Part II

Theory and applications
Chapter 3

Linear theory

3.1 Goal, definitions and results

Objective  In Sections 3.2 and 3.4 of this chapter we shall derive and solve a linearization of the formal equation

\[(N + P) \circ F_\ast = N_\ast. \tag{3.1}\]

In the above equation \(H = N + P\) is the original perturbed Hamiltonian, \(N, N_\ast\) are of the form (2.14), \(P\) is a sufficiently small perturbation, and \(F_\ast\) is the composition of a symplectic transformation \(\Phi_\ast\) and a reparametrization \(\phi_\ast\). All of these objects will be discussed in the present section.

Iteration process  The nonlinear equation (3.1) is solved in a way that is comparable to Newton’s method of finding zeros of nonlinear functions. More precisely, denoting the original unperturbed Hamiltonian and the original perturbation by \(N_0\) and \(P_0\) respectively, we linearize equation (3.1), and insert the solution \(F_0\) of the linearized equation into the original equation. The error made by solving only a linearization causes the right-hand side of equation (3.1) to consist not only of a new unperturbed Hamiltonian \(N_1\), but also of a perturbation \(P_1\), which should be much smaller than the original perturbation \(P_0\):

\[(N_0 + P_0) \circ F_0 = N_1 + P_1.\]

This constitutes one KAM step.

Since we have obtained a Hamiltonian of the same kind, i.e., an unperturbed Hamiltonian plus a small perturbation, the above process can be repeated over and again. This gives sequences of perturbations \(P_0, P_1, P_2, \ldots\) and transformations \(F_0, F_1, F_2, \ldots\). If these sequences converge to zero, respectively to the identity fast enough, the infinite composition \(F_0 \circ F_1 \circ F_2 \circ \cdots\) of transformations converges to a transformation solving the original equation (3.1). The quantitative aspects of a KAM step, the iteration process, and the proof of its convergence are extensively treated in Chapter 7.
Some notation and nomenclature  In this chapter we limit the discussion to a single KAM step. It is then convenient to use the $+$ notation in which the subscript $n + 1$ of the new unperturbed Hamiltonian and perturbation (that is the ones obtained in a KAM step) is replaced by "+$", while the subscript $n$ of the original unperturbed Hamiltonian, perturbation, and transformation is dropped.

The homological equation  The linearization of equation (3.1) is called the homological equation and as we shall see in Section 3.2, it reads

$$N + R + \{N, F\} = \tilde{N}_+.$$  

In this equation, $N$ and $R$ (which is a truncation of the perturbation $P$, to be defined later in this section) are considered given, while we have to solve for a preliminary form $\tilde{N}_+$ of the new unperturbed Hamiltonian $N_+$, and a Hamiltonian $F$, whose time-1-flow gives a symplectic coordinate transformation.

The final form of the new unperturbed Hamiltonian $N_+$ is obtained after reparametrizing $\omega$:

$$N_+ = \tilde{N}_+ \circ \phi.$$  

Later in this section we get back to the reparametrization $\phi$.

The normal $1 : 1 : \ldots : 1$ resonance  For solving the homological equation we need to determine an appropriate class $\mathcal{N}$ of unperturbed Hamiltonians. We want this class to contain the following unperturbed Hamiltonian, which corresponds to a row of uncoupled pendulums:

$$N(\varphi, I, q, p; \omega) = \omega_I I + \frac{1}{2}(q^T q + p^T p).$$  

Here the frequencies normal to the torus

$$\mathcal{T}_\omega = T^n \times \{0\}^n \times \{0\}^m \times \{0\}^m \times \{\omega\}$$  

(3.2)

are all equal. This situation is called the normal $1 : 1 : \ldots : 1$ resonance, and we will see that it complicates the determination of $\mathcal{N}$.

The unfolding theory summarized in Section 3.3 motivates the following choice for $\mathcal{N}$:

Definition 3.1 (The class $\mathcal{N}$ of unperturbed Hamiltonians)  The class $\mathcal{N}$ consists of functions of the following form

$$N(\varphi, I, q, p; \omega) = \omega^T I + \frac{1}{2} \left( \begin{array}{cc} q & p \\ \end{array} \right) \left( \begin{array}{cc} 1 + S_\omega & -A_\omega \\ -A_\omega & I + S_\omega \\ \end{array} \right) \left( \begin{array}{c} q \\ p \end{array} \right),$$

where $A_\omega$ and $S_\omega$ are $\omega$-dependent skew-symmetric and symmetric $m \times m$-matrices, respectively.
Remarks

1. Notice that we have chosen the matrices $A$ and $S$ to be independent of the torus variable $\varphi$. It is possible to drop this condition, see [Bg], but the KAM theory of this thesis does not easily admit such a generalization.

