The Schur transform of a generalized Schur function and operator realizations

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Document Version
Publisher's PDF, also known as Version of record

Publication date:
2005

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

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Chapter 3

Operator realizations

3.1 The general case

An operator colligation is a quadruple \((\mathcal{K}, \mathcal{F}, \mathcal{G}, V)\) consisting of Krein spaces \(\mathcal{K}, \mathcal{F}\) and \(\mathcal{G}\) and a mapping \(V\) from \(\mathcal{K} \oplus \mathcal{F}\) into \(\mathcal{K} \oplus \mathcal{G}\). We call \(\mathcal{K}\) the state space, \(\mathcal{F}\) the inner space, \(\mathcal{G}\) the outer space, and \(V\) the connecting operator of the colligation. The terms input and output are also used instead of inner and outer respectively. We identify the orthogonal sum \(\mathcal{K} \oplus \mathcal{F}\) with the space \(\mathcal{K}F\) consisting of elements \(\begin{pmatrix} x \\ f \end{pmatrix}\) with \(x \in \mathcal{K}\) and \(f \in \mathcal{F}\) and equipped with the inner product

\[
\langle \begin{pmatrix} x \\ f \end{pmatrix}, \begin{pmatrix} y \\ g \end{pmatrix} \rangle_{\mathcal{K} \oplus \mathcal{F}} = \langle x, y \rangle_{\mathcal{K}} + \langle f, g \rangle_{\mathcal{F}}, \quad x, y \in \mathcal{K}, \ f, g \in \mathcal{F}.
\]

With this identification the connecting operator \(V\) has an operator matrix representation of the form

\[
V := \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \left( \begin{array}{c} \mathcal{K} \\ \mathcal{F} \end{array} \right) \to \left( \begin{array}{c} \mathcal{K} \\ \mathcal{G} \end{array} \right),
\]

where \(T \in \text{B}(\mathcal{K}), \ F \in \text{B}(\mathcal{F}, \mathcal{K}), \ G \in \text{B}(\mathcal{K}, \mathcal{G})\) and \(H \in \text{B}(\mathcal{F}, \mathcal{G})\). We call \(T\) the main operator for the colligation. From now on we shall identify the colligation \((\mathcal{K}, \mathcal{F}, \mathcal{G}, V)\) with its connecting operator

\[
\text{(3.1.1)} \quad V := \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \left( \begin{array}{c} \mathcal{K} \\ \mathcal{F} \end{array} \right) \to \left( \begin{array}{c} \mathcal{K} \\ \mathcal{G} \end{array} \right).
\]

We say that the colligation \(V\) is coisometric if \(VV^* = I\), that it is isometric if \(V^*V = I\), and that it is unitary if it is both coisometric and isometric. Two colligations

\[
\text{(3.1.2)} \quad V_1 := \begin{pmatrix} T_1 & F_1 \\ G_1 & H_1 \end{pmatrix} : \left( \begin{array}{c} \mathcal{K}_1 \\ \mathcal{F}_1 \end{array} \right) \to \left( \begin{array}{c} \mathcal{K}_1 \\ \mathcal{G}_1 \end{array} \right)
\]

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and
\[(3.1.3) \quad V_2 := \left( \begin{array}{cc} T_2 & F_2 \\ G_2 & H_2 \end{array} \right) : \left( \begin{array}{c} \mathcal{K}_2 \\ \mathcal{F} \end{array} \right) \rightarrow \left( \begin{array}{c} \mathcal{K}_2 \\ \mathcal{G} \end{array} \right)\]

are said to be \textit{isomorphic} if there exists an isomorphism \( W : \mathcal{K}_1 \rightarrow \mathcal{K}_2 \) such that
\[(3.1.4) \quad T_2 = WT_1 W^{-1}, \quad F_2 = WF_1, \quad G_2 = G_1 W^{-1}, \quad \text{and} \quad H_2 = H_1.\]

