Arithmetic and moduli of elliptic surfaces
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CHAPTER 1

Extremal elliptic surfaces & Infinitesimal Torelli

The paper [34] is based on this chapter and appeared in the Michigan Mathematical Journal.

1. Introduction

An extremal elliptic surface over $\mathbb{C}$ is an elliptic surface such that the rank of the Néron-Severi group equals $h^{1,1}$ and its associated Jacobian surface admits at most finitely many sections.

These surfaces are very useful if one wants to classify all configurations of singular fibers on certain families of elliptic surfaces: given a configuration of singular fibers on an extremal elliptic surface, using so-called deformations of the $j$-map and twisting one can construct other elliptic surfaces with different configurations of singular fibers (terminology from [47, VIII.2] and [48]). For these surfaces, the genus of the base curve and the geometric and arithmetic genus of the surface are the same as for the original surface. In [66] this is done for the case of $K3$ surfaces. In [47, VIII.2] this is done for any elliptic surface.

The classification of singular fibers on a rational elliptic surface has been given more than 10 years ago (see [48], [53], [54]). Recently there has been given a classification of all singular fibers of elliptic $K3$ surfaces with a section (see [66]). From the classification of configurations of singular fibers on rational (see [48]) and on elliptic $K3$ surfaces with a section (see [66]) we know that any configuration can be obtained from an extremal configuration using deformations of the $j$-map and twisting. Whether this is true for arbitrary elliptic surfaces seems to be unknown.

In this chapter we give a complete classification of extremal elliptic surfaces with constant $j$-invariant (Theorem 3.7). From this we deduce that if $\pi : X \to \mathbb{P}^1$ is extremal then either $p_g(X)$ is at most 1 or the $j$-invariant is 0 or 1728. There are some examples of non-trivial families of extremal elliptic surfaces. For example the family of elliptic surfaces associated to $y^2 = x^3 + t^5(t - 1)^5(t - \alpha)^5(t - \beta)^5(t - \gamma)^5$ is such a family. We use this fact to prove the following:

**Theorem 1.1.** Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface without multiple fibers. Assume that $p_g(X) > 1$. Then $X$ satisfies infinitesimal Torelli (cf. Definition 4.1) if and only if $j(\pi)$ is non-constant or $\pi$ is not extremal.

Kii ([31, Theorem 2]) proved infinitesimal Torelli for elliptic surfaces without multiple fibers and non-constant $j$-invariant. Saitō ([58]) proved in a different way infinitesimal Torelli for elliptic surfaces over $\mathbb{P}^1$ without multiple fibers and $j$-invariant not identical 0 or 1728 (and also for large classes of elliptic surfaces over other base curves). The case $p_g(X) = 0$ is trivial, since $H^2(X, \mathbb{C}) = H^{1,1}(X, \mathbb{C})$, the case $p_g(X) = 1$ follows from [55].
For elliptic surfaces with non-constant \( j \)-invariant we will give the following structure theorem:

**Theorem 1.2.** Suppose \( \pi : X \to C \) is an elliptic surface without multiple fibers and non-constant \( j \)-invariant, then the following three statements are equivalent:

1. \( \pi \) is extremal
2. \( j(\pi) : C \to \mathbf{P}^1 \) is unramified outside \( 0, 1728, \infty \);
3. the only possible ramification indices above 0 are 1, 2, 3 and above 1728 are 1, 2;

- and \( \pi \) has no fibers of type \( II, III, IV \) or \( I_0^* \).

4. There exists an elliptic surface \( \pi' : X' \to C \), such that \( j(\pi') = j(\pi) \), the fibration \( \pi' \) has no fibers of type \( II^*, III^* \) or \( IV^* \), at most one fiber of type \( I_0^* \), and \( \pi' \) has precisely \( 2p_g(X) + 4 - 4g(C) \) singular fibers.

We will now present a more precise theorem which moreover gives the possible Mordell-Weil groups for an extremal elliptic surface. Let \( m, n \in \mathbf{Z}_{\geq 1} \) be such that \( mn \) and \( n > 1 \). Let \( X(m, n) \) be the modular curve parameterizing triples \((E, O, P, Q)\), such that \((E, O)\) is an elliptic curve, \( P \in E \) is a point of order \( m \), the point \( Q \in E \) a point of order \( n \) and the group generated by \( P \) and \( Q \) has \( mn \) elements.

If 

\[
(m, n) \not\in \{(1, 2), (2, 2), (1, 3), (1, 4), (2, 4)\}
\]

then there exists a universal family for \( X(m, n) \), which we denote by \( E(m, n) \). Denote by \( j_{m,n} : X(m, n) \to \mathbf{P}^1 \) the map usually called \( j \).

From the results of [69, Sections 4 and 5] it follows that \( E(m, n) \) is an extremal elliptic surface. The following theorem explains how to construct many examples of extremal elliptic surfaces with a given torsion group.

**Theorem 1.3.** Fix \( m, n \in \mathbf{Z}_{\geq 1} \) such that \( m|n \) and \((m, n) \) is not one of the pairs \((1, 1), (1, 2), (2, 2), (1, 3), (1, 4), (2, 4)\). Let \( C \) be a (projective smooth irreducible) curve, let \( j \in \mathbf{C}(C) \) be a non-constant function. Then the following are equivalent

- there exists a unique extremal elliptic surface \( \pi : X \to C \), with \( j(\pi) = j \) and the group of sections has \( \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z} \) as a subgroup
- \( j \) is unramified outside \( 0, 1728, \infty \) and \( j = j_{m,n} \circ h \) for some \( h : C \to X(m, n) \).

An extremal elliptic surface \( \pi : X \to C \) with \( j(\pi) = j \) and \( \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z} \) as a subgroup of the group of section, is one of the \( 2^{2g(C)} \) surfaces with \( j(\pi) = j \) and all singular fibers are of type \( I_v \).

If \( \pi : X \to \mathbf{P}^1 \) is an extremal semi-stable rational elliptic surface then \( X \) is determined by the configuration of singular fibers (see [50, Theorem 5.4]). It seems that this quite special for rational elliptic surfaces. If \( X \) is a K3 surface a similar statement does not hold:

**Theorem 1.4.** There exists pairs of extremal semi-stable elliptic K3 surfaces, \( \pi_i : X_i \to \mathbf{P}^1, (i = 1, 2) \) such that the groups of section of \( \pi_1 \) and \( \pi_2 \) are isomorphic, the configuration of singular fibers of the \( \pi_i \) coincide and \( X_1 \) and \( X_2 \) are non-isomorphic.

This gives a negative answer to [5, Question 0.2] (for a precise formulation of this question see Section 8). The essential ingredient for the proof of Theorem 1.4 comes from [67, Table 2].
This chapter is organized as follows:

Section 2 contains some definitions and several standard facts. In Section 3 we give a list of extremal elliptic surfaces with constant \( j \)-invariant. They behave differently from the non-constant ones. There are exactly 5 infinite families of extremal elliptic surfaces with constant \( j \)-invariant (3 of dimension 1, 1 of dimension 2 and 1 of dimension 3). In Section 4 we explain this different behavior by proving Theorem 1.1. In Section 5 we explain how twisting can reduce the problem of classification. In Section 6 we link the ramification of the \( j \)-map and the number of singular fibers of a certain elliptic surface. This combined with the results of Section 5 gives a proof of Theorem 1.2. Section 7 contains a proof of the version with the description of the group of sections (Theorem 1.3). In Section 8 we prove Theorem 1.4. In Section 9 we give a classification of extremal elliptic surfaces with \( g(C) = p_g(X) = q(X) = 1 \). Section 10 contains a proof of the fact that there exist elliptic surfaces with exactly one singular fiber. It is easy to see that the singular fiber is of type \( I_{12k} \) or \( I_{12k-6}^{*} \), for some \( k > 0 \). In this section we prove that for every positive \( k \), both \( I_{12k} \) and \( I_{12k-6}^{*} \) occur.

2. Definitions and Notation

Assumption 2.1. By a curve we mean a non-singular projective complex connected curve.

By a surface we mean a non-singular projective complex surface.

Definition 2.2. An elliptic surface is a triple \( (\pi, X, C) \) with \( X \) a surface, \( C \) a curve, \( \pi \) a morphism \( X \rightarrow C \), such that almost all fibers are irreducible genus 1 curves and \( X \) is relatively minimal, i.e., no fiber of \( \pi \) contains an irreducible rational curve \( D \) with \( D^2 = -1 \).

We denote by \( j(\pi) : C \rightarrow \mathbb{P}^1 \) the rational function such that \( j(\pi)(P) \) equals the \( j \)-invariant of \( \pi^{-1}(P) \), whenever \( \pi^{-1}(P) \) is non-singular.

A Jacobian elliptic surface is an elliptic surface together with a section \( \sigma_0 : C \rightarrow X \) to \( \pi \). The set of sections of \( \pi \) is an abelian group, with \( \sigma_0 \) as the identity element. Denote this group by \( MW(\pi) \).

By an elliptic fibration on \( X \) we mean that we give a surface \( X \) a structure of an elliptic surface.

Let \( L \) be the line bundle \( [R^1 \pi_* \mathcal{O}_X]^{-1} \). We call \( L \) the fundamental line bundle (terminology from [47]). Let \( \rho(X) \) denote the rank of the Néron-Severi group of \( X \). We call \( \rho(X) \) the Picard number.

We use the line bundle \( L \) only to keep track of some numerical data. Note that \( \deg(L) = p_g(X) + 1 - g(C) = p_a(X) + 1 \). (See [47, Lemma IV.1.1].)

Assumption 2.3. All elliptic surfaces in this thesis are without multiple fibers.

Remark 2.4. To an elliptic surface \( \pi : X \rightarrow C \) we can associate its Jacobian elliptic surface \( \text{Jac}(\pi) : \text{Jac}(X) \rightarrow C \). The Hodge numbers \( h^{p,q} \), the Picard number \( \rho(X) \), the type of singular fibers of \( \pi \) and \( \deg(L) \) are the same for \( \pi \) and its associated Jacobian surface. We have that \( \text{Jac}(\pi) \cong \pi \) if and only if \( \pi \) admits a section.