2. In this thesis the dependence of $A$ and $S$ on $\omega$ is usually real-analytic.

3. Within this chapter it is convenient to consider the matrices $A$ and $S$ as being independent from $\omega$. We often use the abbreviation

$$N_{AS}(\varphi, I, q, p; \omega) = \omega^T I + \frac{1}{2} \begin{pmatrix} q & p \end{pmatrix} \begin{pmatrix} I+S & -A \\ A & I+S \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$  \hspace{1cm} (3.3)

The dynamics of an unperturbed Hamiltonian

Using the abbreviations

$$x := \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{and} \quad W_{AS} := \begin{pmatrix} I+S & -A \\ A & I+S \end{pmatrix},$$

the Hamiltonian equations corresponding to an unperturbed Hamiltonian $N_{AS}$ read

$$\begin{cases}
\dot{\varphi} = \omega, \\
\dot{I} = 0, \\
\dot{x} = -JW_{AS}x,
\end{cases}$$

where $$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

Obviously, the torus $T_\omega$, see (3.2), is invariant under the flow of an unperturbed Hamiltonian. Moreover, the motion on the torus is conditionally periodic with internal or torus frequency vector coinciding with the parameter $\omega$. The linearization of the dynamics normal to the torus is independent of $\varphi$, so $T_\omega$ is a Floquet torus.

The normal frequencies of an unperturbed Hamiltonian

In Lemma 3.6 in Section 3.3 we show that given an unperturbed Hamiltonian $N_{AS}$ there is a linear transformation $T$, which is symplectic and orthogonal, and has the property that in the new coordinates $\tilde{x} = Tx$ the Hamiltonian equations in the normal directions are given by

$$\dot{\tilde{x}} = -J\Lambda_{2m}\tilde{x},$$  \hspace{1cm} (3.4)

where $\Lambda_{2m} = \text{diag}(\lambda_1, \ldots, \lambda_m, \lambda_1, \ldots, \lambda_m)$. This shows that the normal dynamics of $N_{AS}$ is generically elliptic around the origin $x = 0$. We refer to $\lambda = (\lambda_1, \ldots, \lambda_m)$ as the (vector of) normal (or external) frequencies.
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The reparametrization The homological equation will be solved for a preliminary new unperturbed Hamiltonian of the form

\[ \tilde{N}_+ = N_\omega, A_+, S_+ (\varphi, I, x; \omega) := \omega_+^T I + \frac{1}{2} x^T W A_+, S_+ x. \]

where \( \omega_+; A_+; S_+ \) depend analytically on \( \omega; A; S \).

We call \( \tilde{N}_+ \) a preliminary unperturbed Hamiltonian since its internal frequencies, which are \( \omega_+ \), are not yet identical to the parameters \( \omega, A, S \).

A reparametrization \( \varphi : \omega \mapsto \omega^{-1}(\omega) \) is needed to arrange this. In Subsection 7.1.6 we will see that the existence of such a reparametrization follows from the inverse function theorem (Lemma 8.20).

Differential unperturbed Hamiltonian It is convenient to introduce the differential unperturbed Hamiltonian \( \hat{N} = \tilde{N}_+ - N \), which is written as

\[ \hat{N} = N_\hat{\omega}, \hat{A}_+, \hat{S}_+ (\varphi, I, x; \omega) := \hat{\omega}^T I + \frac{1}{2} x^T W \hat{A}_+, \hat{S}_+ x, \]

where the parameters \( (\hat{\omega}; \hat{A}, \hat{S}) \) are given by

\[ (\hat{\omega}; \hat{A}, \hat{S}) := (\omega_+; A_+, S_+) - (\omega; A, S). \]

In terms of the differential unperturbed Hamiltonian, the homological equation is rewritten as

\[ \{F, \tilde{N}\} + \hat{N} = R. \]

