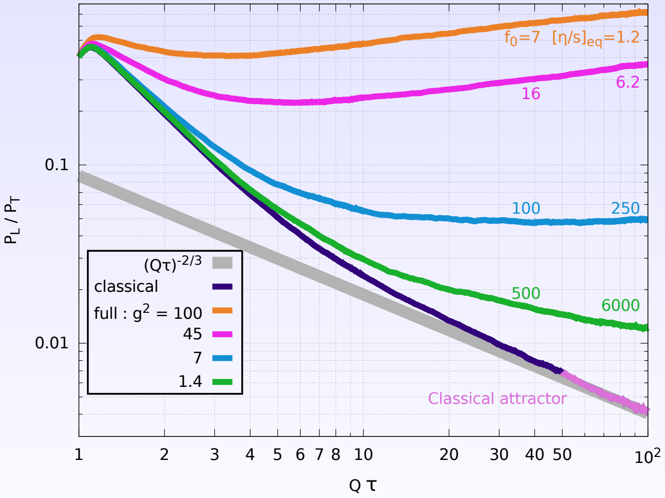

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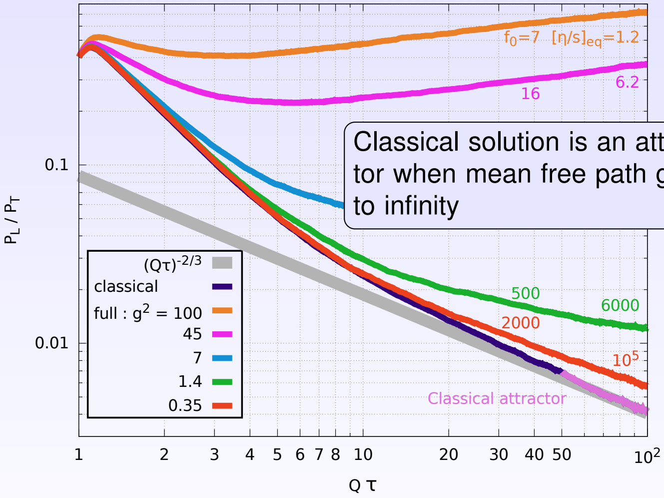


Classical solution is an attractor when mean free path goes to infinity

Summary

Beware of semi-classical approximations!

Thank you for your attention.

Classical limit of the Bose-Einstein distribution

- Assume :

$$\left[a_{\mathbf{k}}, a_{\mathbf{l}}^\dagger\right] = \varepsilon\ (2\pi)^3 2E_{\mathbf{k}}\ \delta(\mathbf{k}-\mathbf{l})$$

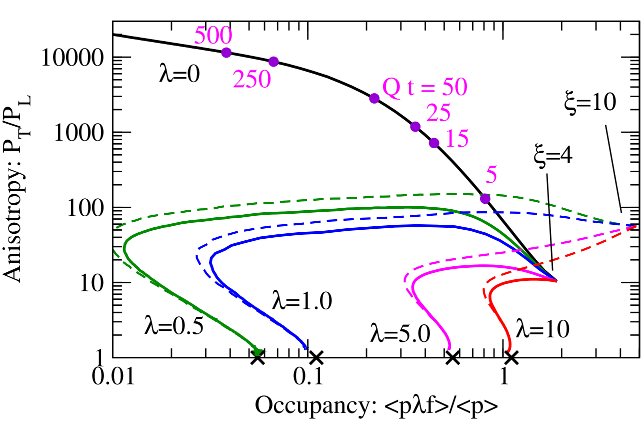
- This leads to :

$$\frac{\mathrm{tr}\left(e^{-\beta H} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\right)}{\mathrm{tr}\left(e^{-\beta H}\right)} = 2E_{\mathbf{k}} \times \mathrm{Volume} \times \frac{\varepsilon}{e^{\beta\,\varepsilon\,E_{\mathbf{k}}} - 1}$$

- In the limit $\varepsilon \to 0$,

$$\frac{\varepsilon}{e^{\beta\,\varepsilon\,E_{\mathbf{k}}} - 1} \approx \frac{1}{\beta E_{\mathbf{k}}} - \frac{\epsilon}{2} + \cdots$$

Kinetic theory results for gluons [Kurkela, Zhu (2015)]



- At $\lambda = g^2 N_c = 0.5$, the classical approximation breaks down for $Q\tau \gtrsim 2$

  The criterion $f \gg 1$ suggests that this approximation should be valid until $Q\tau \approx \alpha_s^{-3/2} \approx 350 \Rightarrow$ criterion too crude