Definition 2.5. An extremal elliptic surface is an elliptic surface such that \( \rho(X) = h^{1,1}(X) \) and \( MW(\text{Jac}(\pi)) \) is finite.
**Definition 2.6.** Let \( \pi : X \to C \) be an elliptic surface. Let \( P \) be a point of \( C \). Define \( v_P(\Delta_P) \) as the valuation at \( P \) of the minimal discriminant of the Weierstrass model, which equals the topological Euler characteristic of \( \pi^{-1}(P) \).

**Proposition 2.7.** Let \( \pi : X \to C \) be an elliptic surface. Then
\[
\sum_{P \in C} v_P(\Delta_P) = 12 \deg(L).
\]
In particular, \( \deg(L) \geq 0 \).

**Proof.** This follows from Noether’s formula (see [7, p. 20]). The precise reasoning can be found in [47, Section III.4]. \( \square \)

**Remark 2.8.** If \( P \) is a point on \( C \), such that \( \pi^{-1}(P) \) is singular then \( j(\pi)(P) \) and \( v_P(\Delta_P) \) behave as follows:

<table>
<thead>
<tr>
<th>Kodaira type of fiber over ( P )</th>
<th>( j(\pi)(P) )</th>
<th>( v_P(\Delta_P) )</th>
<th>number of components</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_0^* ) (( \nu &gt; 0 ))</td>
<td>( \neq \infty )</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>( I_\nu ) (( \nu &gt; 0 ))</td>
<td>( \infty )</td>
<td>( \nu )</td>
<td>( \nu + 1 )</td>
</tr>
<tr>
<td>( I_\nu^* ) (( \nu &gt; 0 ))</td>
<td>( \infty )</td>
<td>( 6 + \nu )</td>
<td>( \nu + 5 )</td>
</tr>
<tr>
<td>( II )</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( IV )</td>
<td>0</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>( IV^* )</td>
<td>0</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>( II^* )</td>
<td>0</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>( III )</td>
<td>1728</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>( III^* )</td>
<td>1728</td>
<td>9</td>
<td>8</td>
</tr>
</tbody>
</table>

For proofs of these facts see [7, p. 150], [73, Theorem IV.8.2] or [47, Lecture 1].

**Definition 2.9.** Let \( X \) be a surface, let \( C \) and \( C_1 \) be curves. Let \( \varphi : X \to C \) and \( f : C_1 \to C \) be two morphisms. Then we denote by \( X \times_C C_1 \) the smooth, relatively minimal model of the ordinary fiber product of \( X \) and \( C_1 \).

**Definition 2.10.** Let \( \pi : X \to C \) be an elliptic surface. We say that \( \pi : X \to C \) is a *semi-stable elliptic surface*, if for all \( p \in C \) we have that \( \pi^{-1}(p) \) is of type \( I_\nu \).

Recall the following theorem.

**Theorem 2.11 (Shioda-Tate ([70, Theorem 1.3 & Corollary 5.3])).** Let \( \pi : X \to C \) be a Jacobian elliptic surface, such that \( \deg(L) > 0 \). Then the Néron-Severi group of \( X \) is generated by the classes of \( \sigma_0(C) \), a non-singular fiber, the components of the singular fibers not intersecting \( \sigma_0(C) \), and the generators of the Mordell-Weil group. Moreover, let \( S \) be the set of points \( P \) such that \( \pi^{-1}(P) \) is singular. Let \( m(P) \) be the number of irreducible components of \( \pi^{-1}(P) \), then
\[
\rho(X) = 2 + \sum_{P \in S} (m(P) - 1) + \text{rank}(\text{MW}(\pi)).
\]

**Definition 2.12.** Suppose \( \pi : X \to C \) is an elliptic surface. Denote by \( T(\pi) \) the subgroup of the Néron-Severi group of \( \text{Jac}(\pi) \) generated by the classes of the fiber, \( \sigma_0(C) \) and the components of the singular fibers not intersecting \( \sigma_0(C) \). Let \( \rho_{tr}(\pi) := \text{rank} T(\pi) \). We call \( T(\pi) \) the *trivial part* of the Néron-Severi group of \( \text{Jac}(\pi) \).
Remark 2.13. In Section 2.7 we give an alternative description for the trivial part of the Néron-Severi group.

Remark 2.14. Suppose \( \pi : X \to \mathbb{P}^1 \) has \( \deg(L) = 0 \), then there are no singular fibers hence \( \rho_{tr} = 2 \).

Definition 2.15. Let \( \pi : X \to C \) be an elliptic surface, define
- \( a(\pi) \) as the number of fibers of type \( II^*, III^*, IV^* \).
- \( b(\pi) \) as the number of fibers of type \( II, III, IV \).
- \( c(\pi) \) as the number of fibers of type \( I_0^* \).
- \( d(\pi) \) as the number of fibers of type \( I_\nu^*, \nu > 0 \).
- \( e(\pi) \) as the number of fibers of type \( I_\nu, \nu > 0 \).

Proposition 2.16. For any elliptic surface \( \pi : X \to C \), not a product, we have
\[
h^{1,1}(X) - \rho_{tr}(\pi) = 2(a(\pi) + b(\pi) + c(\pi) + d(\pi)) + e(\pi) - 2\deg(L) - 2 + 2g(C).
\]

Proof. Recall from [47, Lemma IV.1.1] that
\[
h^{1,1} = 10\deg(L) + 2g(C).
\]
From Kodaira’s classification of singular fibers (see Remark 2.8 and Proposition 2.7) it follows that
\[
\rho_{tr}(\pi) = 2 + 12\deg(L) - 2(a(\pi) + b(\pi) + c(\pi) + d(\pi)) - e(\pi).
\]
Combining these yields the proof.

Corollary 2.17. Let \( \pi : X \to C \) be an elliptic surface with constant \( j \)-invariant, not a product. Then \( \pi \) is extremal if and only if \( \pi \) has \( \deg(L) + 1 \) singular fibers.

Proof. If \( j \) is constant then \( e(\pi) = d(\pi) = 0 \), hence \( a(\pi) + b(\pi) + c(\pi) \) equals the number of singular fibers of \( \pi \). Then apply Proposition 2.16.

3. Constant \( j \)-invariant

In this section we give a list of all extremal elliptic surfaces with constant \( j \)-invariant.

Lemma 3.1. Suppose \( \pi : X \to C \) is an extremal elliptic surface such that \( j(\pi) \) is constant. Then \( g(C) \leq 1 \).

Proof. If \( j(\pi) \) is constant then \( v_P(\Delta_P) \leq 10 \) (see Remark 2.8), for every point \( P \). From this, Proposition 2.7 and Corollary 2.17 it follows that
\[
12\deg(L) = \sum_{P|\pi^{-1}(P) \text{ singular}} v_P(\Delta_P) \leq 10(\deg(L) + 1 - g(C)).
\]
This inequality and the fact that \( \deg(L) \geq 0 \) (see Proposition 2.7) imply that \( g(C) \leq 1 \).

Definition 3.2. An elliptic surface \( \pi : X \to C \), with \( C \) a genus 1 curve, is called a hyperelliptic surface, if \( \pi \) has no singular fibers, \( j(\pi) \) is constant, and \( X \) is not isomorphic to \( C \times \tilde{E} \).
Remark 3.3. In terms of Section 5, an elliptic surface \( \pi : X \to C \), with \( j(\pi) \neq 0, 1728 \) is called a hyperelliptic surface if and only if the associated elliptic curve \( E_1 / C(C) \) is isomorphic to \( E_1^{(f)} \), with \( E_1 / C \) an elliptic curve and \( f \) a function such that the valuation of \( f \) at every place of \( C \) is even, but there is no function \( g \in C(C) \) such that \( f \neq g^2 \). (If \( j(\pi) \) equals 0 or 1728, then one can give a similar description.)

Remark 3.4. The usual definition of a hyperelliptic surface (e.g. [7, page 148]) is different, but equivalent (see [47, Lemma III.4.6.b]), to this one.

Remark 3.5. An hyperelliptic surface does not admit a fibration in hyperelliptic curves. For historical reasons (see [7, page 148]) these surfaces are called hyperelliptic. Beauville [8] calls these surfaces bi-elliptic, because they admit two elliptic fibrations.

Lemma 3.6. Suppose \( \pi : X \to C \) is an extremal elliptic surface such that \( j(\pi) \) is constant. Then one of the following occurs

1. \( g(C) = 1; \deg(L) = 0 \) and \( \pi : X \to C \) is a hyperelliptic (or bi-elliptic) surface.
2. \( g(C) = 0; j(\pi) \neq 0, 1728; \deg(L) = 1. \)
3. \( g(C) = 0; j(\pi) = 0; 1 \leq \deg(L) \leq 5. \)
4. \( g(C) = 0; j(\pi) = 1728; 1 \leq \deg(L) \leq 3. \)

Proof. Suppose \( g(C) = 1. \) Then Corollary 2.17 implies that \( \pi \) has \( \deg(L) \) singular fibers hence

\[
12 \deg(L) = \sum v_P(\Delta_P) \leq 10 \deg(L)
\]

from which it follows that \( \deg(L) = 0. \) Hence \( \pi \) has no singular fibers. Since \( \pi \) has finitely many sections, it follows from the definition that \( \pi : X \to C \) is a so-called hyperelliptic surface.

Suppose \( g(C) = 0. \) If \( \deg(L) = 0 \) then \( \pi \) is a projection from a product, hence there are infinitely many sections. By definition, \( \pi \) is not extremal.

Suppose \( \deg(L) > 0. \) Assume \( j(\pi) \neq 0, 1728. \) Then all singular fibers are of type \( I_0^* \) (see Remark 2.8). Since the Euler characteristic of such a fiber is 6, Proposition 2.7 implies that there are exactly \( 2 \deg(L) \) singular fibers. Applying Corollary 2.17 gives

\[
2 \deg(L) = \deg(L) + 1.
\]

From this we know \( \deg(L) = 1. \)

Suppose \( j(\pi) = 1728. \) In this case all singular fibers are of type \( III, I_0^*, III^* \) (see Remark 2.8). From this it follows that \( v_P(\Delta_P) \leq 9. \) By Proposition 2.7 and Proposition 2.16 we obtain

\[
12 \deg(L) \leq 9(a(\pi) + b(\pi) + c(\pi)) = 9 \deg(L) + 9,
\]

so \( 1 \leq \deg(L) \leq 3. \)

Suppose \( j(\pi) = 0. \) In this case we obtain in a similar way \( 1 \leq \deg(L) \leq 5. \)

Theorem 3.7. Suppose \( \pi : X \to C \) is an elliptic surface with \( j(\pi) \) constant.