Perturbations We restrict ourselves to perturbations that are real-analytic functions in \( (\varphi, I, x; \omega) \). Using the notation \( \mathbb{Z}_0 = \{0, 1, 2, \ldots\} \), such functions can be written as Fourier-Taylor series

\[ P(\varphi, I, x; \omega) = \sum_{k \in \mathbb{Z}_0} \sum_{l \in \mathbb{Z}^n_0} \sum_{a \in \mathbb{Z}^n_0} P_{kla}(\omega) I^l x^a \]

that have a positive radius of convergence (as opposed to infinitely differentiable functions whose series usually have convergence radius zero). The Fourier-Taylor coefficients \( P_{kla}(\omega) \) of real-analytic functions \( P \) are real-analytic in \( \omega \). We have used standard multi-index notation in which for instance \( I^l \) stands for the product

\[ I^l := I_1^{l_1} I_2^{l_2} \cdots I_n^{l_n}. \]

K-truncations \( R \) is the \( K \)-truncation of the perturbation \( P \), which is obtained by truncating the Fourier series of \( P \) at order \( K \in \mathbb{N} \), and taking only terms at most linear in \( I \) or quadratic in \( x \):

\[ R(\varphi, I, x; \omega) = \sum_{k \in \mathbb{Z}_0} \sum_{l \in \mathbb{Z}^n_0} \sum_{a \in \mathbb{Z}^n_0} P_{kla}(\omega) I^l x^a. \] (3.5)

With some abuse of terminology, functions whose Fourier-Taylor series contains only the above terms are also called \( K \)-truncations.
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The generating Hamiltonian Finally, the Hamiltonian $F$ generating the symplectic transformation $\Phi$ is also assumed to be a $K$-truncation:

$$F(\varphi, l, x; \omega) = \sum_{|l| \leq K} e^{i(k,\varphi)} \sum_{2|l|+2|s|\leq 2} F_{ks\varphi}(\omega) l^1 x^s.$$ 

It turns out that in order to solve the homological equation for a differential unperturbed Hamiltonian $\hat{N}$ and $K$-truncation $F$, given an arbitrary unperturbed Hamiltonian $N$ and $K$-truncation $R$, we need to exclude the following resonances:

**Definition 3.2 (K-Resonances)** A $K$-resonance is a relation between the internal and external frequencies $\omega$ and $\lambda$ of the following type

$$\langle k, \omega \rangle + \langle l, \lambda \rangle = 0,$$

where the pair of integer vectors $(k, l)$ are in the resonance set $\mathcal{R}_K$, which is defined as

$$\mathcal{R}_K = \left\{ (k, l) \in \mathbb{Z}^{n+m} : |k| \leq K, |l| \leq 2, \forall i, j \in \{1, 2, \ldots, m\} : (k, l) \neq (0, e_i - e_j) \right\},$$

where $e_i$ and $e_j$ are respectively the $i$th and $j$th standard basis vectors of $\mathbb{R}^m$.

We call the pair $(\omega, \lambda)$ of internal and external frequencies $K$-nonresonant if they do not satisfy (3.6) for all $(k, l) \in \mathcal{R}_K$.

**Remark** We stress here that the normal $1 : 1 : \ldots : 1$ resonance is not a $K$-resonance; the whole point is that we want to solve the homological equation for a class $\mathcal{N}$ including unperturbed Hamiltonians with this resonance.

The main result of this chapter is:

**Theorem 3.3 (Existence, uniqueness, and analyticity of homological solutions)**

Let $N$ be an unperturbed Hamiltonian, and $R$ a $K$-truncation. If the frequencies $(\omega, \lambda)$ are $K$-nonresonant, the homological equation

$$\{F, N\} + \hat{N} = R,$$

(3.8)

can be solved for a $K$-truncation $F$ and a differential unperturbed Hamiltonian $\hat{N}$, such that the Fourier-Taylor coefficients of $F$ and the parameters $(\hat{\omega}; \hat{A}, \hat{S})$ of $\hat{N}$ depend in a real-analytic way on the parameters $(\omega; A, S)$ of $N$ and the coefficients of $R$.

Such a solution $(F, \hat{N})$ is determined uniquely by the condition that the obvious projection of $F$ on the space $\mathcal{N}$ of unperturbed Hamiltonians is zero.

We prove this theorem in Section 3.4.
Remarks

1. Recall that a reparametrization \( \phi \) is needed to make the KAM step complete.

2. Later on, when \( A, S \) are considered as real-analytic families in \( \omega \) instead of as free parameters, \( \hat{\omega} \) and the coefficients of \( F \) and \( N \) are real-analytic in \( \omega \).