The colligation \( V \) is said to be \textit{closely inner connected} if
\[(3.1.5) \quad \mathcal{K} = \text{span} \{ T^n F f \mid f \in \mathcal{F}, \ m = 0, 1, 2, \ldots \},\]
\textit{closely outer connected} if
\[(3.1.6) \quad \mathcal{K} = \text{span} \{ T^n G^* g \mid g \in \mathcal{G}, \ n = 0, 1, 2, \ldots \},\]
and \textit{closely connected} if
\[(3.1.7) \quad \mathcal{K} = \text{span} \{ T^n F f, T^n G^* g \mid f \in \mathcal{F}, \ g \in \mathcal{G}, \ m, n = 0, 1, 2, \ldots \},\]
or equivalently, for some neighborhood \( \Omega \) of the origin,
\[(3.1.8) \quad \mathcal{K} = \text{span} \{ (I - wT)^{-1} F f \mid f \in \mathcal{F}, \ w \in \Omega \},\]
\[(3.1.9) \quad \mathcal{K} = \text{span} \{ (I - zT^*)^{-1} G^* g \mid g \in \mathcal{G}, \ z \in \Omega \},\]
and
\[(3.1.10) \quad \mathcal{K} = \text{span} \{ (I - wT)^{-1} F f, (I - zT^*)^{-1} G^* g \mid f \in \mathcal{F}, \ g \in \mathcal{G}, \ w, z \in \Omega \},\]
respectively. The terms “inner” and “outer” as used here refer to the inner and outer spaces \( \mathcal{F} \) and \( \mathcal{G} \) of the colligation. Conditions (3.1.5), (3.1.6), and (3.1.7) (or (3.1.8), (3.1.9), and (3.1.10)) are called \textit{minimality conditions}. From now on we shall use the terms \textit{minimal isometric colligation}, \textit{minimal coisometric colligation}, and \textit{minimal unitary colligation} to refer to a closely inner connected isometric colligation, a closely outer connected coisometric colligation, and a closely connected unitary colligation respectively.

The \textit{characteristic function} \( S_V(z) \) of the colligation \( V \) in (3.1.1) is the function defined by
\[(3.1.11) \quad S_V(z) = H + zG(I_{\mathcal{K}} - zT)^{-1}F,\]
where we take \( \Omega(S_V) \), the domain of \( S_V(z) \), to be the component of the origin in the complex plane such that the operator \( I_{\mathcal{K}} - zT \) is invertible. From (3.1.4) and (3.1.11) we see that if two colligations \( V_1 \) and \( V_2 \) are isomorphic,
3.1. The general case

then they give rise to the same characteristic function. On the other hand, assume that \( S_{V_1}(z) = S_{V_2}(z) \) in a neighborhood of the origin for two colligations (3.1.2) and (3.1.3) such that \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are Pontryagin spaces. If the two colligations are of the same type, that is, both are minimal isometric, minimal coisometric, or minimal unitary, then \( V_1 \) and \( V_2 \) are isomorphic (see [10, Theorem 2.1.3]).

Given \( S(z) \), the question arises if there exists a colligation \( V \) such that

\[
S(z) = S_V(z)
\]

in a neighborhood of the origin. If such a colligation exists, the identity (3.1.12) is called a realization of \( S(z) \) induced by the colligation \( V \). The realization is called minimal isometric, minimal coisometric, or minimal unitary if the corresponding colligation has the same property.

Recall here that \( S_\kappa(\mathbb{C}^n, J) \) denotes the class of \( J \)-generalized Schur functions with \( \kappa \) negative squares. We set

\[
S_\kappa^0(\mathbb{C}^n, J) = S_\kappa(\mathbb{C}^n, J) \cap A_n^0
\]

where \( A_n^0 \) is the set of all \( n \times n \) matrix functions which are holomorphic in a neighborhood of the origin in the complex plane, and let

\[
S^0(\mathbb{C}^n, J) = \cup_{\kappa \in \mathbb{N}} S_\kappa^0(\mathbb{C}^n, J),
\]

where \( \mathbb{N} \) is the set of natural numbers.

**Theorem 3.1.1** Let \( S(z) \in S^0(\mathbb{C}^n, J) \).

(i) There exist a Pontryagin space \( \mathcal{K} \) and a colligation

\[
V := \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \mathcal{K} \oplus \mathbb{C}^n \to \mathcal{K} \oplus \mathbb{C}^n
\]

such that the characteristic function

\[
s_V(z) = H + zG^*(I-zT)^{-1}F
\]

associated with \( V \) coincides with \( S(z) \) in a neighborhood of \( z = 0 \).

(ii) In this representation \( V \) can always be chosen such that either,

(a) \( V \) is minimal coisometric in the space \( \mathcal{K} \oplus \mathbb{C}^n \), or

(b) \( V \) is minimal isometric in the space \( \mathcal{K} \oplus \mathbb{C}^n \), or

(c) \( V \) is minimal unitary in the space \( \mathcal{K} \oplus \mathbb{C}^n \).

In these cases the operator \( V \) in the representation for \( s(z) \) is uniquely determined up to an isomorphism of the space \( \mathcal{K} \) and \( \text{ind}_-(\mathcal{K}) = \text{sq}_-(S) \).

