Preliminaries

2.1 Notation

Given \( p \) scalar-valued variables \( z_1, z_2, \ldots, z_p \), we define \( z := \text{col}(z_1, z_2, \ldots, z_p) \) and let \( \text{diag}\{z\} \) represent the diagonal matrix with the \( i \)th diagonal entry equal to the \( i \)th element of \( z \). We denote by \( S_z \) the support of \( z \), which is the set of indices that correspond to the nonzero entries of \( z \), and by \( \|z\|_0 \) the 0-norm of \( z \), which is the number of elements in \( S_z \). We also let \( |z| \) denote its Euclidean norm and \( |z|_\infty \) its infinity norm. When \( z \) is a scalar, \( |z| \) represents its absolute value. The vector \( \mathbf{1}_p \) and \( \mathbf{0}_p \) represent the \( p \)-dimensional vectors with all elements equal to 1 and 0, respectively. Given a matrix \( A \), \( A_{ij} \) represents its \( i \)th row and \( a_{ij} \) represents its element in the \( i \)th row and \( j \)th column. The cardinality of a set \( S \) is denoted by \( |S| \). For two sets \( S \) and \( M \), we let \( S \setminus M = \{ x \in S \mid x \notin M \} \) represent the complement of \( M \) in \( S \). Given a signal \( s \) mapping \( \mathbb{R}_0^+ \) to \( \mathbb{R}^n \), we define \( |s|_1 := \sup_{t \in \mathbb{R}_0^+} |s(t)|_\infty \) and say that \( s \) is bounded if \( |s|_\infty \) is finite.

2.2 Graph-theoretic notions

For a network with \( n \) nodes, let its topology be represented by an undirected and connected graph \( G = \{V, E\} \), with \( V = \{1, 2, \ldots, n\} \) being the set of nodes and \( E \subseteq V \times V \) the set of edges, where \( \{i, j\} \in E \), or equivalently, node \( i \) is a neighbour of node \( j \), means that node \( i \) can receive information from node \( j \) and vice versa. We denote the set of neighbors of node \( i \) by \( N_i \) and let \( d_i = |N_i| \).

The adjacency matrix \( A \) of \( G \) is defined as \( a_{ij} = 1 \) if node \( j \) is the neighbor of node \( i \) and \( a_{ij} = 0 \) otherwise. For an undirected graph \( G \), we can assign arbitrary orientations to the edges such that each edge \( \{i, j\} \in E \) has a head and a tail. The edge-node incidence matrix \( B \in \mathbb{R}^{m \times n} \) of \( G \), with \( m = |E| \), is defined as \( b_{ij} = 1 \) if \( j \) is the head node of the edge \( i \in E \) and \( b_{ij} = -1 \) if \( j \) is the tail node. The Laplacian matrix \( L \) of \( G \) is an \( n \times n \) matrix given by \( l_{ij} = -a_{ij} \) for \( j \neq i \) and \( l_{ii} = \sum_{j \in N_i} a_{ij} = d_i \). Since \( G \) is undirected, it is well-known that \( L = B^\top B \). The incidence matrix can be decomposed as the head incidence matrix \( B_+ \in \mathbb{R}^{m \times n} \) and the tail incidence matrix \( B_- \in \mathbb{R}^{m \times n} \), which are given
We also let $R$ denote the signless edge-node incidence matrix with $r_{ij} = |b_{ij}|$. It is easy to verify that $B = B_+ + B_-$ and $R = B_+ - B_-$. Let $d = [d_1, d_2, \ldots, d_n]^\top$ and $D = \text{diag}(d)$. The matrix $A + D$ is called the signless Laplacian matrix. When $G$ is undirected, $A + D = R^\top R$. Hence, $A + D$ is positive semi-definite and all its eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ are real and nonnegative.

A path $P_{ij}$ from node $i$ to node $j$ is a sequence of nodes and edges such that each successive pair of nodes in the sequence is adjacent. The length of a path is the number of edges in the path. The distance between node $i$ and $j$ is the length of the shortest path from $i$ to $j$. We denote by $D_G$ the diameter of $G$, which is the maximum distance between any two nodes.