Then \( \pi \) is extremal if and only if either \( C \) is a curve of genus 1 and \( \text{Jac}(X) \) is a hyperelliptic surface or \( C \cong \mathbb{P}^1 \) and \( \text{Jac}(\pi) \) has a model isomorphic to one of the following:
• \((j(\pi) = 0)\) \(y^2 = x^3 + f(t)\) where \(f(t)\) comes from the following table (the left hand side indicates the positions of the singular fibers)

<table>
<thead>
<tr>
<th>(\text{III}^*)</th>
<th>(\text{IV}^*)</th>
<th>(p_g)</th>
<th>(f(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, (\infty)</td>
<td>(\infty)</td>
<td>0</td>
<td>(t)</td>
</tr>
<tr>
<td>1, 0, (\infty)</td>
<td>0, (\infty)</td>
<td>1</td>
<td>(t^5(1 - t)^2)</td>
</tr>
<tr>
<td>(\alpha), 0, 1, (\infty)</td>
<td>0, (\infty)</td>
<td>2</td>
<td>(t^5(1 - t - \alpha)^3)</td>
</tr>
<tr>
<td>(\beta), (\alpha), 0, 1, (\infty)</td>
<td>(\alpha), 0, 1, (\infty)</td>
<td>3</td>
<td>(t^5(1 - t - \alpha)^5(t - \beta)^4)</td>
</tr>
<tr>
<td>0, 1, (\infty), (\alpha), (\beta), (\gamma)</td>
<td>(0, 1, \infty)</td>
<td>4</td>
<td>(t^5(1 - t - \alpha)^5(t - \beta)^5(t - \gamma)^5)</td>
</tr>
</tbody>
</table>

where \(\alpha, \beta, \gamma \in \mathbb{C} - \{0, 1\}\), pairwise distinct.

• \((j(\pi) = 1728)\) \(y^2 = x^3 + g(t)x\) where \(g(t)\) comes from the following table

<table>
<thead>
<tr>
<th>(\text{III}^*)</th>
<th>(\text{IV}^*)</th>
<th>(p_g)</th>
<th>(g(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, (\infty)</td>
<td>(\infty)</td>
<td>0</td>
<td>(t)</td>
</tr>
<tr>
<td>1, 0, (\infty)</td>
<td>0, (\infty)</td>
<td>1</td>
<td>(t^3(1 - t)^2)</td>
</tr>
<tr>
<td>0, 1, (\infty), (\alpha)</td>
<td>(0, 1, \infty)</td>
<td>2</td>
<td>(t^3(1 - t - \alpha)^3)</td>
</tr>
</tbody>
</table>

where \(\alpha \neq 0, 1\).

• \((j(\pi) \neq 0, 1728)\) \(y^2 = x^3 + at^2x + t^3\), with singular fibers of type \(I_0^*\) at \(t = 0\) and \(t = \infty\)

**Proof.** The above list follows directly from Corollary 2.17 and Lemma 3.6. Since all cases are very similar, we discuss only the case \(j(\pi) = 0\) and \(p_g = 2\). In this case \(\text{Jac}(\pi)\) has a Weierstrass model isomorphic to

\[
y^2 = x^3 + f(t)
\]

with \(f\) a polynomial such that \(13 \leq \deg(f) \leq 18\), and \(v_p(f) \leq 5\) for all finite \(P\). At all zeros of \(f\) there is a singular fiber. If \(\deg(f) < 18\) then the fiber over \(t = \infty\) is also singular.

If \(\pi\) is extremal then from Corollary 2.17 it follows that \(\pi\) has exactly 4 singular fibers. Assume that the fibers with the highest Euler characteristic are over \(t = \infty, 0, 1\). Since \(5 + 5 + 5 + 3 = 5 + 5 + 4 + 4\) are the only two ways of writing 18 as a sum of four positive integers smaller then 6, we obtain that after applying an isomorphism, if necessary, \(f\) equals either

\[
t^5(1 - t - \alpha)^3 \text{ or } t^5(1 - t - \alpha)^4.
\]

\[\square\]

**Remark 3.8.** Note that all extremal elliptic surfaces with constant \(j\)-invariant and \(p_g(X) > 1\) have moduli.
4. Infinitesimal Torelli

In the previous section we gave examples of families of elliptic surfaces with maximal Picard number. In this section we prove that these surfaces are counterexamples to infinitesimal Torelli. Moreover we give a complete solution for infinitesimal Torelli for Jacobian elliptic surfaces over $\mathbb{P}^1$.

Suppose that $X$ is a smooth complex algebraic variety. Then the first order deformations of $X$ are parameterized by $H^1(X, \Theta_X)$, with $\Theta_X$ the tangent bundle of $X$. The isomorphism $H^{p,q}(X, \mathbb{C}) = H^q(X, \Omega^p)$ and the contraction map $\Theta_X \otimes \Omega_X^p \to \Omega_X^{p-1}$ give a cup product map:

$$H^1(X, \Theta_X) \otimes H^{p,q}(X, \mathbb{C}) \to H^{p-1,q+1}(X, \mathbb{C}).$$

From this one obtains the infinitesimal period map

$$\delta_k : H^1(X, \Theta_X) \to \oplus_{p+q=k} \text{Hom}(H^{p,q}(X, \mathbb{C}), H^{p-1,q+1}(X, \mathbb{C})).$$

The (holomorphic) map $\delta_k$ is closely related to the period map. Assume that $\varphi : X \to B$ is a proper, smooth, surjective holomorphic map between complex manifolds having connected fibers, and that for all $t \in B$ the vector space $H^k(X_t, \mathbb{C})$ carries a Hodge structure of weight $k$, with $X_t := \varphi^{-1}(t)$. Fix a point $0 \in B$. Let $U$ be a small simply connected open neighborhood of 0.

Define

$$\mathcal{P}^{p,k} : U \to \text{Grass} \left( \sum_{i \geq p} H^{i,k-i}(X_0, \mathbb{C}) \right)$$

$$t \mapsto (\oplus_{i \geq p} H^{i,k-i}(X_t, \mathbb{C})) \subset H^k(X_0, \mathbb{C}),$$

via the identification $H^k(X_0, \mathbb{C}) \cong H^k(X_1, \mathbb{C})$. (Note that $U$ is simply connected.)

If the Kodaira-Spencer map $\rho_{U,0} : T_{U,0} \to H^1(X, \Theta_X)$ is injective then the differential at 0 of the period-map $\oplus_p \mathcal{P}^{p,k}$ is injective if and only if $\delta_k$ is injective. (See [58, Section 2] or [82, Chapter 10].)

Note that if $X$ is an Jacobian elliptic surface with base $\mathbb{P}^1$ then $\delta_k$, for $k \neq 2 = \dim X$ is the zero-map.

**Definition 4.1.** We say that $X$ satisfies *infinitesimal Torelli* if and only if $\delta_{\dim(X)}$ is injective.

If $X$ is a rational surface, then $H^{1,1}(X, \mathbb{C}) = H^2(X, \mathbb{C})$ hence the image of $\oplus_p \mathcal{P}^{p,2}$ is a point, while the moduli space of rational elliptic surfaces has positive dimension, so infinitesimal Torelli does not hold for rational elliptic surfaces. If $X$ is a $K3$ surface, then infinitesimal Torelli follows from [55]. This means that we handled the case $p_g(X) \leq 1$. This section will focus on the case $p_g(X) > 1$.

There is an easy sufficient condition of Kiñ([31]), Lieberman, Peters and Wilsker ([42]) for checking infinitesimal Torelli for manifolds with divisible canonical bundle. The following result is a direct consequence of [42, Theorem 1].

**Theorem 4.2.** Let $X$ be a compact Kähler $n$-manifold, with $p_g(X) > 1$. Let $\mathcal{L}$ be a line bundle such that

1. $\mathcal{L}^\otimes k = \Omega_X^k$ for some $k > 0$.
2. the linear system corresponding to $\mathcal{L}$ has no fixed components of codimension 1.
(3) \( H^0(X, \Omega_X^{n-1} \otimes L) = 0 \).

Then \( \delta_n \) is injective.

We want to apply the above theorem, when \( X \) is an elliptic surface. We take \( L \) to be the line bundle \( \mathcal{O}_X(F) \), where \( F \) is the class of a smooth fiber.

**Lemma 4.3.** Let \( \pi : X \to \mathbf{P}^1 \) be an elliptic surface. Assume that \( X \) is not birational to a product \( C \times \mathbf{P}^1 \). Then for \( n > 0 \) we have

\[
\dim H^0(X, \mathcal{O}_{\mathbf{P}^1}(nF)) = \begin{cases} 
n - 1 + \max(0, n + d + 1) & \text{if } j(\pi) \text{ is constant,} 
n - 1 + \max(0, n + d + 1) & \text{if } j(\pi) \text{ is not constant,} 
\end{cases}
\]

where \( d = \deg(L) - \# \{ P \in C(\mathbf{C}) | \pi^{-1}(P) \text{ singular} \} \).

**Proof.** By [58, Prop. 4.4 (I)] we know that \( \pi_*\Omega^1_X \cong \Omega^1_{\mathbf{P}^1} \) if \( j(\pi) \) is not constant, which gives the first case.

If \( j(\pi) \) is constant then we have the following exact sequence (by [58, Prop. 4.4 (II)]):

\[ 0 \to \Omega^1_{\mathbf{P}^1} \to \pi_*\Omega^1_X \to \mathcal{O}_{\mathbf{P}^1}(d) \to 0. \]

Tensoring with \( \mathcal{O}_{\mathbf{P}^1}(n) \) gives

\[
\dim H^0(X, \Omega^1_X(nF)) = \dim H^0(\mathbf{P}^1, \Omega^1_{\mathbf{P}^1}(n)) + \dim H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n + d)) = n - 1 + \max(0, n + d + 1),
\]

using that \( \dim H^1(\mathbf{P}^1, \Omega^1_{\mathbf{P}^1}(n)) = 0. \)

**Corollary 4.4.** Suppose \( \pi : X \to \mathbf{P}^1 \) is an elliptic surface (cf. Assumption 2.3), such that \( p_g(X) > 1 \). Suppose that \( j(\pi) \) is non-constant or \( \pi \) is not extremal then \( X \) satisfies infinitesimal Torelli.