For the proof of this theorem we refer to [21, Theorem 5.2, Corollary 5.4 and Theorem 5.6]. The theorem still holds when \( \mathbb{C}^n \) is replaced by \( \mathbb{C}^n_J \) for some \( n \times n \) signature operator \( J \) (see [10, Theorems 2.2.1, 2.2.2, 2.3.1] and [22, Theorem 2.2]).
3.2 Canonical colligations

In this section we consider the special case of the colligation

\[ V := \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} K \\ F \end{pmatrix} \to \begin{pmatrix} K \\ G \end{pmatrix} \]

in which \( F = G = C \) for some signature matrix \( J \). We introduce the canonical colligations in a form which is convenient for our purposes. These colligations play a crucial role in this thesis. Although we can adapt the method of proof of [10, Sections 2.2, 2.3], we prefer to use the results in these sections to give much simpler proofs.

**Theorem 3.2.1** Let \( S(z) \in S^0(C^n, J) \).

(i) The operators

\[
T_S : P(S) \to P(S), \quad T_S k(z) = \frac{k(z) - k(0)}{S(z) - S(0)},
\]

\[
F_S : C^n \to P(S), \quad F_S \alpha(z) = \frac{S(z) - S(0)}{z} \alpha,
\]

\[
G_S : P(S) \to C^n, \quad G_S k = k(0),
\]

\[
H_S : C^n \to C^n, \quad H_S \alpha = S(0) \alpha
\]

for all \( k \in P(S), \alpha \in C^n \), are bounded operators.

(ii) The colligation

\[ V_S = \begin{pmatrix} T_S & F_S \\ G_S & H_S \end{pmatrix} : \begin{pmatrix} P(S) \\ C^n \end{pmatrix} \to \begin{pmatrix} P(S) \\ C^n \end{pmatrix} \]

is \( J \)-coisometric, that is,

\[ V_S \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} V_S^* = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}, \]

and minimal.

(iii) \( S(z) \) has the corresponding minimal \( J \)-coisometric realization:

\[ S(z) = H_S + zG_S(I - zT_S)^{-1}F_S \text{ near } z = 0. \]

(3.2.1)

The colligation \( V \) in the theorem is unique and is called the canonical \( J \)-coisometric colligation for \( S(z) \) and the realization (3.2.1) is called the canonical \( J \)-coisometric realization of \( S(z) \).

**Proof of Theorem 3.2.1.** To prove part (i), let \( K_S(z, w) \) and \( H_S(z, w) \) be the kernels defined in (2.1.2) and (2.3.1) respectively and let \( \iota : H(S) \to P(S) \) be the inclusion mapping of Theorem 2.3.5. Let \( T, F, G, \) and \( H \) be
the bounded operators defined in [10, Theorem 2.2.1], that is,

\[
T : \mathcal{H}(S) \to \mathcal{H}(S), \quad Th(z) = \frac{h(z) - h(0)}{z},
\]

\[
F : \mathbb{C}^k_0 \to \mathcal{H}(S), \quad F\alpha(z) = \frac{S(z) - S(0)}{z}\alpha,
\]

\[
G : \mathcal{H}(S) \to \mathbb{C}^k_0, \quad Gh = h(0),
\]

\[
H : \mathbb{C}^k_0 \to \mathbb{C}^k_0, \quad H\alpha = S(0)\alpha
\]

for all \( h \in \mathcal{H}(S) \) and \( \alpha \in \mathbb{C}^k_0 \). The colligation

\[
V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \begin{pmatrix} \mathcal{H}(S) \\ \mathbb{C}^k_0 \end{pmatrix} \begin{pmatrix} \mathcal{H}(S) \\ \mathbb{C}^k_0 \end{pmatrix}
\]

is minimal coisometric and \( S(z) = S_V(z) \) near \( z = 0 \). Let \( \nu \) be the identity operator from \( \mathbb{C}^n \) onto \( \mathbb{C}^n \) as linear spaces. Then

\[
T_S = \nu T^{-1} = \nu T^*, \quad F_S = \nu F, \quad G_S = \nu^{-1}G, \quad \text{and} \quad H_S = \nu^{-1}H, \nu.
\]

Since \( T, F, G, \nu \), and \( \nu \) are bounded operators, (i) follows immediately.