The symplectic transformation \( \Phi \) being the time-1-flow of the \( K \)-truncation \( F \) is then of the following form:

\[
\Phi : \begin{pmatrix} \phi \\ l \\ x \end{pmatrix} \mapsto \begin{pmatrix} U(\phi; \omega) \\ V(\phi, l, x; \omega) \\ W(\phi, x; \omega) \end{pmatrix}
\]  

(3.9)

where \( W \) is affine in \( x \), and where the summands of \( V \) are either at most linear in \( l \) or quadratic in \( x \), like in a \( K \)-truncation, though the Fourier series is not necessarily truncated at \( K \).

The map \( \Phi \) is composed with the reparametrization \( \phi \) to give \( F \):

\[
F = \phi \circ \Phi.
\]

The reparametrization obviously has no influence on the structure properties (3.9). One also easily verifies that the structure of the transformation is preserved under composition. In fact, such transformations form a group, see [P1] and [Mo2].

3.2 Derivation of the homological equation

We linearize equation (3.1) and obtain an expression for the new perturbation.

The linearization

1. The perturbation \( P \) is \( K \)-truncated to obtain \( R \). This gives the following equation:

\[
( N + R ) \circ F = N_+.
\]

2. Recall that the transformation \( F \) is a composition \( \phi \circ \Phi \) of an \( \omega \)-dependent symplectic transformation \( \Phi \), which is taken as the time-1-flow of a Hamiltonian \( F \):

\[
\Phi = X^1_F,
\]

and a reparametrization \( \phi \), which we deal with later. We know from Differential Geometry (see for instance [A2], Section 40) that

\[
\frac{\partial H \circ X^1_F}{\partial t} = \{ H, F \} \circ X^1_F,
\]

and expand \( ( N + R ) \circ \Phi \) to first order in \( R, F \).
Lemma 3.4 (Derivation of the homological equation, [P1, Mo2]) \[ \{F, N\} + \hat{N} = R. \]

Regarding the above linearization the homological equation reads

\[ \{F, N\} + \hat{N} = R. \] (3.10)

Supposing this equation can be solved for a differential unperturbed Hamiltonian \( \hat{N} = N_+ - N \) and \( F \), the new perturbation becomes

\[ P_+ = \int_{t=0}^{1} ((1-t)\hat{N} + tR, F) \circ X_t^F \, dt \quad (P - R) \circ \Phi. \]

Proof. We repeat the argument from [P1]. Using Taylor’s formula with remainder (see for instance [Ap1], Section 7.5) to expand \( N \circ \Phi \) to first and \( R \circ \Phi \) to zeroth order in \( F \) gives

\[
(N + R) \circ \Phi = N \circ X_t^F + R \circ X_t^F
\]

\[
= N + \{N, F\} + \int_0^1 (1-t)\{\{N, F\}, F\} \circ X_t^F \, dt +
\]

\[
R + \int_0^1 \{R, F\} \circ X_t^F \, dt
\]

\[
= N + R + \{N, F\} + \int_0^1 (1-t)\{N, F\} + R, F \} \circ X_t^F \, dt. \quad (3.11)
\]

The latter integral is “quadratic” in \( R \) and \( F \) and will be part of the new perturbation term \( P_+ \).

Regarding our linearization, the point is to find \( F \) such that

\[ \hat{N}_+ := N + R + \{N, F\} \]

(3.12)

is indeed a preliminary unperturbed Hamiltonian. Equivalently, setting \( \hat{N} = \hat{N}_+ - N \), this amounts to solving the homological equation

\[ \{F, N\} + \hat{N} = R \]

in \( F \) and \( \hat{N} \), when the unperturbed Hamiltonian \( N \) and the truncated perturbation \( R \) are given. Suppose for the moment that such a solution exists, we rewrite the first argument of the Poisson bracket in the integral in (3.11) to

\[ (1-t)\{N, F\} + R = (1-t)\hat{N} + tR. \]

(3.13)

Taking together equations (3.11,3.12,3.13) gives

\[ H \circ \Phi = (N + R + (P - R)) \circ \Phi \]

\[ = \hat{N}_+ + \int_0^1 \{((1-t)\hat{N} + tR, F) \circ X_t^F \, dt + (P - R) \circ \Phi. \]

Note that the integral is quadratic in \( R, F \) and \( \hat{N} \). □
3.3 Results from unfolding theory

We use the unfolding theory of symplectic matrices to justify the choice of the class \( \mathcal{N} \) of unperturbed Hamiltonians: When we forget about the torus variables \((\varphi, I)\), a universal unfolding of the \( 1 : 1 : \ldots : 1 \) resonance gives an appropriate class of unperturbed Hamiltonians. Such an unfolding is given here, as well as a linear algebra result that helps to generalize the result. A discussion of unfolding theory, containing proofs of all results stated in this section, is postponed to Chapter 6.