**Proof.** Let \( F \) be a smooth fiber of \( \pi \). Note that \( \mathcal{O}(F)^{\otimes (p_g(X)-1)} = \Omega^2_X \) (see [70, Theorem 2.8]) and the linear system \( |F| \) is the elliptic fibration, hence without base-points.

We claim that \( \dim H^0(X, \Omega^1(F)) = 0. \) If this is not the case then by Lemma 4.3 \( j(\pi) \) is constant and \( \pi \) has at most \( \deg(L) + 1 \) singular fibers. From Lemma 2.17 it follows that \( \pi \) is extremal. Apply now Theorem 4.2 with \( L = \mathcal{O}(F) \).

The following technical result states, more or less, that if we have a family of surfaces satisfying Infinitesimal Torelli and the generic member has a large Picard number, then the base of this family has small dimension. The main idea in the proof is that a high Picard number gives a severe restriction on the image of \( \pi_t \) restricted to this family”. Joseph Steenbrink pointed out to me that the following result is a direct consequence of [12, Theorem 1.1].

**Proposition 4.5.** Let \( \varphi : \mathcal{X} \to \Delta \) be a family of smooth surfaces with \( \Delta \) a small polydisc. For any \( t \in \Delta \) denote by \( X_t \) the fiber over \( t \). Assume that for all \( t, t' \in \Delta \) such that \( t \neq t' \), we have that \( X_t \not\cong X_{t'} \). Moreover, assume that the Kodaira-Spencer map \( \rho_{\Delta,t} : T_{\Delta,t} \to H^1(X_t, \Theta_{X_t}) \) is injective for all \( t \in \Delta \).

Let \( r \) be the Picard number of a generic member of the family \( \varphi \). Suppose that for some \( t \) we have \( p_g(X_t) > 1 \) and

\[ \dim \Delta > \frac{1}{2} p_g(X_t)(h^{1,1}(X_t, \mathbf{C}) - r). \]
or \( p_g(X_t) = 1 \) and
\[
\dim \Delta > (h^{1,1}(X_t, \mathbb{C}) - r).
\]
Then for no \( t \), the surface \( X_t \) satisfies infinitesimal Torelli.

**Corollary 4.6.** Let \( \varphi : X \to \Delta \) be a non-trivial family of surfaces such that \( \rho(X_t) = h^{1,1}(X_t, \mathbb{C}) \) for all \( t \). Then for no \( t \) the surface \( X_t \) satisfies infinitesimal Torelli.

**Corollary 4.7.** Let \( \pi : X \to \mathbb{P}^1 \) be an extremal elliptic surface without multiple fibers, with constant \( j \)-invariant and \( p_g(X) > 1 \). Then \( X \) does not satisfy infinitesimal Torelli.

**Proof.** From Remark 3.8 it follows that \( X \) is a member of positive dimensional family of surfaces with \( \rho(X) = h^{1,1}(X, \mathbb{C}) \). The fact that the Kodaira-Spencer map is injective for these families is well-known. \( \square \)

**Proof of Proposition 4.5.** We start with some reduction steps. Fix a base point \( 0 \in \Delta \). Denote \( X \) the fiber over \( 0 \). It suffices to show that
\[
H^1(X, \Theta_X) \xrightarrow{\delta_2} \text{Hom}(H^{2,0}(X, \mathbb{C}), H^{1,1}(X, \mathbb{C})) \oplus \text{Hom}(H^{1,1}(X, \mathbb{C}), H^{0,2}(X, \mathbb{C}))
\]
is not injective. Using Serre duality one can show that this is equivalent to show that
\[
H^1(X, \Theta_X) \xrightarrow{\delta'_2} \text{Hom}(H^{2,0}(X, \mathbb{C}), H^{1,1}(X, \mathbb{C}))
\]
is not injective (see [58, Section 3]). Since \( \rho_{\Delta,0} \) is injective, it suffices to show that the dimension of the image of \( \delta'_2 \circ \rho_{\Delta,0} \) is less than the dimension of \( \Delta \).

Since \( \Delta \) is simply connected there is a natural isomorphism
\[
\psi_t : H^2(X, \mathbb{C}) \cong H^2(X_t, \mathbb{C}),
\]
for all \( t \in \mathbb{C} \). Let \( \Lambda := \cap_{t \in \Delta} \psi_t^{-1}(NS(X_t)) \). Then \( \Lambda \) is a rank \( r \) lattice.

Let \( T(X_t) \) be the orthogonal complement (with respect to the cup-product) of \( \psi_t(\Lambda) \) in \( H^2(X_t, \mathbb{Z}) \). Then \( T(X_t, \mathbb{C}) := T(X_t) \otimes \mathbb{C} \) carries a sub-Hodge structure. From
\[
\psi_t(\Lambda) \subset NS(X_t) = H^{1,1}(X_t, \mathbb{C}) \cap H^2(X, \mathbb{Z})
\]
we obtain that the Hodge structure on \( H^2(X_t, \mathbb{C}) \) is determined by the Hodge structure on \( T(X_t, \mathbb{C}) \). We consider the variation of the Hodge structure on \( T(X_t) \) (cf. [24, Section 6]).

One easily shows that the composed map \( \delta'_2 \circ \rho_{\Delta,0} \) factors
\[
T_{\Delta,0} \to \text{Hom}(T^{2,0}(X, \mathbb{C}), T^{1,1}(X, \mathbb{C})) \to \text{Hom}(H^{2,0}(X, \mathbb{C}), H^{1,1}(X, \mathbb{C})).
\]
This follows almost immediately from the fact that \( \Lambda \) has a pure Hodge structure of type \((1,1)\), hence the Hodge structure does not vary.

The above factorization implies that for any value of \( p_g(X) > 0 \)
\[
\dim \text{Im}(\delta'_2 \circ \rho_{\Delta,0}) \leq \dim T^{2,0}(X, \mathbb{C}) \cdot \dim T^{1,1}(X, \mathbb{C}) = p_g(X)(h^{1,1}(X, \mathbb{C}) - r).
\]
If \( p_g(X) > 1 \) then by Griffiths’ transversality we obtain that
\[
\dim \text{Im}(\delta'_2 \circ \rho_{\Delta,0}) \leq \frac{1}{2} \dim T^{2,0}(X, \mathbb{C}) \cdot \dim T^{1,1}(X, \mathbb{C}) = \frac{1}{2} p_g(X)(h^{1,1}(X, \mathbb{C}) - r),
\]
4. INFINITESIMAL TORELLI

Theorem 4.8. Let \( \pi : X \to \mathbb{P}^1 \) be an elliptic surface (cf. Assumption 2.3). Assume that \( p_g(X) > 1 \). Then \( X \) satisfies infinitesimal Torelli if and only if \( j(\pi) \) is non-constant or \( \pi \) is not extremal.

Proof. Combine Corollary 4.4 and Corollary 4.7.

Remark 4.9. Note that the hyperelliptic surfaces form a family of elliptic surfaces with \( p_g(X) = 0 \), so they do not satisfy infinitesimal Torelli.

Remark 4.10. Chakiris ([15, Section 4]) gave different formulae for the dimension of \( H^0(X, \Omega^1(nF)) \). He used them to deduce a formula for \( \dim H^1(X, \Theta_X) \), which he used to prove that generic global Torelli holds. Even with the use of these incorrect formulae his proof of generic global Torelli seems to remain valid, after a small modification. His formulae would imply that infinitesimal Torelli holds for any Jacobian elliptic surface over \( \mathbb{P}^1 \). The same erroneous formulae leads Beauville ([10, p. 13]) to state in a survey paper on Torelli problems that infinitesimal Torelli holds for an arbitrary Jacobian elliptic surface. Theorem 4.8 shows instead that this is true only under the condition that \( j(\pi) \) is not constant or \( \pi \) is not extremal.

The argument used in Corollary 4.4 to prove that several elliptic surfaces satisfy infinitesimal Torelli, relies heavily on \( C = \mathbb{P}^1 \). Saitō [58] proved, using other techniques, that if \( C \) is an arbitrary smooth curve and if \( \pi : X \to C \) is an elliptic surface with non-constant \( j \)-invariant then \( X \) satisfies infinitesimal Torelli.

We give now some example of Jacobian elliptic surfaces over curves of positive genus for which infinitesimal Torelli does not hold. Hence the \( j \)-invariant is constant.

Lemma 4.11. Let \( \varphi : \mathcal{X} \to \mathcal{B} \) be a family of elliptic surfaces with \( p_g(X_0) > 1 \), constant \( j \)-invariant and \( s \) singular fibers. Let \( g \) be the genus of the base curve of a generic member of this family. Suppose that for all \( t \) we have that \( \{ t' \mid X_t \cong X_{t'} \} \) is zero-dimensional and

\[
\dim \mathcal{B} > (s - \deg(L) + g - 1)p_g.
\]

Then there is no \( t \) such that \( X_t \) satisfies infinitesimal Torelli.

Proof. Note that \( \rho(X_t) \geq \rho_{tr}(\pi) \) for all \( t \in \mathcal{B} \) and

\[
h^{1,1}(X) - \rho_{tr}(\pi) = 2s - 2\deg(L) - 2 + 2g.
\]

Apply now Proposition 4.5.

We now give two examples where the conditions of Lemma 4.11 are satisfied. The first example covers the case where the \( j \)-invariant is constant, but arbitrary. The second example covers the case where \( j = 0 \) or \( j = 1728 \).

Example 4.12. Let \( \psi : \mathcal{X}^* \to \mathcal{B} \) be a maximal-dimensional family of Jacobian elliptic surfaces over a base curve of a fixed positive genus \( g \), having \( 2\deg(L) > 0 \) singular fibers of type \( I_0^* \). Then \( \mathcal{B} \) has dimension \( 3g - 3 + 2\deg(L) + 1 \).