We now show that \( V_S \) is \( J \)-coisometric. For \( \alpha, \beta \in \mathbb{C}^n \) we have

\[
\langle \nu \alpha, \beta \rangle_{\mathbb{C}^n} = \beta^* J \alpha = \langle \alpha, J \beta \rangle_{\mathbb{C}^n}
\]

and so \( \nu^* = J \). Similarly, \( \nu^{-*} = J \). It therefore follows that

\[
V_S \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} V_S^* = \begin{pmatrix} T_S & F_S \\ G_S & H_S \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} T_S^* & G_S^* \\ F_S^* & H_S^* \end{pmatrix}
\]

\[
= \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix} \begin{pmatrix} T & F \\ G & H \end{pmatrix} \begin{pmatrix} \nu^* & 0 \\ 0 & \nu \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix} \begin{pmatrix} T^* & G^* \\ F^* & H^* \end{pmatrix} \begin{pmatrix} \nu^* & 0 \\ 0 & \nu^{-*} \end{pmatrix}
\]

\[
= \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix} \begin{pmatrix} T^* & G^* \\ F^* & H^* \end{pmatrix} \begin{pmatrix} \nu^* & 0 \\ 0 & \nu^{-*} \end{pmatrix}
= \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}
\]

To prove minimality we use the fact that \( \mathcal{P}(S) = \nu \mathcal{H}(S) \) (see Theorem 2.3.5). Now

\[
\mathcal{P}(S) = \nu \mathcal{H}(S) = \overline{\text{span}} \{ \nu^\ell G^\ell \alpha \mid \alpha \in \mathbb{C}^k_0, \ \ell = 0, 1, 2, \ldots \}
\]

\[
= \overline{\text{span}} \{ \nu^\ell T_S^\ell \nu G^\ell J \alpha \mid J \alpha \in \mathbb{C}^n, \ \ell = 0, 1, 2, \ldots \}
\]

\[
= \overline{\text{span}} \{ \nu^\ell T_S^\ell G_S^\ell \beta \mid \beta \in \mathbb{C}^n, \ \ell = 0, 1, 2, \ldots \}
\]

Here the closure in the first three lines is taken in \( \mathcal{H}(S) \) and that in the last line is taken in \( \mathcal{P}(S) \). Lastly, for \( w \in \Omega(S) \cap \Omega(S_V) \) we have that

\[
S_{V_S}(w) = H_S + wG_S (I - wT_S)^{-1} F_S = \nu^{-1} \left[ H + wG (I - wT_S)^{-1} F \right] \nu
\]

\[
= H + wG (I - wT_S)^{-1} F = S_V(w) = S(z).
\]
**Theorem 3.2.2** Let $S(z) \in S^0(C^n, J)$.

(i) The operators

$$
\begin{align*}
\tilde{T}_S^* : \mathcal{P}(S^\#) &\to \mathcal{P}(S^\#), & \tilde{T}_S^* k(z) &= \frac{k(z) - k(0)}{z}, \\
\tilde{F}_S^* : \mathcal{P}(S^\#) &\to C^n, & \tilde{F}_S^* k &= k(0), \\
\tilde{G}_S^* : C^n &\to \mathcal{P}(S^\#), & \tilde{G}_S^* \beta(z) &= \frac{S^\#(z) - S^\#(0)}{z} \beta, \\
\tilde{H}_S^* : C^n &\to C^n, & \tilde{H}_S^* \beta &= S^\#(0) \beta
\end{align*}
$$

for all $k \in \mathcal{P}(S^\#), \beta \in C^n$, are bounded operators.

(ii) Set $T_S = (T_S^\#)^*, \tilde{F}_S = (F_S^\#)^*, \tilde{G}_S = (G_S^\#)^*,$ and $\tilde{H}_S = (H_S^\#)^*$. The colligation

$$
\tilde{V}_S = \begin{pmatrix} \tilde{T}_S^* & \tilde{F}_S^* \\ \tilde{G}_S^* & \tilde{H}_S^* \end{pmatrix} : \left( \mathcal{K}(S^\#) C^n \right) \to \left( \mathcal{K}(S^\#) C^n \right)
$$

is $J$-isometric, that is,

$$
\tilde{V}_S^* \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \tilde{V}_S = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix},
$$

and minimal.

(iii) $S(z)$ has the corresponding minimal isometric realization:

$$
(3.2.2) \quad S(z) = \tilde{H}_S + z\tilde{G}_S (I - z\tilde{T}_S)^{-1}\tilde{F}_S \text{ near } z = 0.
$$

The colligation $\tilde{V}$ in the theorem is unique and is called the *canonical $J$-isometric colligation* for $S(z)$ and the realization (3.2.2) is called the canonical $J$-isometric realization of $S(z)$.

