Chapter 4

Multiscale Modeling of Vascular Trees


4.1 Introduction

The vasculature is a complex tree-like network of vessels. Computing the blood flow in such networks requires a lot of computational resources. Different approaches exist for reducing the complexity of vascular flow computations [FQV09b; PV09]. In general, one seeks to avoid solving the 3D Navier–Stokes equations (which in addition might be coupled with the solid mechanics of the vessel walls, tracer transport, turbulence) in parts of the domain, where simplifying assumptions can be made for the flow profile and/or the geometry. The full flow model is resolved where a detailed analysis of the flow field is of interest or cannot be neglected or simplified. This would be typically the case in large vessels with curvature or obstructions, junctions and other situations of circulating or separating flow. Reduced models can be derived in many different ways. For instance, by assuming a simplified symmetric shape of the vessels such as thin cylinders, the flow equations can be integrated over the vessel cross section, resulting, e.g., in simplified PDEs and a reduced geometric dimension (e.g., one-dimensional (1D)). 1D models can also be obtained by means of asymptotic analysis invoking the assumption that the radius is much smaller than the length of the vessel [Bar+66]. Considering thin, compliant vessels, 1D models can also be derived from conservation principles, using different assumptions on the geometry and the physics [HL73; PV09; BFU07; Bla+09]. Often, the vasculature is modeled by a ‘0D’ lumped parameter network, where the flow through the vessels (and even organs) is described as electric networks in terms of resistances, capacities and inductances [PV09; BCF13; Vig+10]. 0D models can be derived either by averaging 1D or 3D models or from conversation laws [PV09].
Geometric multiscale modeling is concerned with coupling models of different dimensions. Multiscale models can couple a 3D Navier–Stokes system in a part of a large vessel where the flow is expected to exhibit strong 3D features with a 1D model describing the flow in smaller peripheral vessels, or with a 0D model providing boundary conditions for the truncated part of the vasculature, or both [FQV09b; BFU07; Urq+06]. Usually, the domain is first decomposed, then the model is reduced within the separate subdomains, finally the resulting models of different orders are coupled. Such multiscale models result in coupled systems which can be difficult to solve numerically. Often, the compartments with different models are separated and solved for with an iteration scheme. Questions of well-posedness and stability arise.

In this work, to simplify its complex structure, the vascular network is modeled as thin tube structures. Tube structures are domains which are tree-like sets of thin cylinders (or thin rectangles in the two-dimensional setting). The ratio of the diameters of cylinders to their heights (or ratio of the sides of rectangles) is a small parameter $\varepsilon$. The method of asymptotic partial decomposition of a domain (MAPDD) allows to reduce essentially the computer resources needed for the numerical solution of such problems [Ber+19]. This new method combines the full-dimensional description in some neighborhoods of bifurcations and a reduced-dimensional description of the connecting tubes, prescribing some special junction conditions at the interfaces between these 3D and 1D submodels (see Blanc et al. [Bla+99], Panasenko [Pan98a], Panasenko [Pan98c], and Panasenko and Pileckas [PP15]). With the approach presented here, the reduced-order compartments enter directly the full-dimensional model. In contrast to the ‘reduce first, then couple’ approach outlined above, here the subdomains remain fully coupled and are reduced in order subsequently (‘couple first, then reduce’). As a consequence, the complete multiscale model can be solved efficiently at once, nested iteration schemes and difficult to solve, coupled systems of different order ODEs/DAEs can be avoided. Furthermore, well-posedness and error estimates are covered by the theory. On the other hand, the approach is so far limited to rigid tubes.

Junction conditions for the steady-state Stokes equations and generalizations to the unsteady Navier–Stokes equations were constructed in Bertoglio et al. [Ber+19]. In the present chapter the theoretical results are summarized, the numerical implementation is discussed and a validation study for a 2D test case is presented.