We want to find examples satisfying the conditions of Lemma 4.11. Easy combinatorics show that if \( g > 1 \) then the conditions \( p_g > 1 \) and

\[
\dim \mathcal{B} > (s - \deg(L) + g - 1)p_g
\]
of Lemma 4.11 hold if and only if \((g, p_g)\) is one of \((1, 2), (2, 2), (3, 2), (4, 3)\). In all these cases any member of the family \(\psi\) is a counterexample to Infinitesimal Torelli. It is easy to construct examples with these invariants. Fix \(j_0 \in \mathbb{C}\). Fix \(A, B \in \mathbb{C}\) such that the elliptic curve associated to \(y^2 = x^3 + Ax + B\) has \(j\)-invariant \(j_0\). Fix a curve \(C\) of genus \(g\). Take \(f \in K(C)^*\) such that \(f\) has an odd valuation at precisely \(2 \deg(L)\) places of \(C\). Then the elliptic surface

\[ y^2 = x^3 + Af^2x + Bf^3 \]

has the above mentioned invariants.

**Example 4.13.** Let \(\psi : \mathcal{X} \to \mathcal{B}\) be a maximal-dimensional family of Jacobian elliptic surfaces over a base curve of a fixed positive genus \(g\), such that almost all fibers have constant \(j\)-invariant 0 or 1728 and \(s\) singular fibers. Then \(\mathcal{B}\) has dimension \(3g - 3 + s\). Using \(p_g = \deg(L) + g - 1\) and some easy combinatorics we obtain that the condition \(\dim \mathcal{B} > (s - \deg(L) + g - 1)p_g\) of Lemma 4.11 holds if and only if

\[ s < \frac{(\deg(L))^2 - g^2 + 5g - 4}{\deg(L) + g - 2}. \]

From Noether’s condition, the smallest \(s\) that is possible is \(\lceil \frac{6}{5} \deg(L) \rceil\). This implies that

\[ \deg(L) < -3g + 6 + \sqrt{4g^2 - 11g + 16}. \]

All combinations of \(g\) and \(p_g\) satisfying \(\deg(L) > 0\), \(p_g > 1\) and (1) are mentioned in the table below. In this table \(s_{\text{max}}\) denotes the number of singular fibers such that an elliptic surface with constant \(j\)-invariant and at most \(s_{\text{max}}\) singular fibers satisfy (1). In particular, elliptic surfaces with these invariant do not satisfy Infinitesimal Torelli by Lemma 4.11. One can find examples with these invariants in a similar way as above. The columns with \(\lceil \frac{6}{5} \deg(L) \rceil\) and \(\lceil \frac{4}{3} \deg(L) \rceil\) denote the minimal number of singular fibers for an elliptic surface with \(j\)-invariant 0 or 1728, whenever this number is at most \(s_{\text{max}}\).

<table>
<thead>
<tr>
<th>(g(C))</th>
<th>(p_g(X))</th>
<th>(\deg(L))</th>
<th>(s_{\text{max}})</th>
<th>(\lceil \frac{6}{5} \deg(L) \rceil)</th>
<th>(\lceil \frac{4}{3} \deg(L) \rceil)</th>
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### 5. Twisting

In this section we study the behavior of \(h^{1,1}(X) - \rho_{tr}(X)\) under twisting, when \(\pi : X \to \mathbb{P}^1\) is a Jacobian elliptic surface. We are mostly interested in the case that \(j(\pi)\) is not constant.

Given a Jacobian elliptic surface \(\pi : X \to C\), we can associate an elliptic curve in \(\mathbb{P}^2_{\mathbb{C}(C)}\) corresponding to the generic fiber of \(\pi\). This induces a bijection on isomorphism classes of Jacobian elliptic surfaces and elliptic curves over \(\mathbb{C}(C)\).

Two elliptic curves \(E_1\) and \(E_2\) are isomorphic over \(\mathbb{C}(C)\) if and only if \(j(E_1) = j(E_2)\) and the quotients of the minimal discriminants of \(E_1/\mathbb{C}(C)\) and \(E_2/\mathbb{C}(C)\) is a 12-th power (in \(\mathbb{C}(C)^*\)).
Assume that $E_1, E_2$ are elliptic curves over $\mathbb{C}(C)$ with $j(E_1) = j(E_2) \neq 0, 1728$. Then one shows easily that $\Delta(E_1)/\Delta(E_2)$ equals $u^6$, with $u \in \mathbb{C}(C)^*$. Hence $E_1$ and $E_2$ are isomorphic over $\mathbb{C}(C)(\sqrt{u})$. We call $E_2$ the twist of $E_1$ by $u$, denoted by $E_1^{(u)}$. Actually, we are not interested in the function $u$, but in the places at which the valuation of $u$ is odd.

**Definition 5.1.** Let $\pi : X \rightarrow C$ be a Jacobian elliptic surface. Fix $2n$ points $P_i \in C(C)$. Let $E/C(C)$ be the Weierstrass model of the generic fiber of $\pi$.

A Jacobian elliptic surface $\pi' : X' \rightarrow C$ is called a (quadratic) twist of $\pi$ by $(P_1, \ldots, P_n)$ if the Weierstrass model of the generic fiber of $\pi'$ is isomorphic to $E^{(f)}$, where $E^{(f)}$ denotes the quadratic twist of $E$ by $f$ in the above mentioned sense and $f \in \mathbb{C}(C)$ is a function such that $v_{P_i}(f) \equiv 1 \mod 2$ and $v_Q(f) \equiv 0 \mod 2$ for all $Q \notin \{P_i\}$.

The existence of a twist of $\pi$ by $(P_1, \ldots, P_{2n})$ follows directly from the fact that $\text{Pic}^{0}(C)$ is 2-divisible.

If we fix $2n$ points $P_1, \ldots, P_{2n}$ then there exist precisely $2^{2g(C)}$ twists by $(P_1)_{i=1}^{2n}$.

If $P$ is one of the $2n$ distinguished points, then the fiber of $P$ changes in the following way (see [47, V.4]).

$$I\nu \leftrightarrow I_\nu^{*} \ (\nu \geq 0) \quad II \leftrightarrow IV^{*} \quad III \leftrightarrow III^{*} \quad IV \leftrightarrow II^{*}$$

The fiber-types with a * have a higher Euler characteristics then the fiber-type one obtains after twisting that fiber (see Remark 2.8).

After fixing a base curve $C$, the quantities $p_0(X)$ and $p_a(X)$ are increasing functions of the Euler characteristic of $X$. This motivates us to consider a special class of twists:

**Definition 5.2.** A $\ast$-minimal twist of $\pi$ is a twist $\tilde{\pi} : \tilde{X} \rightarrow C$ such that none of the fibers are of type $II^{*}, III^{*}, IV^{*}$ or $I_\nu^{*}$ and at most 1 fiber is of type $I_0^{*}$.

Later on we will introduce another notion of minimality: the twist for which $h_{1,1}(X) - \rho_{tr}(\pi)$ is minimal. It turns out that this is a twist which might have several *-fibers.

A *-minimal twist of an elliptic curve need not be unique for two reasons: first of all, if the *-minimal configuration contains a $I_0^{*}$-fiber one might move the $I_0^{*}$-fiber. Secondly, when fixing the points tot twist with, there are $2^{2g}$ possibilities for the function to twist with.

One can easily see that the configuration of the singular fibers of any two *-minimal twists of the same surface are equal.

**Lemma 5.3.** Let $\pi : X \rightarrow C$ be a Jacobian elliptic surface. Let $P_i, i = 1 \ldots 2n$ be points of $C$. Let $\pi' : X' \rightarrow C$ be a twist by $(P_i)$. Then

$$h_{1,1}(X') - \rho_{tr}(\pi') = h_{1,1}(X) - \rho_{tr}(\pi) + \sum_{i=1}^{2n} c_{P_i}$$

with

$$c_{P_i} = \begin{cases} 
1 & \text{if } \pi^{-1}(P_i) \text{ is of type } I_0, IV^{*}, III^{*} \text{ or } II^{*}, \\
0 & \text{if } \pi^{-1}(P_i) \text{ is of type } I_0^{*}, \text{ with } \nu > 0, \\
-1 & \text{if } \pi^{-1}(P_i) \text{ is of type } II, III, IV, \text{ or } I_0^{*}.
\end{cases}$$

**Proof.** Suppose $\pi^{-1}(P_i)$ is of type $I_0$. Then $\pi^{-1}(P_i)$ is of type $I_0^{*}$. The Euler characteristic of this fiber is 6, so this point causes $h_{1,1}$ to increase by 5. An $I_0^{*}$ fiber has 4 components not intersecting the zero-section. Hence $\rho_{tr}$ increases by 4.
The other fiber types can be done similarly. \hfill \Box

The notation $a(\pi), b(\pi), \ldots, e(\pi)$ is introduced in Section 2.

**Lemma 5.4.** Given a Jacobian elliptic surface $\pi : X \to C$ with non-constant $j$-invariant. There exist finitely many twists $\pi'$ of $\pi$ such that the non-negative integer $h^{1,1} - \rho_{tr}$ is minimal under twisting. These twists are characterized by $b(\pi') = c(\pi') = 0$, i.e., there are no fibers of type II, III, IV or $I_0^*$.

**Proof.** It is easy to see that there are at most finitely many twists with $c(\pi') = 0$. Hence it suffices to show that $b(\pi') = c(\pi') = 0$ if $h^{1,1}(X') - \rho_{tr}(\pi')$ is minimal under twisting. From Lemma 5.3 it follows that it suffices to show that for any elliptic surface there exists a twist with $b(\pi') = c(\pi') = 0$.

Consider a $\ast$-minimal twist $\tilde{\pi} : \tilde{X} \to C$. Note that $e(\tilde{\pi}) > 0$ (otherwise the $j$-invariant would be constant.)

Suppose $b(\tilde{\pi}) + c(\tilde{\pi})$ is even. Twist by all points with a fiber of type II, III, IV or $I_0^*$. The new elliptic surface has $b = c = 0$.