### 2.3 Bipartite Graphs

In Chapter 5 we show that the ability to recover the biases from relative measurements depends on whether the measurement graph is bipartite or not. Hence, in this subsection we give a short introduction on bipartite graphs and their matrix properties. A graph $G$ is bipartite if the vertex set $V$ can be partitioned into two sets $V_+$ and $V_-$ in such a way that no two vertices from the same set are adjacent. The sets $V_+$ and $V_-$ are called the colour classes of $G$ and $(V_+, V_-)$ is a bipartition of $G$. For a bipartite graph, the following result holds:

**Theorem 2.1.** Asratian et al. (1998) A graph $G$ is bipartite if and only if $G$ has no cycle of odd length.

An algebraic characterization of bipartite graphs is provided next.

**Lemma 2.1.** An undirected and connected graph $G$ is bipartite if and only if the signless incidence matrix $R$ does not have full column rank. Moreover, if $G$ is bipartite, then any $n - 1$ columns of $R$ are linearly independent.

**Proof.** To prove the first part, suppose that $Rv = 0$ for some nonzero vector $v \in \mathbb{R}^n$. It is easy to see that $|v_i| = |v_j| = a$ for every $i, j \in V$, where $a > 0$. In
fact, for every pair \((r, s)\) of adjacent nodes, we must have \(v_r = -v_s\), otherwise \(Rv \neq 0\). Since the graph is connected and since \(v\) must be nonzero, we obtain the claim. Note that this also shows that any two nodes \(i, j \in V\) with \(v_i = v_j\) should not be adjacent. Let \(V_+/V_-\) contain the nodes corresponding to the entries of \(v\) with value \(a/ -a\), then any node in \(V_+/V_-\) should not be adjacent to other nodes in \(V_+/V_-\). This implies that \(G\) is bipartite and \((V_+, V_-)\) is a bipartition of \(G\). Conversely if \(G\) is bipartite, there exists a bipartition \((V_+, V_-)\) of \(G\). By letting the elements of \(v\) corresponding to \(V_+\) and \(V_-\) be \(a\) and \(-a\), respectively, with \(a \neq 0\), we have \(Rv = 0\), which shows that \(R\) does not have full column rank.

For the second part, we prove it by contradiction. Suppose there exist some dependent columns of \(R\) and let the index set of these columns be \(S \subset V\), with \(|S| \leq n - 1\), then there should exist a nonzero vector \(v \in \mathbb{R}^{|S|}\) such that \(R_Sv = 0\) where \(R_S\) is the matrix whose columns are those indexed by \(S\). The latter implies the existence of a nonzero vector \(\tilde{v}\), whose nonzero entries are given by \(v\), and satisfies \(R\tilde{v} = 0\). However, from the proof of the first part, the absolute values of all the elements of \(\tilde{v}\) should be equal to each other. Hence, \(v\) must be the zero vector, which is a contradiction.

The if and only if part of the statement above is also provided in (Bapat, 2010, Lemma 2.17). We provide the proof here, since it is used in proving the second part of the statement as well as in other parts of the paper.

For later use, by the proof of Lemma 2.1, we note that

\[
Rv = 0_n \iff \exists a \in \mathbb{R} \text{ s.t. } v_i = \begin{cases} 
a & i \in V_+ 
-a & i \in V_-
\end{cases}
\tag{2.1}
\]

for a bipartite graph with bipartition \((V_+, V_-)\).

**Lemma 2.2.** Cvetković et al. (2007) The smallest eigenvalue of the signless Laplacian matrix \(A + D\) of an undirected and connected graph is equal to zero if and only if the graph is bipartite. In case the graph is bipartite, zero is a simple eigenvalue.

### 2.4 COMPRESSED SENSING

In the field of compressed sensing or sparse signal recovery, one of the most important problems is how to find the sparsest solution from the number-
deficient measurements. Formally, consider the following linear equation

\[ y = Fx \]  

(2.2)

where \( x \in \mathbb{R}^n \) is the vector of unknown variables, \( y \in \mathbb{R}^p \) is the vector of known values, and \( F \in \mathbb{R}^{p \times n} \) is a matrix defining the linear relation from \( x \) to \( y \). It is assumed that \( p < n \), thus equation (2.2) is under-determined. It is then of interest to find solutions \( x \) such that \( \|x\|_0 \ll n \), and in particular to seek for the sparsest solution of (2.2). Let us define the set of \( k \)-sparse vectors as

\[ \mathcal{W}_k := \{ x \in \mathbb{R}^n \mid \|x\|_0 \leq k \}. \]  

(2.3)

The following result provides a sufficient condition under which the solution of (2.2) can be uniquely determined.