**Proof of Theorem 3.2.2.** We apply Theorem 3.2.1 with $S(z)$ replaced by $S^\#(z)$. Let

$$
W = \begin{pmatrix} \tilde{T}_S^* & \tilde{G}_S^* \\ \tilde{F}_S^* & \tilde{H}_S^* \end{pmatrix} : \left( \mathcal{K}(S^\#) C^n \right) \to \left( \mathcal{K}(S^\#) C^n \right)
$$

be the minimal $J$-coisometric colligation given in Theorem 3.2.1 and define $\tilde{V}$ by $\tilde{V} = W^*$. Then $\tilde{V}$ is a minimal $J$-isometric colligation, $\Omega(S_{\tilde{V}})$ is the set of complex conjugates of the numbers in $\Omega(S_W)$, and

$$
S^\#(z) = S_W(z) = \tilde{H}_S^* + z\tilde{G}_S^* (I_{\mathcal{K}(S^\#)} - z\tilde{T}_S)^{-1}\tilde{G}_S^*
$$

for all $z \in \Omega(S^\#) \cap \Omega(S_W)$. Replacing $z$ by its complex conjugate and taking adjoints, we get (3.2.2). The formulas for $\tilde{T}_S^*, \tilde{F}_S^*, \tilde{G}_S^*,$ and $\tilde{H}_S^*$ hold by construction.
**Theorem 3.2.3** Let $S(z) \in S^0(\mathbb{C}^n, J)$.

(i) The operators

$$A_S : \mathcal{D}(S) \to \mathcal{D}(S), \quad A_S \left( \frac{h}{k} \right)(z) = \left( \frac{h(z) - h(0)}{zk(z) - S^\#(z)Jh(0)} \right),$$

$$B_S : \mathbb{C}^n \to \mathcal{D}(S), \quad B_S \alpha(z) = \left( \frac{S(z) - S(0)}{z} \right) \alpha,$$

$$C_S : \mathcal{D}(S) \to \mathbb{C}^n, \quad C_S \left( \frac{h}{k} \right) = h(0),$$

$$D_S : \mathbb{C}^n \to \mathbb{C}^n, \quad D_S \alpha = S(0) \alpha$$

for all $\left( \frac{h}{k} \right) \in \mathcal{D}(S), \ \alpha \in \mathbb{C}^n$, are bounded operators.

(ii) Their adjoints are given by

$$A_S^* \left( \frac{h}{k} \right)(z) = \left( \frac{zh(z) - S(z)Jk(0)}{k(z) - k(0)} \right), \quad B_S^* \left( \frac{h}{k} \right)(z) = k(0),$$

and for $\alpha \in \mathbb{C}^n$,

$$(C_S^* \alpha)(z) = \left( \frac{J - S(z)JS^\#(0)}{S^\#(z) - S^\#(0)} \right) \alpha, \quad D_S^* \alpha = S(0)^* \alpha.$$

(iii) The colligation

$$U_S = \begin{pmatrix} A_S & B_S \\ C_S & D_S \end{pmatrix} : \begin{pmatrix} \mathcal{D}(S) \\ \mathbb{C}^n \end{pmatrix} \to \begin{pmatrix} \mathcal{D}(S) \\ \mathbb{C}^n \end{pmatrix}$$

is $J$-unitary on the space $\mathcal{D}(S) \oplus \mathbb{C}^n$, that is,

$$U_S \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} U_S^* = U_S^* \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} U_S = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix},$$

and minimal.

(iv) $S(z)$ has the minimal unitary realization

$$S(z) = D_S + zC_S(I - zT_S)^{-1}B_S \text{ near } z = 0.$$
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Proof of Theorem 3.2.3. To prove part (i) of the theorem, we let \( \omega : K(S) \to D(S) \) be the unitary map defined in Theorem 2.3.5 and let \( A, B, C, \) and \( D \) be the bounded operators defined in [10, Theorem 2.3.1], that is,

\[
A : K(S) \to K(S), \quad A \begin{pmatrix} h \\ k \end{pmatrix}(z) = \begin{pmatrix} \frac{h(z) - h(0)}{z} \\ zk(z) - S^\#(z)h(0) \end{pmatrix},
\]

\[
B : \mathbb{C}_j^n \to K(S), \quad B\beta(z) = \begin{pmatrix} \frac{S(z) - S(0)}{z} \\ I - S^\#(z)S^\#(0)^* \end{pmatrix} \beta,
\]

\[
C : K(S) \to \mathbb{C}_j^n, \quad C \begin{pmatrix} h \\ k \end{pmatrix} = h(0),
\]

