CHAPTER 4

Elliptic $K3$ surfaces with Mordell-Weil rank 15

1. Introduction

The Mordell-Weil rank $r$ of a Jacobian elliptic surface $\pi : X \rightarrow C$ is defined as the rank of the group of sections of $\pi$. If $X$ is a $K3$ surface, then it follows easily that $C = \mathbb{P}^1$. If one works over a field of characteristic 0, then it is well known that $0 \leq r \leq 18$. (In positive characteristic we know that $0 \leq r \leq 20$.)

By a result of Cox [17] there exists a Jacobian elliptic $K3$ surface defined over $\mathbb{C}$ with any given Mordell-Weil rank $r$, with $r$ an integer, $0 \leq r \leq 18$. Actually, using a similar reasoning as in [17] one can show there are infinitely many $18 - r$-dimensional families of Jacobian elliptic $K3$ surfaces defined over $\mathbb{C}$, with Mordell-Weil rank $r$. The examples constructed in the proof of Cox are not explicit: the existence of such examples follows from properties of a so-called period map.

Kuwata [41] has given a list of explicit Weierstrass equations for elliptic $K3$ surfaces defined over $\mathbb{Q}$ with Mordell-Weil rank $r$ (over $\mathbb{Q}$) for any $r$ between 0 and 18, except for the case $r = 15$.

The aim of this chapter is to produce an explicit three-dimensional family of elliptic $K3$ surfaces with Mordell-Weil rank 15.

**Theorem 1.1.** Let $K$ be an algebraically closed field, with $\text{char}(K) \neq 2, 3$. Let $E_{a,b,c}$ be the curve defined over $K(s)$ given by the Weierstrass equation

$$y^2 = x^3 + A_{a,b,c}(s)x + B_{a,b,c}(s),$$

with

$$A_{a,b,c}(s) = 4a^3b^3((b-a)cs^8 + (2ac + 2bc + 4ab)s^4 + (b-a)c$$

and

$$B_{a,b,c} = 16a^5b^5s^2((b-a)s^8 + 2(b+a)s^4 + (b-a)).$$

Then for a general $(a, b, c) \in K^3$ this defines an elliptic $K3$ surface with 24 fibers of type $I_1$ and Mordell-Weil rank at least 15. Moreover if $K = \mathbb{C}$, then a generic member of this family has Mordell-Weil rank 15.

2. Construction

Let $K$ be an algebraically closed field of characteristic different from 2 and 3.

In this chapter we use very often the notion of twisting an elliptic surface by $2n$ points. For more information on this see Section 1.5. We recall that if $P$ is one of the $2n$ distinguished points, then the fiber of $P$ changes in the following way:

$$I_\nu \leftrightarrow I_\nu^* (\nu \geq 0) \quad II \leftrightarrow IV^* \quad III \leftrightarrow III^* \quad IV \leftrightarrow II^*$$

and the type of fiber in any point different from the $2n$ distinguished point remains the same.

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Let $\pi : X \to C$ be a Jacobian elliptic surface, $P_1, \ldots, P_{2n} \in C$ points. Let $\tilde{\pi} : \tilde{X} \to C$ be a twist by the $P_i$. Let $\varphi : C_1 \to C$ be a double cover ramified at the $P_i$, such that the minimal models of base-changing $\varphi$ and $\tilde{\varphi}$ by $\pi$ are isomorphic. Denote this model by $\pi_1 : X_1 \to C_1$.

Recall that
\[
\text{rank}(MW(\pi_1)) = \text{rank}(MW(\pi)) + \text{rank}(MW(\tilde{\pi})).
\]

Moreover, the singular fibers change as follows

\begin{tabular}{|c|c|c|c|c|c|}
\hline
Fiber of $\pi$ at $P_i$ & & $I_0$ or $I_0^*$ & $II$ or $IV^*$ & $III$ or $III^*$ & $IV$ or $IV^*$ \\
\hline
Fiber of $\pi_1$ at $\varphi^{-1}(P_i)$ & $I_2$ & $IV$ & $I_0$ & $IV^*$ & \\
\hline
\end{tabular}

The following result will be used several times. It is a direct consequence of the Shioda-Tate formula 1.2.11.