**Lemma 2.3.** Given an integer \( s \geq 0 \), let \( 2s \leq p \), and assume that any matrix made of \( 2s \) columns of \( F \) is full column rank. If \( x \in \mathcal{W}_s \) is a solution of (2.2), then there exists no other solution of (2.2) in \( \mathcal{W}_s \).

**Remark 2.1.** Under the assumptions of the lemma, the solution \( x \in \mathcal{W}_s \) of (2.2) is also the solution to

\[ \min_{x \in \mathbb{R}^n} \|x\|_0 \]  

s.t. \( y = Fx \),

(2.4)

that is, the sparsest solution to (2.2). The proof of Lemma 2.3 descends from (Hayden et al., 2016, Lemma 1).

However, solving \( x \) from (2.2) under the assumption that \( \|x\|_0 \leq s \) is cumbersome when \( s \) is not small, as it requires to combinatorially search for \( s \) columns of \( F \) whose span contains \( y \). A typical way to avoid this exhaustive search is to change the problem into the following \( \ell_1 \)-norm optimization problem

\[ \min_{x \in \mathbb{R}^n} \|x\|_1 \]  

s.t. \( y = Fx \),

(2.5)

where \( \|x\|_1 = \sum_{i=1}^n |x_i| \) denotes the 1 norm of \( x \), the vector \( y \) is known from (2.2) and the objective function and the constraint are both convex. Problem (2.5) can be solved by linear programming Rauhut (2010). The \( \ell_1 \)-norm minimization may return a solution \( \bar{x} \) different from the solution \( x \) of (2.2). The
following definition and result characterize the relation between the matrix $F$, the equation (2.2) and the $\ell_1$-norm minimization problem.

**Definition 2.1 (Nullspace Property).** A matrix $F \in \mathbb{R}^{p \times n}$ is said to satisfy the nullspace property of order $s$, with $s$ being a positive integer, if for any set $S \subset V = \{1, 2, \ldots, n\}$ with $|S| \leq s$ and any nonzero vector $v$ in the null space of $F$, the condition below holds

$$
\|v_S\|_1 < \|v_{S_c}\|_1,
$$

(2.6)

where $v_S \in \mathbb{R}^{|S|}$ and $v_{S_c} \in \mathbb{R}^{|S_c|}$ are subvectors of $v$ whose elements are indexed by $S$ and $S_c$, respectively, and $S_c = V \setminus S$.

The null space property is usually difficult to verify and a more restrictive but more conveniently checkable condition known as restricted isometry property is considered (Rauhut, 2010, p. 8). Yet, in the special cases that are of interest to us the null space property can be easily confirmed, and we will persist with it in the sequel.

**Theorem 2.2.** (Rauhut, 2010, Theorem 2.3) Every vector $x \in \mathcal{W}_s$ is the unique solution of the $\ell_1$-norm minimization problem (2.5), with $y = Fx$, if and only if $F$ satisfies the null space property of order $s$.

We highlight the role of this theorem explicitly in connection with the equation (2.2). For a given $y \in \mathbb{R}^p$, let $x \in \mathbb{R}^n$ be a solution of (2.2). Assume that $\|x\|_0 \leq s$ and $F$ satisfies the null space property of order $s$, with $0 < s < n$. By Theorem 2.2, $x$ is the unique solution of (2.5), with $y = Fx$. Stated directly, there exists a unique solution $\tilde{x}_s$ of (2.5), with $y = Fx$, and it satisfies $\tilde{x}_s = x$. Hence, under the given condition of $s$-sparsity of the vector $x$ solution of (2.2) and the null space property of order $s$ of the matrix $F$, solving the optimization problem (2.5), with $y = Fx$, univocally returns $x$. 