Properties of the class \( \mathcal{N} \) of unperturbed Hamiltonians  Recall that an important problem in this chapter is to find an appropriate class \( \mathcal{N} \) of unperturbed Hamiltonians that contains the linearized Hamiltonian (2.14) of a row of uncoupled identical pendulums:

\[
N_0(\varphi, I, q, p; \omega) = \omega^2 I + \frac{1}{2}(q^\top q + p^\top p),
\]

all of whose normal frequencies equal 1. The class \( \mathcal{N} \) must be structurally stable, in the sense that perturbation by a sufficiently small \( K \)-truncation \( R \) of any member \( N \in \mathcal{N} \) can be transformed back to \( \mathcal{N} \) by a transformation \( F \) that is close to the identity and analytically dependent on \( N \) and \( R \).

Focusing on the linear part of the normal dynamics  Within the class of integrable normal linear Hamiltonians (where by definition the normal dynamics is decoupled from the position on the torus) it seems sensible to focus on the linear dynamics normal to the torus, and investigate what happens to this dynamics under perturbations by small linear Hamiltonian fields.

Unfoldings  A natural way to do so is to consider a universal unfolding of

\[
H_{1:1: \ldots : 1} = \frac{1}{2}(q^\top q + p^\top p).
\]

Roughly speaking, this is a family of quadratic Hamiltonians depending analytically on parameters and reducing to \( H_{1:1: \ldots : 1} \) for a specific choice of parameter values. Such a family is called versal if it is structurally stable under small perturbations in the class of parametrized quadratic Hamiltonians. A versal unfolding is called universal if the number of parameters is minimal.

Theorem 3.5 (A universal unfolding of the \( 1 : 1 : \ldots : 1 \) resonance)  A universal unfolding of the Hamiltonian \( H_{1:1: \ldots : 1} \) of the \( 1 : 1 : \ldots : 1 \) resonance is given by

\[
H_{AS}^R(q, p) = \frac{1}{2} \left( \begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} 1 + S & -A \\ A & 1 + S \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \right),
\]

where \( A \) is a skew-symmetric and \( S \) a symmetric \( m \times m \) matrix. These matrices play the role of parameters. The dimension of the parameter space is \( m^2 \).
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Remarks
1. The superscript $l$ indicates the particular universal unfolding that will be obtained in Chapter 6.
2. The above universal unfolding of the $1 : 1 : \ldots : 1$ resonance is considerably more complicated than the obvious one for the Hamiltonian

$$H(q, p) = \sum_{j=1}^{m} \frac{1}{2} \lambda_j (q_j^2 + p_j^2),$$

where $|\lambda_i| \neq |\lambda_j|$ whenever $i \neq j$.

In this case a universal unfolding is given by

$$H_\mu(q, p) = \sum_{j=1}^{m} \frac{1}{2} (\lambda_j + \mu_j) (q_j^2 + p_j^2),$$

where $\mu = (\mu_1, \ldots, \mu_m)$ is an $m$-vector of unfolding parameters, see for instance [BHT, Sec. 2b].

3. The following lemma gives the normal frequencies $\lambda = (\lambda_1, \ldots, \lambda_m)$ and is of some technical value:

**Lemma 3.6 (Diagonalizability of the universal unfolding)**

Any member $H_{l,A,S}^{lc}$ of the unfolding in Theorem 3.5 can be diagonalized by a linear coordinate transformation $T$, which depends on $(A, S)$ and is symplectic as well as orthogonal. In coordinates $\tilde{x} = T^T x$ the Hamiltonian $H_{A,S}^{lc}(x)$ reads

$$H_{A,S}^{lc}(T\tilde{x}) = \frac{1}{2} (\tilde{x}, \Lambda_{2m}\tilde{x}), \quad \Lambda_{2m} := \text{diag}(\lambda_1, \ldots, \lambda_m, \lambda_1, \ldots, \lambda_m).$$