**Theorem 2.1 ([70, Theorem 10.3]).** Let $\pi : X \to \mathbb{P}^1$ be a rational Jacobian elliptic surface, then the rank of the Mordell-Weil group is $8$ minus the number of irreducible components of singular fibers not intersecting the identity component.

Consider the following construction:

**Construction 2.2.** Let $\pi : X \to \mathbb{P}^1$ be a Jacobian elliptic surface whose singular fibers are three fibers of type $I_1$ and one fiber of type $III^*$. Let $f \in K(t)$ be a function of degree two, such that the fibers of $\pi$ over the critical values of $f$ are non-singular.

Let $\alpha, \beta \in \mathbb{P}^1$ be the two distinct points such that $f(\alpha) = f(\beta)$ is the point which fiber is of type $III^*$. Let $g$ be a degree 4 cyclic covering, with only ramification over $\alpha, \beta$. Let $\varphi : Y \to \mathbb{P}^1$ be the non-singular relatively minimal model of the fiber product $X \times_{\mathbb{P}^1} \mathbb{P}^1$ with respect to $\pi$ and $f \circ g : \mathbb{P}^1 \to \mathbb{P}^1$.

**Proposition 2.3.** The Mordell-Weil rank of $\varphi$ (of Construction 2.2) is at least 15, and is precisely 15 if and only if the rank of the twist of $\pi$ by the two critical values of $f$ is 0.

**Proof.** The assumptions imply that $X$ is a rational surface and hence using Theorem 2.1 we have that $\text{rank}(MW(\pi)) = 1$. Let $\pi_1 : X_1 \to \mathbb{P}^1$ be the fiber product $X \times_{\mathbb{P}^1} \mathbb{P}^1$ with respect to $f : \mathbb{P}^1 \to \mathbb{P}^1$ and $\pi$. Let $\tilde{\pi} : \tilde{X} \to \mathbb{P}^1$ be the twist of $\pi$ by the two critical values of $f$. Then by (5) and Theorem 2.1
\[
\text{rank}(MW(\pi_1)) = \text{rank}(MW(\pi)) + \text{rank}(MW(\tilde{\pi})) = 1 + \text{rank}(MW(\tilde{\pi})).
\]
Note that \( \pi_1 \) has two fibers of type \( III^* \) and six fibers of type \( I_1 \). Let \( P_1 \) and \( P_2 \) be the points with a fiber of type \( III^* \).

Let \( g_2 : \mathbb{P}^1 \to \mathbb{P}^1 \) be the degree two function, with critical values \( P_1 \) and \( P_2 \). Define \( \pi_2 : X_2 \to \mathbb{P}^1 \) to be the non-singular relatively minimal model of the fiber product \( X \times_{\mathbb{P}^1} \mathbb{P}^1 \) with respect to \( \pi_1 \) and \( g_2 \).

Let \( \tilde{\pi}_1 \) be the twist of \( \pi_1 \) by \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \). Then \( \tilde{\pi}_1 \) has two fibers of type \( III \) and 6 fibers of type \( I_1 \), hence the corresponding surface is rational and by Theorem 2.1 \( \tilde{\pi}_1 \) has Mordell-Weil rank 6. From this it follows that \( \text{rank}(MW(\pi_2)) = 7 + \text{rank}(MW(\tilde{\pi})) \).

Furthermore, \( \pi_2 \) has two fibers of type \( I_0^* \), and 12 fibers of type \( I_1 \).

Let \( \tilde{\pi}_2 \) be the twist of \( \pi_2 \) by the two points with fiber of type \( I_0^* \). Then \( \tilde{\pi}_2 \) has 12 fibers of type \( I_1 \) and the corresponding surface is rational with Mordell-Weil rank 8. So

\[
\text{rank}(MW(\varphi)) = \text{rank}(MW(\pi_2)) + \text{rank}(MW(\tilde{\pi}_2)) = 15 + \text{rank}(MW(\tilde{\pi})).
\]

\( \square \)

**Remark 2.4.** If we suppose that \( \text{rank}(MW(\tilde{\pi})) = 0 \), then it is relatively easy to find explicit generators for \( MW(\varphi) \). In that case the pull-backs of the generators of \( MW(\pi) \), \( MW(\tilde{\pi}_1) \), \( MW(\tilde{\pi}_2) \) generate a subgroup of \( MW(\varphi) \) of index \( 2^m \), for some \( m \geq 0 \). Since all these three surfaces are rational, we can take a specific Weierstrass model for these surfaces such that all Mordell-Weil groups are generated by polynomials of degree at most 2. (See [53].)