Suppose $b(\tilde{\pi}) + c(\tilde{\pi})$ is odd. Twist by all points with fibers of type II, III, IV or $I_0^*$ and one point with a fiber of type $I_0$ or $I_0^\ast$, with $\nu > 0$ (such a fiber exists because the $j$-invariant is not constant). The new elliptic surface has $b = c = 0$. \hfill \Box

**Remark 5.5.** The classification (in [67]) of extremal (Jacobian) elliptic $K3$ surfaces is a classification of the root lattices corresponding to the singular fibers. In general one cannot decide which singular fibers correspond to these lattices, since each of the pairs $(I_1, II), (I_2, III)$ and $(I_3, IV)$ give rise to the same lattice $(A_0, A_1, A_2)$. From the Lemma above it follows that this problem does not occur when $\pi$ is extremal.

**Proposition 5.6.** Let $\pi : X \to C$ be a $\ast$-minimal twist with non-constant $j$-invariant and fundamental line bundle $L$. Let $\tilde{\pi} : \tilde{X} \to C$ be a twist for which $h^{1,1} - \rho_{tr}$ is minimal. Then

$$h^{1,1}(\tilde{X}) - \rho_{tr}(\tilde{X}) = 2g(C) - 2\deg(L) - 2 + \#\{\text{singular fibers for } \pi\}.$$  

**Proof.** From Proposition 2.16 and the Lemmas 5.4 and 5.3 we have

$$\deg(\tilde{L}) = \deg(L) + (d(\tilde{\pi}) + a(\tilde{\pi}) - c(\pi))/2, d(\tilde{\pi}) + e(\tilde{\pi}) = e(\pi), a(\tilde{\pi}) = b(\pi).$$

This yields

$$h^{1,1}(\tilde{X}) - \rho_{tr}(\tilde{X}) = 2g(C) - 2\deg(L) - 2 + 2(a(\tilde{\pi}) + d(\tilde{\pi})) + e(\tilde{\pi}) = 2g(C) - 2\deg(L) - 2 + b(\pi) + c(\pi) + e(\pi).$$

Finally note that $a(\pi) = d(\pi) = 0$. \hfill \Box

**Corollary 5.7.** Let $\pi : X \to C$ be an elliptic surface with $j(\pi)$ non-constant, then $\pi$ is extremal if and only if $\pi$ has no fibers of type II, III, IV or $I_0^*$ and the $\ast$-minimal twist of its Jacobian $\tilde{\pi} : \tilde{X} \to C$ has $2\deg(L) + 2 - 2g(C)$ singular fibers.

6. Configurations of singular fibers

In order to apply the results of the previous section, we need to know which elliptic surfaces have a $\ast$-minimal twist with $2\deg(L) + 2 - 2g(C)$ singular fibers.

We need the following definition:
Definition 6.1. A function \( f : C \to \mathbb{P}^1 \) is called of \((3,2)\)-type if the ramification indices of the points in the fiber of 0 are at most 3, and in the fiber of 1728 are at most 2.

Proposition 6.2. Let \( \pi : X \to C \) be a (Jacobian) elliptic surface with \( j(\pi) \) non-constant, such that \( \pi \) is a \(*\)-minimal twist. Then \( j(\pi) \) is of \((3,2)\)-type and unramified outside 0, 1728, \( \infty \) if and only if there are 2\( \deg(L) \) + 2 - 2\( g(C) \) singular fibers.

Proof. Denote by

- \( n_2 \) the number of fibers of \( \pi \) of type II.
- \( n_3 \) the number of fibers of \( \pi \) of type III.
- \( n_4 \) the number of fibers of \( \pi \) of type IV.
- \( n_6 \) the number of fibers of \( \pi \) of type \( I_0^* \).
- \( m_\nu \) the number of fibers of \( \pi \) of type \( I_\nu \). (\( \nu > 0 \))

Let \( r = \sum \nu m_\nu \). The ramification of \( j(\pi) \) is as follows (using [47, Lemma IV.4.1]).

Above 0 we have \( n_2 \) points with ramification index 1 modulo 3, we have \( n_4 \) point with index 2 modulo 3 and at most \((r - n_2 - 2n_4)/3\) points with index 0 modulo 3. In total, we have at most \( n_2 + n_4 + (r - n_2 - 2n_4)/3\) points in \( j(\pi)^{-1}(0) \).

Above 1728 we have \( n_3 \) points with index 1 modulo 2 and at most \((r - n_3)/2\) points with index 0 modulo 2. So \( j(\pi)^{-1}(1728) \) has at most \( n_3 + (r - n_3)/2 \) points.

Above \( \infty \) we have \( \sum m_\nu \) points.

Collecting the above gives

\[
\#j(\pi)^{-1}(0) + \#j(\pi)^{-1}(1728) + \#j(\pi)^{-1}(\infty) \leq \frac{2}{3}n_2 + \frac{1}{2}n_3 + \frac{1}{3}n_4 + \frac{5}{6}r + \sum m_\nu
\]

with equality if and only if \( j(\pi) \) is of \((3,2)\)-type.

Hurwitz’ formula implies that

\[
r + 2 - 2g(C) \leq \#j(\pi)^{-1}(0) + \#j(\pi)^{-1}(1728) + \#j(\pi)^{-1}(\infty)
\]

with equality if and only if there is no ramification outside 0, 1728 and \( \infty \).

So

\[
r \leq 12g(C) - 12 + 4n_2 + 3n_3 + 2n_4 + 6 \sum m_\nu
\]

holds and Proposition 2.7 implies that

\[
\sum in_i + r = 12\deg(L).
\]

Substituting gives

\[
2\deg(L) + 2 - 2g(C) \leq n_2 + n_3 + n_4 + n_6 + \sum m_\nu = \#\{p \in C \mid \pi^{-1}(p) \text{ singular}\}
\]

with equality if and only if \( j(\pi) \) is unramified outside 0, 1728 and \( \infty \) and \( j(\pi) \) is of \((3,2)\)-type.

This enables us to prove

Theorem 6.3. Suppose \( \pi : X \to C \) is an elliptic surface with non-constant \( j \)-invariant, then the following three are equivalent

1. \( \pi \) is extremal
2. \( j(\pi) \) is of \((3,2)\)-type, unramified outside 0, 1728, \( \infty \) and \( \pi \) has no fibers of type II, III, IV or \( I_0^* \).
3. The minimal twist \( \pi' \) of \( \text{Jac}(\pi) \) has 2\( \deg(L) + 2 - 2g(C) \) singular fibers.

\( \square \)
Proof. Apply Proposition 6.2 to Corollary 5.7.

Remark 6.4. Frédéric Mangolte pointed out to me that the equivalence of (1) and (2) was already proved in [51].

Remark 6.5. Given a function $f : C \to \mathbb{P}^1$ of $(3,2)$-type, unramified outside 0, 1728, $\infty$, then there exists a Jacobian elliptic surface $\pi : X_1 \to C$, with $j(\pi_1) = t$. (For example one can take the elliptic surface associated to $y^2 + xy = x^3 - 36/(t - 1728)x - 1/(t - 1728)$.) After base changing $\pi_1$ by $f$, and then twisting away all the $II$, $III$, $IV$ and $I^{*}_0$ fibers gives the desired surface.

Remark 6.6. Consider functions $f : C \to \mathbb{P}^1$ up to automorphisms of $C$. If we fix the ramification indices above 0, 1728, $\infty$ and demand that $f$ is unramified at any other point, then there are only finitely many $f$ with that property. Any small deformation of $f$ in $Mor_d(C, \mathbb{P}^1)$, the moduli space of morphisms $C \to \mathbb{P}^1$ of degree $d$, has more critical values.

So the $j$-invariants of extremal elliptic surfaces lie form a discrete set in $Mor_d(C, \mathbb{P}^1)$. From Lemma 5.4 it follows that to any $j$-invariant there correspond only finitely many extremal elliptic surfaces. In particular, extremal elliptic surfaces over $\mathbb{P}^1$ with geometric genus $n$ and non-constant $j$-invariant, form a discrete set in the moduli space of elliptic surfaces over $\mathbb{P}^1$ with geometric genus $n$.

Suppose that $\pi : X \to \mathbb{P}^1$ has $2p_g(X) + 4$ singular fibers of type $I_\nu$ and no other singular fibers. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a cyclic morphism ramified at two points $P$ such that $\pi^{-1}(P)$ is singular.

Fastenberg ([21, Theorem 2.1]) proved that then the base changed surface has Mordell-Weil rank 0. In fact, she proved that the base-changed surface is extremal. The first surface is also extremal (by Proposition 5.7). A slightly more general variant is the following.

Example 6.7. Suppose $\pi : X \to C$ is an extremal elliptic surface. Let $f : C' \to C$ be a finite morphism.

Then the base-changed elliptic surface $\pi' : X' \to C'$ is extremal if $f$ is not ramified outside the set of points $P$, such that $\pi^{-1}(P)$ is multiplicative or potential multiplicative.

In that case the composition $j' : C' \to C \xrightarrow{j} \mathbb{P}^1$ is not ramified outside 0, 1728 and $\infty$ and the ramification indices above 0 and 1728 are at most 3 and 2. Moreover there are no fibers of type $II$, $III$, $IV$ or $I^*_0$.

An easy calculation shows that all elliptic surfaces mentioned in [21, Theorem 1] are either extremal elliptic surfaces or have a twist which is extremal. Moreover these surfaces have no fibers of type $I^*_0$, hence all elliptic surfaces for which her results hold lie discretely in the moduli spaces mentioned above.

7. Mordell-Weil groups of extremal elliptic surfaces

It remains to classify which Mordell-Weil groups can occur. To this we can give only a partial answer to this. Note first of all, that a Mordell-Weil group of an extremal elliptic surface is isomorphic to $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with $m|n$. 

Let \( m, n \in \mathbb{Z}_{\geq 1} \) be such that \( m|n \) and \( n > 1 \). Recall from Section 1 that \( X(m, n) \) is the modular curve parameterizing triples \( ((E, O), P, Q) \), such that \( (E, O) \) is an elliptic curve, \( P \in E \) is a point of order \( m \) and \( Q \in E \) a point of order \( n \).

If
\[
(m, n) \not\in \{(1, 2), (2, 2), (1, 3), (1, 4), (2, 4)\}
\]
then there exists a universal family for \( X(m, n) \), which we denote by \( E(m, n) \). Denote by \( j_{m,n} : X(m, n) \to \mathbb{P}^1 \) the map usually called \( j \).

From the results of [69, Sections 4 and 5] it follows that \( E(m, n) \) is an extremal elliptic surface. The following theorem explains how to construct many examples of extremal elliptic surfaces with a given torsion group.