\[
D : \mathbb{C}_j^n \to \mathbb{C}_j^n, \quad D\beta = S(0)\beta,
\]

for all \( \begin{pmatrix} h \\ k \end{pmatrix} \in K(S) \) and \( \beta \in \mathbb{C}_j^n \), with minimal canonical unitary colligation

\[
U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} K(S) \\ \mathbb{C}_j^n \end{pmatrix} \begin{pmatrix} K(S) \\ \mathbb{C}_j^n \end{pmatrix}
\]

such that \( S(z) = S_U(z) \) near \( z = 0 \) and adjoints given by

\[
A^* : K(S) \to K(S), \quad A^* \begin{pmatrix} h \\ k \end{pmatrix}(z) = \begin{pmatrix} \frac{zh(z) - S(z)k(0)}{k(z) - k(0)} \\ z \end{pmatrix},
\]

\[
B^* : K(S) \to \mathbb{C}_j^n, \quad B^* \begin{pmatrix} h \\ k \end{pmatrix} = k(0),
\]

\[
C^* : \mathbb{C}_j^n \to H(S), \quad C^*\beta(z) = \begin{pmatrix} \frac{I - S(z)S(0)^*}{z} \\ \frac{S^\#(z) - S^\#(0)}{z} \end{pmatrix} \beta,
\]

\[
D^* : \mathbb{C}_j^n \to \mathbb{C}_j^n, \quad D\beta = S(0)^*\beta.
\]

Let \( \nu \) be the identity mapping from \( \mathbb{C}_j^n \) onto \( \mathbb{C}_j^n \) as linear spaces. Then

\[
A_S = \omega A_\omega^{-1} = \omega A_\omega^*, \quad B_S = \omega B\nu, \quad C_S = \nu^{-1}C\omega^* \quad \text{and} \quad D_S = \nu^{-1}D\nu.
\]

Since all the operators involved are bounded, (i) follows immediately.

We now prove the formulas for the adjoints in part (ii). Let \( \begin{pmatrix} h \\ k \end{pmatrix} \in D(S) \).

From \( A_S = \omega A_\omega^* \) we get

\[
A_S^* \begin{pmatrix} h \\ k \end{pmatrix}(z) = \omega A^* \omega^* \begin{pmatrix} h \\ k \end{pmatrix}(z) = \omega A^* \begin{pmatrix} h \\ Jk \end{pmatrix}(z)
= \omega \begin{pmatrix} \frac{zh(z) - S(z)Jk(0)}{Jk(z) - Jk(0)} \\ \frac{zh(z) - S(z)Jk(0)}{k(z) - k(0)} \end{pmatrix} = \begin{pmatrix} zh(z) - S(z)Jk(0) \\ k(z) - k(0) \end{pmatrix},
\]
which is the desired formula. From $B_S = \omega B$ and $\nu^* = J$ we get

$$B^*_S \left( \frac{h}{k} \right) (z) = JB^* \omega^* \left( \frac{h}{k} \right) (z) = JB^* \left( \frac{h}{Jk} \right) (z) = k(0).$$

From $C_S = \nu^{-1}C \omega^*$ and $\nu^{-*} = J$ we get for $\alpha \in \mathbb{C}^n$

$$C^*_S \alpha(z) = \omega \left( \frac{I - S(z)S(0)^*}{S(z)^* - S(0)^*} \right) J \alpha(z)$$

$$= \omega \left( \frac{J - S(z)JS(0)^*}{J^* - S(0)^*} \right) J \alpha(z)$$

$$= \left( \frac{J - S(z)JS(0)^*}{S(z)^* - S(0)^*} \right) \alpha(z).$$

Lastly $D_S = \nu^{-1}D \nu$ implies that $D^*_S = JD^*J$ and so

$$D^*_S = JJS(0)^*JJ = S(0)^*.$$

This completes the proof of part (ii).