**Remark 2.5.** In the case \( K = \mathbb{C} \) there exists another proof. Since \( Y \) and \( \tilde{X} \) are both \( K3 \) surfaces, and there exists a finite map between them, the Picard numbers of both surfaces coincide (see [29, Corollary 1.2]). From an easy exercise using Kodaira’s classification of singular fibers it follows that the configuration of singular fibers of \( \varphi \) is the one mentioned in the Theorem. By Kodaira’s classification of singular fibers and the Shioda-Tate formula 1.2.11 we conclude

\[
2 + 15 + \text{rank}(MW(\tilde{\pi})) = \rho(X) = \rho(Y) = 2 + \text{rank}(\varphi).
\]

Proposition 2.3 enables us to prove the main theorem of this chapter.

**Proof of Theorem 1.1.** Let \( c \in K^* \) such that \( c^2 \neq -1 \). Then the rational elliptic surface \( E'_c \) associated to the Weierstrass equation

\[
y^2 = x^3 + t^3(t - c)x + t^5
\]

has a fiber of type \( III^* \) and three fibers of type \( I_1 \). One easily shows that if \( E'_c \cong E'_c \), then \( c^2 = c^2 \). (If \( E'_c \cong E'_c \), then there exist be an automorphism \( h : \mathbb{P}^1 \to \mathbb{P}^1 \) fixing 0 and \( \infty \), and a constant \( \lambda \in K \), verifying \( h(t)^3(h(t) - c) = \lambda^4t^3(t - c') \) and \( h(t)^5 = \lambda^6t^5 \).

This implies that \( \lambda^2 = 1 \) and \( c' = \lambda c \).)

Let \( a \neq b \) and

\[
f_{a,b}(s) = \frac{4abs}{(a - b)s^2 - 2(a + b)s + a - b}.
\]

The critical values of \( f_{a,b} \) are \( a \) and \( b \), and \( f^{-1}(0) = \{0, \infty\} \). Hence by Proposition 2.3 the elliptic surface associated to the Weierstrass equation

\[
y^2 = x^3 + f_{a,b}(s^4)^3(f_{a,b}(s^4) - c)x + f_{a,b}(s^4)^5
\]

satisfies the properties stated in the theorem. After a coordinate change, which clears denominators, we obtain the equation of \( E_{a,b,c} \).
This family contains a three-dimensional sub-family of non-isomorphic elliptic surfaces, because it is a finite base change of a three-dimensional family of non-isomorphic elliptic surfaces.

Assume now that $K = \mathbb{C}$. Let $U \subset M_2$ be the set of elliptic surfaces with non-constant \( j \)-invariant. (Notation from Chapter 2.)

Suppose that a generic twist of $E'_c$ would have positive Mordell-Weil rank. Then the constructed family $E_{a,b,c}$ would map to a 3-dimensional component $C$ of $NL_{18}$, moreover the general member of the family $E_{a,b,c}$ has non-constant $j$-invariant, hence $\dim C \cap U = 3$. From Theorem 2.1.1 it follows that $\dim NL_{18} \cap U \leq 2$, a contradiction. From Proposition 2.3 it follows that the generic member of $E_{a,b,c}$ has Mordell-Weil rank precisely 15. \( \square \)

**Remark 2.6.** Kuwata [41] gave explicit examples defined over $\mathbb{Q}$. Theorem 1.1 does not suffice to conclude that there exist examples of $K3$ surfaces defined over $\mathbb{Q}$ with Mordell-Weil rank 15. One can show that the elliptic surface

$$y^2 = x^3 + 2(t^8 + 14t^4 + 1)x + 4t^2(t^8 + 6t^4 + 1)$$

has Mordell-Weil rank 15 over $\overline{\mathbb{Q}}$.

The methods used to prove this result are completely different from the methods we use in this chapter. We intend to develop this in a future publication [36].