It is well-known that a universal unfolding versally unfolds any member of the unfolding (apart from a trivial reparametrization). In our case, a much stronger result holds: Extending the universal unfolding $H_{A,S}^{lc}$ of the $1 : 1 : \ldots : 1$ resonance to the globally defined family

$$H = \{ H_{A,S}^{lc}(q, p) : A^T = -A, S^T = S \},$$

(hence no restrictions on the norms of $A$ and $S$) we have the following extension of Theorem 3.5:

**Theorem 3.7 (An extension of the unfolding result)** Any element of the global family $H$ for which the frequencies $\lambda$ satisfy the nonresonance conditions

$$\frac{\lambda_i}{\lambda_j} \neq \frac{0}{-\lambda_i}, \quad i, j \in \{1, \ldots, m\},$$

has a versal unfolding contained in $H$ itself.

Note that we want $\lambda$ to satisfy a similar property. However, for proving Theorem 3.3 on existence, uniqueness and analyticity of homological solutions, it is convenient to make one step from Theorem 3.7 back to linear algebra.
Some definitions  We first give an explicit representation of the linear complement to $\mathcal{H}$:

$$
\mathcal{H}^\perp = \left\{ \begin{pmatrix} \frac{q}{p} \\ S_1 \\ -S_2 \end{pmatrix} : S_1 = S_1^T, S_2 = S_2^T \right\}, \quad (3.15)
$$

and define the adjoint operator $\text{ad}_H$ of $H$ as

$$
\text{ad}_H := \{\cdot, H\}.
$$

**Theorem 3.8 (Des Pudels Kern, [Go])**  Suppose that $H \in \mathcal{H}$, and that its frequencies are $\lambda = (\lambda_1, \ldots, \lambda_m)$. We then have

1. The $4m^2$ eigenvalues of the adjoint operator $\text{ad}_H$ are
   $$
   \pm i(\lambda_i \pm \lambda_j) \text{ with } i, j \in \{1, \ldots, m\}.
   $$

2. The adjoint operator $\text{ad}_H$ respects the splitting $\mathcal{H} \oplus \mathcal{H}^\perp$.

Supposing moreover that the frequencies $\lambda$ of $X_{AS}$ are nonzero and do not contain $1 : -1$ resonances, i.e., satisfy the following nonresonance conditions $\lambda_i + \lambda_j \neq 0$ for all $i, j \in \{1, \ldots, m\}$, we have

3. The kernel of the adjoint operator of $H$ is contained in $\mathcal{H}$.

4. The adjoint operator $\text{ad}_H$ is invertible on $\mathcal{H}^\perp$.

Using this theorem, it is easy to prove Theorem 3.3. We recall that proofs of the above results as well as background and examples to unfolding theory will be given in Chapter 6.

### 3.4 Solving the homological equation

We prove Theorem 3.3 on existence, uniqueness and analyticity of homological equations, using Theorem 3.8.

**The adjoint operator “conserves degrees and wave vector”**  Let us apply the adjoint operator

$$
\text{ad}_N = \{\cdot, N\},
$$

where $N$ is an unperturbed Hamiltonian (see Definition 3.1), to a Fourier-Taylor monomial:

$$
\text{ad}_N\left(e^{i(k, \omega)l}x^\sigma\right) = e^{i(k, \omega)l} \left( i\langle k, \omega \rangle x^\sigma + \sum_{\ell-1}^{2m} \hat{a}_{\ell} x^{\sigma-\ell}( -JW_A x) \right).
$$

We see that the image under the adjoint operator of a Fourier-Taylor monomial is a sum of monomials with the same wave vector $k$ and degrees in $l$ and $x$ as the
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original monomial. Since $\alpha_{N}$ is linear, we can decompose $F$, $\tilde{N}$, and $R$ into finite sums of terms with a specific wave vector $k$, and of a specific degree in $x$, and solve the homological equation for the various wave vectors and degrees separately. This also implies that $F$ can indeed be taken $K$-truncated, just like $R$. The advantage of decomposing the homological equation is that for each subequation we can suppress the multi-index notation, which gets rather intractable.

The proof of Theorem 3.3 distinguishes the cases of degree 0, 1, and 2 in $x$.