We now show that the colligation $U_S$ is $J$-unitary. We have

$$U_S \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} U^*_S = \begin{pmatrix} A_S & B_S \\ C_S & D_S \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A^*_S & C^*_S \\ B^*_S & D^*_S \end{pmatrix}$$

$$= \begin{pmatrix} \omega & 0 \\ 0 & \nu^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \omega^* & 0 \\ 0 & \nu \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \omega^* & 0 \\ \nu^{-*} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \omega & 0 \\ 0 & \nu^{-1} \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} \omega^* & 0 \\ 0 & J \end{pmatrix}$$

$$= \begin{pmatrix} \omega & 0 \\ 0 & \nu^{-1} \end{pmatrix} \begin{pmatrix} \omega^* & 0 \\ 0 & J \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}.$$

Hence

$$U_S \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} U^*_S = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}.$$

That

$$U^*_S \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} U_S = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$$

can be proved similarly.
To prove minimality we use the fact that $D(S) = \omega K(S)$ (see Theorem 2.3.5).

\[
D(S) = \omega K(S) \\
= \overline{\text{span} \{ A^\ell B \alpha, A^{\ast \ell} C^* \beta \mid \alpha, \beta \in \mathbb{C}^n, \ell, m = 0, 1, 2, \ldots \}} \\
= \overline{\text{span} \{ (\omega^* A^\ell_B \omega)^m \omega^* B \nu^{-1} \alpha, (\omega^* A^\ell_B \omega)^m \omega^* C^* \nu^* \beta \mid \alpha, \beta \in \mathbb{C}^n, m, \ell = 0, 1, 2, \ldots \}} \\
= \overline{\text{span} \{ \omega^* A^\ell_B \gamma, \omega^* A^{\ast \ell} C^* \delta \mid \gamma, \delta \in \mathbb{C}^n, m, \ell = 0, 1, 2, \ldots \}} \\
= \text{span} \{ A^\ell_B \gamma, A^{\ast \ell} C^* \delta \mid \gamma, \delta \in \mathbb{C}^n, m, \ell = 0, 1, 2, \ldots \}.
\]

Here the first three closures are taken in the space $K(S)$ while the last one is taken in $D(S)$.

Lastly, for $w \in \Omega(S) \cap \Omega(S_{U_S})$ we have that

\[
S_{U_S}(w) = D_S + w C_S (I - w A_S)^{-1} B_S = \nu^{-1} [D + w C (I - w A)^{-1} B] \nu \\
= \nu [D + w C (I - w A)^{-1} B = S_U(w) = S(w).
\]

\[\blacktriangleleft\]

### 3.3 The scalar case

Here we consider a special case of the colligation $V$ in (3.1.1) in which $\mathcal{F} = \mathcal{G} = \mathbb{C}$. In this case $V$ takes the form

\[
V = \begin{pmatrix} T & u \\ (\cdot, v)_K & \gamma \end{pmatrix} : \begin{pmatrix} K \\ \mathbb{C} \end{pmatrix} \rightarrow \begin{pmatrix} K \\ \mathbb{C} \end{pmatrix},
\]

where $u, v \in K$ with $F 1 = u$, $G^* 1 = v$ and $\gamma \in \mathbb{C}$. To show that the left lower entry of $V$ in (3.3.1) has the indicated form, consider $f \in K$. Then

\[
G f = \langle G f, 1 \rangle_K = \langle f, G^* 1 \rangle_K = \langle f, v \rangle_K
\]

so that $G = (\cdot, v)_K$. The characteristic function $s_V(z)$ of $V$ is given by

\[
s_V(z) = \gamma + z \langle (I_K - zT)^{-1} u, v \rangle.
\]

The domain of $s_V(z)$ is taken to be the component of the origin in the complex plane consisting of $z$ where the operator $I_K - zT$ is invertible. The colligation $V$ in (3.3.1) has adjoint

\[
V^* := \begin{pmatrix} T^* & v \\ (\cdot, u)^*_K & \gamma^* \end{pmatrix} : \begin{pmatrix} K \\ \mathbb{C} \end{pmatrix} \rightarrow \begin{pmatrix} K \\ \mathbb{C} \end{pmatrix}
\]
3.3. The scalar case

and so the fact that $V$ is coisometric can be expressed in terms of the entries $T, u, v, \gamma$ of $V$ by the three equations

\begin{align}
(3.3.2) & \quad TT^* + \langle \cdot, u \rangle u = I_K, \\
(3.3.3) & \quad \langle v, v \rangle + |\gamma|^2 = 1, \\
(3.3.4) & \quad Tv + \gamma^* u = 0.
\end{align}

That $V$ is isometric can be expressed by the three equations

\begin{align}
(3.3.5) & \quad T^* T + \langle \cdot, v \rangle v = I_K, \\
(3.3.6) & \quad \langle u, u \rangle + |\gamma|^2 = 1, \\
(3.3.7) & \quad T^* u + \gamma v = 0.
\end{align}

To express the fact that $V$ is unitary in terms of its entries, one needs the six equations (3.3.2)–(3.3.7).