**Theorem 7.1.** Fix \( m, n \in \mathbb{Z}_{\geq 1} \) such that \( m|n \) and \( (m, n) \) is not one of the pairs \( (1, 1), \) \( (1, 2), (2, 2), (1, 3), (1, 4), (2, 4) \). Let \( C \) be a (projective smooth irreducible) curve, let \( j \in C(C) \) be a non-constant function. Then the following are equivalent

- there exists a unique extremal elliptic surface \( \pi : X \to C \), with \( j(\pi) = j \) and the group of sections has \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \) as a subgroup
- \( j \) is unramified outside \( 0, 1728, \infty \) and \( j = j_{m,n} \circ g \) for some \( g : C \to X_m(n) \).

An extremal elliptic surface \( \pi : X \to C \) with \( j(\pi) = j \) and \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \) is a subgroup of the group of section, is one of the \( 2^{2g} \) elliptic surfaces with \( j(\pi) = j \) and all singular fibers are of type \( I_\nu \).

**Proof.** Let \( \pi : X \to C \) be an elliptic surface, such that \( MW(\pi) \) has a subgroup isomorphic to \( G := \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \). Then \( j : C \to \mathbb{P}^1 \) can be decomposed in \( g : C \to X(m, n) \) and \( j_{m,n} : X(m, n) \to \mathbb{P}^1 \), and \( X \) is isomorphic to \( E(m, n) \times_{X(m, n)} C \).

Conversely, for any base change \( \pi' \) of \( \varphi_{m,n} : E(m, n) \to X(m, n) \), the group \( MW(\pi') \) has \( G \) as a subgroup.

Moreover, since \( \varphi_{m,n} \) has only singular fibers of type \( I_\nu \), the same holds for \( \pi' \). An application of Theorem 6.3 concludes the proof.

**Remark 7.2.** Let \( n = 2 \) and \( m \leq 2 \). Then any elliptic surface such that \( j(\pi) = j_{m,n} \circ g \), has \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \) as a subgroup of \( MW(\pi) \).

## 8. Uniqueness

Artal Bartolo, Tokunaga and Zhang ([5]) expect that an extremal elliptic surface is determined by the configuration of the singular fibers and the Mordell-Weil group. More precise, they raise the following question:

**Question 8.1.** Suppose \( \pi_1 : X_1 \to \mathbb{P}^1 \) and \( \pi_2 : X_2 \to \mathbb{P}^1 \) are extremal semi-stable elliptic surfaces, such that \( MW(\pi_1) \cong MW(\pi_2) \) and the configurations of singular fibers of \( \pi_1 \) and \( \pi_2 \) are the same.

Are \( X_1 \) and \( X_2 \) isomorphic, and if so, is there then an isomorphism that respects the fibration and the zero section?

By [50, Theorem 5.4] this is true in the case where \( X_1 \) and \( X_2 \) are rational elliptic surfaces.

In the case where \( X_1 \) and \( X_2 \) are \( K3 \) surfaces the answer is the following theorem.
Theorem 8.2. There exists precisely 19 pairs \((\pi_1 : X_1 \to \mathbb{P}^1, \pi_2 : X_2 \to \mathbb{P}^1)\) of extremal elliptic K3 surfaces, such that \(\pi_1\) and \(\pi_2\) have the same configuration of singular fibers, \(\text{MW}(\pi_1)\) and \(\text{MW}(\pi_2)\) are trivial, and \(X_1\) and \(X_2\) are not isomorphic. Of these pairs 13 are semi-stable. There is an unique pair \((\pi_1 : X_1 \to \mathbb{P}^1, \pi_2 : X_2 \to \mathbb{P}^1)\) of extremal elliptic K3 surfaces, such that \(\pi_1\) and \(\pi_2\) have the same configuration of singular fibers, \(\text{MW}(\pi_1) = \text{MW}(\pi_2) = \mathbb{Z}/2\mathbb{Z}\) and \(X_1\) and \(X_2\) are not isomorphic, which is not semi-stable.

All configurations are listed below in Table 1.

Proof. From [67, Table 2] there exist 19 pairs of surfaces \((X_1, X_2)\) such that the transcendental lattices of \(X_1\) and \(X_2\) lie in distinct \(SL_2(\mathbb{Z})\)-orbits, they admit elliptic fibrations \(\pi_i : X_i \to \mathbb{P}^1\) such that \(\text{MW}(\pi_1) = \text{MW}(\pi_2) = 0\) and the contributions of the singular fibers as sub-lattices of the Néron-Severi lattice coincide. This is not yet enough to conclude that the configuration of singular fibers coincides, but from Remark 5.5 we know that for extremal elliptic surfaces the sub-lattices determine the singular fibers. Since the transcendental lattices are in different \(SL_2(\mathbb{Z})\)-orbits, the surfaces are not isomorphic.

The rest of the statement follows from the same Table. \(\square\)

In Table 1 we list 20 (extremal) configurations of singular fibers, such that the configuration of singular fibers plus the Mordell-Weil group does not determine the K3 surfaces up to isomorphism. The list is complete with this property.

The column SZ-number indicates the number in the list of Shimada and Zhang, see [67, Table 2]. The numbers between square brackets indicate which \(I_\nu\) fibers occur, all other fiber types are in the usual Kodaira notation.

In Table 2 we list the (extremal) configurations \(C\) with the property that there are at least two extremal elliptic K3 surfaces \(\pi_i : X_i \to \mathbb{P}^1\) such that \(C(\pi_i) = C\) and \(\text{MW}(\pi_i) \not\sim \text{MW}(\pi_j)\) for \(i \neq j\). For any other configuration of singular fibers on a (Jacobian) extremal elliptic K3 surface the configuration of singular fibers determines the Mordell-Weil group. In all cases, it turns out that not more than two different (finite) Mordell-Weil groups occur. The only intersection of both list is the case \([1, 1, 1, 2, 5, 14]\). There are three surfaces admitting a fibration with this configuration of singular fibers. Two of these have a trivial Mordell-Weil group, one has \(\mathbb{Z}/2\mathbb{Z}\) as Mordell-Weil group.

Remark 8.3. From these surfaces one should be able to construct other pairs of extremal elliptic surfaces with isomorphic Mordell-Weil groups, and the same configuration of singular fibers, such that the geometric genus is higher then 1.

Start with two non-isomorphic extremal elliptic K3 surfaces with the same configuration of singular fibers and the same Mordell-Weil group. Then the \(j\)-invariant of both surfaces are unequal modulo automorphism of \(\mathbb{P}^1\).

We can base-change both surfaces in such a way that the base-changed surfaces remain extremal (cf. Example 6.7), they have the same configuration of singular fibers and their \(j\)-invariants are unequal modulo an automorphism of \(\mathbb{P}^1\).

The configuration of singular fibers gives restrictions on the possibilities for the torsion part of the Mordell-Weil group. One can hope that this is sufficient to prove that the Mordell-Weil groups are isomorphic.

Note that the base-changed surfaces are not isomorphic, since a surface which is not a K3 surface has at most one elliptic fibration.
### 8. Uniqueness

<table>
<thead>
<tr>
<th>Number</th>
<th>SZ-number</th>
<th>configuration</th>
<th>Mordell-Weil group</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18</td>
<td>[1, 2, 4, 5, 5, 7]</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>23</td>
<td>[1, 2, 3, 6, 7]</td>
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</tr>
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<td>3</td>
<td>36</td>
<td>[1, 1, 3, 5, 6]</td>
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</tr>
<tr>
<td>4</td>
<td>40</td>
<td>[1, 1, 2, 5, 7, 8]</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>71</td>
<td>[1, 1, 3, 5, 11]</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>73</td>
<td>[1, 1, 2, 4, 5, 11]</td>
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</tr>
<tr>
<td>7</td>
<td>74</td>
<td>[1, 1, 2, 3, 6, 11]</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
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</tr>
<tr>
<td>9</td>
<td>77</td>
<td>[1, 1, 1, 3, 7, 11]</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>78</td>
<td>[1, 1, 1, 2, 8, 11]</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>92</td>
<td>[1, 1, 2, 2, 5, 13]</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>100</td>
<td>[1, 1, 1, 2, 5, 14]</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>109</td>
<td>[1, 1, 1, 2, 2, 17]</td>
<td>0</td>
</tr>
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<td>120</td>
<td>[1, 2, 6, 7] + $I^*_1$</td>
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<tr>
<td>15</td>
<td>125</td>
<td>[1, 2, 5, 9] + $I^*_1$</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>134</td>
<td>[1, 1, 2, 13] + $I^*_1$</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>148</td>
<td>[1, 2, 3, 10] + $I^*_2$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>18</td>
<td>227</td>
<td>[2, 3, 4, 7] + $IV^*$</td>
<td>0</td>
</tr>
<tr>
<td>19</td>
<td>267</td>
<td>[1, 2, 5, 7] + $III^*$</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>277</td>
<td>[1, 1, 2, 11] + $III^*$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1. Configurations of singular fibers not determining the surface.

<table>
<thead>
<tr>
<th>Number</th>
<th>SZ-number</th>
<th>configuration</th>
<th>Mordell-Weil groups</th>
</tr>
</thead>
<tbody>
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<td>26</td>
<td>[1, 1, 3, 3, 8, 8]</td>
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<td>2</td>
<td>42</td>
<td>[1, 1, 2, 2, 9, 9]</td>
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<td>3</td>
<td>54</td>
<td>[1, 1, 1, 10, 10]</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>62</td>
<td>[1, 1, 1, 6, 6, 10]</td>
<td>0</td>
</tr>
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<td>5</td>
<td>83</td>
<td>[1, 1, 3, 3, 4, 12]</td>
<td>$\mathbb{Z}/3\mathbb{Z}$</td>
</tr>
<tr>
<td>6</td>
<td>84</td>
<td>[1, 2, 2, 3, 4, 12]</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>7</td>
<td>87</td>
<td>[1, 1, 2, 2, 6, 12]</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
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<td>9</td>
<td>100</td>
<td>[1, 1, 1, 2, 5, 14]</td>
<td>0</td>
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<td>103</td>
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<td>107</td>
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<td>111</td>
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<td>0</td>
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<tr>
<td>13</td>
<td>241</td>
<td>[1, 1, 2, 12] + $IV^*$</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>276</td>
<td>[1, 1, 3, 10] + $III^*$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2. Configurations of singular fibers not determining the Mordell-Weil group.
Remark 8.4. ("[5]-case 49") In [5] it is proven that there are two non-isomorphic elliptic surfaces $\pi : X_i \to \mathbb{P}^1$ with $MW(\pi_i) = \mathbb{Z}/5\mathbb{Z}$ and singular fibers $2I_1, 2I_2, 2I_5, I_{10}$, where isomorphic means that the isomorphism respects the fibration. In [67] it is proven that $X_1$ and $X_2$ are isomorphic as surfaces.