3.4.1 Degree zero in $x$

**Notation** We first introduce a notation that partially gets around the multi-index notation. The coefficients $F_{k,0}^{j}$ for the $\text{deg} = 1$ terms are defined by

$$F_{k,0}^{j} := F_{k,j,0},$$

and $F_{k,0}^{j}$ is the $n$-vector whose coefficients are $F_{k,0}^{j}$, $j = 1, \ldots, n$, i.e.,

$$F_{k,0}^{j} = \left( F_{10,k}^{j} F_{20,k}^{j} \cdots F_{n0,k}^{j} \right).$$

In this notation the $\text{deg}_{x} = 0$ part of $F$ is:

$$\sum_{|I| \leq K} e^{i(k,\omega)} \left( F_{00} + (F_{k,0}^{j})^\top I \right).$$

**Homological equation** Using the same notation for the coefficients in the series of $R$, the homological equation (3.8) for $\text{deg} = 0$ and wave vector $k$ reads

$$i(k,\omega) \left( F_{00} + (F_{k,0}^{j})^\top I \right) + \delta_{k,0} \omega^\top I = \left( R_{00} + (R_{k,0}^{j})^\top I \right),$$

(3.17)

where $\delta$ is Kronecker’s delta. The cases $k = 0$ and $k \neq 0$ are now treated separately:

**Wave vector zero** We may assume $R_{00} = 0$ since constants in a Hamiltonian have no influence on the dynamics. Analogously we take

$$F_{00} = 0.$$  

(3.18)

The remaining part of the $K$-truncation $F$ with $\text{deg} = 0$, $k = 0$ is in the space $\mathcal{N}$ of unperturbed Hamiltonians, and therefore taken to be zero by the hypothesis in Theorem 3.3:

$$F_{00}^{i} = 0, \quad i = 1, 2, \ldots, n.$$  

(3.19)

This fixes a unique solution:

$$\omega_{i} = R_{00}^{i}, \quad i = 1, 2, \ldots, n.$$  

(3.20)
Nonzero wave vector Provided the following particular $K$-nonresonance condition holds
\[ \langle k, \omega \rangle \neq 0 \text{ for } 0 < |k| \leq K, \]
a quick look at equation (3.17) confirms that the kernel of $\text{ad}_N$ contains no deg$_x = 0$ terms with $0 < |k| \leq K$. The unique solution is therefore:
\[ F_{k00} = \frac{R_{k00}}{i(k, \omega)}, \quad F_{k0} = \frac{R_{k0}}{i(k, \omega)}. \] (3.21)
The coefficients of $F$ are evidently analytic in $\omega$ and in the coefficients of $R$. This finishes the proof of Theorem 3.3 for terms of degree zero in $x$.

3.4.2 Degree one in $x$

Notation We define $F_{k0}^i$ as the $2m$-vector with components
\[ F_{k0}^i := F_{k0,i}, \]
and likewise for $R$. In this notation the deg$_x = 1$ part of $F$ is
\[ \sum_{i \neq k} e^{i \langle k, \phi \rangle} (F_{k0}^i)^\top x. \]
Notice that since unperturbed Hamiltonians do not contain terms linear in $x$, the hypothesis for uniqueness in Theorem 3.3 is automatically satisfied for deg$_x = 1$.

Homological equation For wave vector $k$ and deg$_x = 1$ the homological equation reads
\[ (F_{k0}^i)^\top \left( i(k, \omega) I - JW_{AS} \right) x = (R_{k0}^i)^\top x. \] (3.22)
This equation can be solved uniquely for all $R_{k0}^i$ if and only if the matrix $(i(k, \omega) I - JW_{AS})$ is invertible, which is the case precisely when all eigenvalues of $JW_{AS}$ are unequal to $i(k, \omega)$, $|k| \leq K$. Recalling that the eigenvalues of $JW_{AS}$ are $\pm i \lambda_1, \ldots, \pm i \lambda_m$, we have invertibility precisely if
\[ \lambda_j \neq \pm (k, \omega), \ |k| \leq K. \] (3.23)
These conditions are satisfied when the $K$-nonresonance conditions are. The homological equation is uniquely solved by
\[ (F_{k0}^i)^\top = (R_{k0}^i)^\top \left( i(k, \omega) I - JW_{AS} \right)^{-1}. \] (3.24)
For analytic dependence of the coefficients $F_{k0}^i$ on $\omega$ and on the elements of $A$ and $S$, we use the fact that the elements of the inverse of a matrix depend analytically on the elements of the original matrix (provided the inverse exists). This finishes the proof of Theorem 3.3 for terms of degree one in $x$. 