Assume $s(z)$ is a holomorphic function in a neighborhood of the origin such that $s(z) = s_V(z)$ and let

\[ s(z) = \sigma_0 + \sigma_1 z + \sigma_2 z^2 + \cdots + \sigma_n z^n + \cdots, \quad \sigma_n = s^{(n)}(0)/n!, n = 1, 2, \ldots \]

be its Taylor expansion at 0. Then

\[
\begin{align*}
\sigma(z) &= \gamma + z\langle (I_K - zT)^{-1} u, v \rangle \\
&= \gamma + z\langle (I_K + zT + z^2 T^2 + \cdots + z^n T^n + \cdots) u, v \rangle \\
&= \gamma + z\langle u, v \rangle + z^2 \langle T u, v \rangle + \cdots + z^n \langle T^{n-1} u, v \rangle + \cdots
\end{align*}
\]

and so $\sigma_0 = \gamma$ and $\sigma_n = \langle T^{n-1} u, v \rangle$, $n = 1, 2, \ldots$.

We now state a lemma that we shall need in later chapters. We include the proof of this lemma as given in [1].

**Lemma 3.3.1** Let $s(z)$ be the characteristic function of a minimal coisometric, minimal isometric, or minimal unitary colligation of the form (3.3.1):

\[ s(z) = \gamma + z\langle (I_K - zT)^{-1} u, v \rangle, \]

and assume $\gamma = s(0) \neq 0$. Then the function $s_i(z)$ defined by $s_i(z) = s(z)^{-1}$ is the characteristic function of the colligation

\[ v_i = \begin{pmatrix} T_i & u_i \\ \langle \cdot, v_i \rangle_{K'} & \gamma_i \end{pmatrix} : (K' \oplus \mathbb{C}) \to (K' \oplus \mathbb{C}), \]

where $K'$ is the anti-space of the Krein space $K$.

\[ T_i = T - \frac{\langle \cdot, v \rangle_{K'}}{\gamma} u, \quad u_i = \frac{u}{\gamma}, \]

\[ v_i = \frac{v}{\gamma^*}, \quad \gamma_i = \frac{1}{\gamma}, \]
that is,

\[ s_i(z) = \gamma_i + z\langle (I - zT_i)^{-1}u_i, v_i \rangle. \]

Moreover, \( v_i \) is minimal coisometric, minimal isometric, or minimal unitary respectively.

**Proof** Set \( x = (1 - zT_i)^{-1}u_i \). Then the following equalities can be verified, one after the other:

\[
\begin{align*}
x &= u_i + zT_ix = \frac{u}{\gamma} + zTx + z\langle x, v_i \rangle' u, \\
(1 - zT)x &= \left( \frac{1}{\gamma} + z\langle x, v_i \rangle' \right) u, \\
x &= \left( \frac{1}{\gamma} + z\langle x, v_i \rangle' \right) (1 - zT)^{-1}u, \\
\langle x, v_i \rangle' &= -\frac{1}{\gamma} \left( \frac{1}{\gamma} + z\langle x, v_i \rangle' \right) \langle (1 - zT)^{-1}u, v \rangle, \\
\langle x, v_i \rangle' &= -\frac{1}{\gamma} \frac{\langle (1 - zT)^{-1}u, v \rangle}{\gamma + z\langle (1 - zT)^{-1}u, v \rangle},
\end{align*}
\]

and hence

\[ s_i(z) = \frac{1}{\gamma} + z\langle x, v_i \rangle' = \frac{1}{s(z)}. \]

The coisometry and isometry properties of \( v_i \) follow from that of \( V \) via the formulas (3.3.2)–(3.3.4) and (3.3.5)–(3.3.7) respectively. The unitarity of \( v_i \) follows from that of \( V \) via all the six formulas. Indeed, if they hold for \( T, u, v, \gamma \) then they also hold with \( T, u, v, \gamma \) and the inner product \( \langle \cdot, \cdot \rangle \) replaced by \( T_i, u_i, v_i, \gamma \) and the inner product \( \langle \cdot, \cdot \rangle' \). The verification of these equalities is straightforward and therefore omitted. The relations

\[
\text{span} \{v, T^sv, \cdots, T^{sn}v\} = \text{span} \{v_i, T_i^sv_i, \cdots, T_i^{sn}v_i\}, \quad n = 0, 1, \ldots,
\]

and

\[
\text{span} \{u, Tu, \cdots, T^mu\} = \text{span} \{u_i, T_iu_i, \cdots, T_i^mu_i\}, \quad n = 0, 1, \ldots,
\]

readily imply the minimality statements. \( \blacksquare \)