This case is a bit special: In the same paper it is proven that for any other pair of (semi-stable) extremal elliptic $K3$ surfaces with $\#MW(\pi) > 4$ and the same singular fibers configuration, there exists an isomorphism which respects the fibration. ([5, Theorem 0.4])

9. Extremal elliptic surfaces with $p_g = 1, q = 1$

A Jacobian elliptic surface, not a product, with $q = 1$ needs to have a genus 1 base curve. This implies that for an extremal elliptic surface with $p_g = 1, q = 1$ we have $\deg(L) = 1$.

The minimal twist of an extremal elliptic surface with $p_g = 1, q = 1$ has two singular fibers.

All possible pairs of fiber types such that the sum of the Euler characteristics is 12, are given in the following table.

<table>
<thead>
<tr>
<th>$I_{11}$</th>
<th>$I_{10}$</th>
<th>$I_9$</th>
<th>$I_8$</th>
<th>$I_7$</th>
<th>$I_6$</th>
<th>$I_{10}$</th>
<th>$I_9$</th>
<th>$I_8$</th>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Several of these surfaces are already described in the literature. See the above mentioned references. We prove in this section that all these possibilities actually occur except for $I_7, I_5$.

Remark 9.1. Note that the 6 configurations with two $I_\nu$ fibers are already extremal. The configurations with one additive and one multiplicative fiber are not extremal. In the case $I_6 I_0^*$ an extremal twist has one singular fiber of type $I_6^*$ and $\deg(L) = 1$. All other combinations of one additive and one multiplicative fiber have as extremal twist an elliptic surface, such that the degree of $L$ is 2.

Proposition 9.2. All these configurations occur as total configuration of singular fibers on a Jacobian elliptic surface with genus 1 base curve, except $I_7 I_5$.

Proof. This follows from the following lemmas. Note that the existence of elliptic surfaces over $\mathbb{P}^1$ with the below mentioned singular fibers follows from [54].

Lemma 9.3. The configurations with $I_k I_{12-k}$ occur for $k = 2, 4, 6$.

Proof. Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface with two $III$ fibers, a fiber of type $I_{k/2}$ and a fiber of type $I_{6-k/2}$, and no other singular fibers. Let $\phi : C \to \mathbb{P}^1$ be a degree two cover ramified at the four points where the fiber of $\pi$ is singular. Then $\pi' : X \times_{\mathbb{P}^1} C \to C$ has two fibers of type $I_6^*$, a fiber of type $I_k$ and a fiber of type $I_{12-k}$. Twisting by the two points with $I_0^*$ fibers gives the desired configuration. \(\square\)

Lemma 9.4. The configurations $I_8 IV$ and $I_6 I_0^*$ occur.

Proof. For the first, let $\pi : X \to \mathbb{P}^1$ be an elliptic surface with two $III$ fibers, a fiber of type $II$ and a fiber of type $I_4$, and no other singular fibers. Let $\phi : C \to \mathbb{P}^1$ be a degree two cover ramified at the four points where the fiber of $\pi$ is singular. Then
\pi' : X \times_{\mathbf{P}^1} C \to C \text{ has two fibers of type } I_{0}^* \text{ a fiber of type } I_8 \text{ and a fiber of type } IV.

For the second, let \( \pi : X \to \mathbf{P}^1 \) be an elliptic surface with three \( III \) fibers and a fiber of type \( I_3 \), and no other singular fibers. Let \( \varphi : C \to \mathbf{P}^1 \) be a degree two cover ramified at the four points where the fiber of \( \pi \) is singular. Then \( \pi' : X \times_{\mathbf{P}^1} C \to C \) has three fibers of type \( I_{0}^* \) and a fiber of type \( I_6 \). Twisting by two points with a \( I_{0}^* \) fiber gives the desired configuration.

For the other four configurations we have a different strategy. We simply show that the \( j \)-map with the right ramification indices exists. This is equivalent to give the monodromy representation.

**Lemma 9.5.** The configurations \( I_{11} I_1, I_9 I_3, II I_{10}, III I_9 \) occur.

**Proof.** For the two configurations of type \( I_\mu I_\nu \) we need to find curves \( C \) and functions \( j : C \to \mathbf{P}^1 \) of degree 12 such that above \( \infty \) there are two point with ramification indices \( \mu \) and \( \nu \), all points above 0 have ramification index 3 and all points above 1728 have ramification index 2. (see [47, Lemma IV.4.1].)

By the Riemann existence theorem it suffices to give two permutations \( \sigma_0, \sigma_1 \) in \( S_{12} \), such that \( \sigma_0 \) is the product of 6 disjoint 2-cycles, \( \sigma_1 \) the product of 4 disjoint 3-cycles, and \( \sigma_0 \sigma_1 \) is a product of a \( \mu \)-cycle and a \( \nu \) cycle, and the subgroup generated by \( \sigma_0 \) and \( \sigma_1 \) is transitive. (See [49, Corollary 4.10].)

For \( I_1 I_{11} \), we use

\[(1 2 3)(4 5 6)(7 8 9)(10 11 12) \ast (1 3)(2 4)(5 7)(8 10)(9 11)(6 12) = (3 2 5 8 11 7 6 10 9 12 4)\]

For \( I_3 I_9 \), we use

\[(1 2 3)(4 5 6)(7 8 9)(10 11 12) \ast (1 6)(4 9)(3 7)(2 10)(5 12)(8 11)(1 4 7)(2 11 9 5 10 8 12 6)\]

Similarly, the existence of \( II I_{10} \), follows from

\[(1 2)(3 4)(5 6)(7 8)(9 10) \ast (1 4 7)(2 5 8)(3 6 9) = (1 3 5 7 2 6 10 9 4 8)\]

and the existence of \( III I_9 \), follows from

\[(2 3)(4 5)(6 7)(8 9) \ast (1 4 7)(2 5 8)(3 6 9) = (1 5 9 2 4 6 8 3 7)\].

**Lemma 9.6.** The configuration \( I_7 I_5 \) does not occur.

**Proof.** A computer search learned us that the permutations needed for the existence of \( I_7 I_5 \) do not exist.

**Corollary 9.7.** Let \( k_i \) be positive integers such that \( \sum k_i = 12 \), with \( i \geq 2 \), and if \( i = 2 \) then \( (k_1, k_2) \neq (7, 5) \) or \( (5, 7) \). Then there exist a curve \( C \) of genus 1, and an elliptic surface \( \pi : X \to C \) such that the configuration of singular fibers of \( \pi \) is \( \sum I_{k_i} \).

**Proof.** Use the monodromy representation as in [48, Remark after Corollary 3.5].
10. Elliptic surfaces with one singular fiber

In the previous section we proved that there exists an elliptic surface with an $I_6$ and an $I_0^*$ fiber. Twisting by the points with a singular fiber yields an elliptic surface with one singular fiber, of type $I_6^*$.

In this section we will prove that for fixed $\deg(L)$ there are exactly two possible configurations of one singular fibers that can be realized as an elliptic surface.

**Proposition 10.1.** Suppose $\pi : X \to C$ is an elliptic surface with one singular fiber. The fiber is of type $I_{12k}^* - 6$ or $I_{12k}$ for some $k \in \mathbb{Z}_{>0}$, and $g(C) \geq k$ in the first case and $g(C) \geq k + 1$ in the second. Conversely, for each $k > 0$ there exists an elliptic surface $\pi_k : X_k \to C$ and $\pi_k' : X_k' \to C$, such that $\pi_k$ and $\pi_k'$ have precisely one singular fiber. Moreover the singular fiber of $\pi_k$ is of type $I_{12k}^*$ and the singular fiber of $\pi_k'$ is of type $I_{12k}^* - 6$.

**Proof.** Since the Euler characteristic of the singular fiber is $12\deg(L)$, the only possible configurations are $I_{12k}^* - 6$ or $I_{12k}$ for some $k \in \mathbb{Z}_{>0}$, and $g(C) \geq k$ in the first case and $g(C) \geq k + 1$ in the second. Conversely, for each $k > 0$ there exists an elliptic surface $\pi_k : X_k \to C$ and $\pi_k' : X_k' \to C$, such that $\pi_k$ and $\pi_k'$ have precisely one singular fiber. Moreover the singular fiber of $\pi_k$ is of type $I_{12k}^*$ and the singular fiber of $\pi_k'$ is of type $I_{12k}^* - 6$.

Fix a rational elliptic surface $\pi : X \to \mathbb{P}^1$ with 3 fibers of type $III$, and one fiber of type $I_3$. (The existence follows from [54].)

Fix $k$ a positive integer. Take a curve $C$ such that $\varphi : C \to \mathbb{P}^1$, has degree $4k - 2$, and is ramified at the four points which have the singular fibers, and above such a point there is exactly one point.

The base change $\pi' : X \times_{\mathbb{P}^1} C \to C$ has 3 fibers of type $I_0^*$ and one fiber of type $I_{12k}^* - 6$. Twisting by all four points with a singular fiber yields an elliptic surface with one singular fiber and this fiber is of type $I_{12k}^* - 6$.

If we replace $4k - 2$ by $4k$, we obtain an elliptic surface with one fiber of type $I_{12k}$.

For any elliptic surface with only one singular fiber and that fiber is of type $I_{12k}^* - 6$, we have the following: the $j$-map $C \to \mathbb{P}^1$ has degree $12k - 6$, one point above $\infty$, at most $4k - 2$ points above 0, and at most $6k - 3$ points above 1728. This implies that the base curve has genus at least $k$. A similar argument shows that in the case that the only singular fiber is of type $I_{12k}$, the base curve has genus at least $k + 1$. \qed