3.4.3 Degree two in $x$

**Notation** We denote the wave vector $k$ and $\text{deg}_x = 2$ part of $F$ by

$$\frac{1}{2}e^{ik\varphi}x^\top F_kx,$$

where $F_k$ is a symmetric $2m \times 2m$ matrix, and use the same notation for $R$. 

**Homological equation** The $\text{deg}_x = 2$ homological equation reads

$$\left\{ \frac{1}{2}e^{ik\varphi}x^\top F_kx, \omega^\top I + \frac{1}{2}x^\top W_{A,S}x \right\} + \delta_{i0} \frac{1}{2}x^\top W_{A,S-i}x = e^{i(k\varphi)}\frac{1}{2}x^\top R_kx.$$

Computing the Poisson bracket of the first argument with the first term of the second argument gives the following equation

$$e^{i(k\varphi)}\left( i\langle k, \omega \rangle \frac{1}{2}x^\top F_kx + \left\{ \frac{1}{2}x^\top F_kx, \frac{1}{2}x^\top W_{A,S}x \right\} \right) + \delta_{i0} \frac{1}{2}x^\top W_{A,S-i}x = e^{i(k\varphi)}\frac{1}{2}x^\top R_kx,$$

where the Poisson bracket now involves only derivatives with respect to $x$. Again we consider the cases $k = 0$ and $k \neq 0$ separately:

**Wave vector zero** The homological equation now reads

$$\left\{ \frac{1}{2}x^\top F_0x, \frac{1}{2}x^\top W_{A,S}x \right\} + \frac{1}{2}x^\top W_{A,S-i}x = \frac{1}{2}x^\top R_0x,$$

Provided $\frac{1}{2}x^\top W_{A,S}x$ has no zero frequencies nor opposite frequencies, i.e., assuming the inequalities

$$\lambda_i \neq -\lambda_j, \ i, j = 1, \ldots, m,$$

which are part of the $K$-nonresonance conditions, Theorem 3.8 tells that the adjoint operator $\text{ad} \frac{1}{2}x^\top W_{A,S}x$ respects the splitting $\mathcal{H} \oplus \mathcal{H}^\perp$ and is invertible on $\mathcal{H}^\perp$.

We split the perturbation term $\frac{1}{2}x^\top R_0x$ into

$$\frac{1}{2}x^\top R_0x = \mathcal{H}\left(\frac{1}{2}x^\top R_0x\right) + \mathcal{H}^\perp\left(\frac{1}{2}x^\top R_0x\right),$$

i.e., into a part in the kernel $\mathcal{H}$ and a part in the image $\mathcal{H}^\perp$ of the adjoint operator of the $1:1: \ldots:1$ resonance. The unique solution is now given by

\[
\frac{1}{2}x^\top W_{A,S-i}x = \mathcal{H}\left(\frac{1}{2}x^\top R_0x\right) \quad (3.25)
\]

\[
\frac{1}{2}x^\top F_0x = \left(\text{ad} \frac{1}{2}x^\top W_{A,S}x\right)^{-1} \mathcal{H}^\perp\left(\frac{1}{2}x^\top R_0x\right). \quad (3.26)
\]
Nonzero wave vector The homological equation for this case is
\[
\left( i\langle k, \omega \rangle \frac{1}{2} x^\top F_k x + \left\{ \frac{1}{2} x^\top F_k x, \frac{1}{2} x^\top W_{A,S} x \right\} \right) = \frac{1}{2} x^\top R_k x.
\]
Using the first part of Theorem 3.8, we find that the linear operator
\[
\left( i\langle k, \omega \rangle + \text{ad}_{\frac{1}{2} x^\top W_{A,S} x} \right)
\]
has eigenvalues
\[
i \left( \langle k, \omega \rangle \pm \lambda_i \pm \lambda_j \right), \quad i, j = 1, \ldots, m.
\]
If the $K$-nonresonance conditions (Definition 3.2) are satisfied, these eigenvalues are nonzero for $0 < |k| \leq K$, and the linear operator is invertible. In the present finite-dimensional situation, and away from zero eigenvalues, the inverse depends analytically on the original operator. This finishes the proof of Theorem 3.3. \qed