First, in Section 4.2, the full dimensional flow problems will be introduced, followed by a description of the MAPDD model and a summary of the theoretical results in Section 4.3. The focus of this chapter is the confirmation of the theoretical error estimates by means of numerical examples in Section 4.4.

4.2 The full dimensional fluid flow problem in a tube structure

In this section we will introduce the full dimensional fluid flow problem in a tube structure. Further its solution will be approximated using partial dimension reduction.
4.2. THE FULL DIMENSIONAL FLUID FLOW PROBLEM IN A TUBE STRUCTURE

4.2.1 Thin tube structure domain

Let us remind the definition of a thin tube structure [Pan98b; Pan05; PP15], and graphically exemplified in Figure 4.2.1.

![Diagram](https://via.placeholder.com/150)

Figure 4.2.1: Illustration of the computational domain for $N = 2$ and $M = 1$.

Let $O_1, O_2, \ldots, O_N$ be $N$ different points in $\mathbb{R}^n$, $n = 2, 3$, and $e_1, e_2, \ldots, e_M$ be $M$ closed segments each connecting two of these points (i.e. each $e_j = O_i O_k$, where $i_j, k_j \in \{1, \ldots, N\}$, $i_j \neq k_j$). All points $O_i$ are supposed to be the ends of some segments $e_j$. The segments $e_j$ are called edges of the graph. The points $O_i$ are called nodes. Any two edges $e_j$ and $e_i$, $i \neq j$, can intersect only at the common node. A node is called vertex if it is an end point of only one edge. Assume that the set of vertices is $O_{N_1+1}, O_{N_1+2}, \ldots, O_N$, where $N_1 < N$. Denote $\mathcal{B} = \bigcup_{j=1}^M e_j$ the union of edges, and assume that $\mathcal{B}$ is a connected set. The graph $\mathcal{G}$ is defined as the collection of nodes and edges. Let $e$ be some edge, $e = O_i O_j$. Consider two Cartesian coordinate systems in $\mathbb{R}^n$. The first one has the origin in $O_i$ and the axis $O_i x_1(e)$ has the direction of the ray $[O_i O_j]$; the second one has the origin in $O_j$ and the opposite direction, i.e., $O_j x_1(e)$ is directed over the ray $[O_j O_i]$. With every edge $e_j$ we associate a bounded domain $\sigma_j \subset \mathbb{R}^{n-1}$ having a $C^2$-smooth boundary $\partial \sigma_j$, $j = 1, \ldots, M$. For every edge $e_j = e$ and associated $\sigma_j = \sigma(e)$ we denote by $B_{\varepsilon}(e)$ the cylinder

$$B_{\varepsilon}(e) = \{ x(e) \in \mathbb{R}^n : x_1(e) \in (0, |e|), \frac{x_1(e)}{\varepsilon} \in \sigma(e) \},$$

where $x(e) = (x_2(e), \ldots, x_n(e))$, $|e|$ is the length of the edge $e$ and $\varepsilon > 0$ is a small parameter. Notice that the edges $e_j$ and Cartesian coordinates of nodes and vertices $O_j$, as well as the domains $\sigma_j$, do not depend on $\varepsilon$. Denoting $\sigma_{\varepsilon}(e) = \{ x(e) \in \mathbb{R}^{n-1} : \frac{x(e)}{\varepsilon} \in \sigma(e) \}$ we can write $B_{\varepsilon}(e) = (0, |e|) \times \sigma_{\varepsilon}(e)$. Let $\omega^1, \ldots, \omega^N$ be bounded independent
of $\varepsilon$ domains in $\mathbb{R}^n$ with Lipschitz boundaries $\partial \omega^j$; introduce the nodal domains $\omega^j_\varepsilon = \{x \in \mathbb{R}^n : \frac{x - O_j}{\varepsilon} \in \omega^j\}$. Denote $d = \max_{1 \leq j \leq N} \text{diam} \omega^j$. By a tube structure we call the following domain $$B_\varepsilon = \left( \bigcup_{j=1}^{M} B^{(\varepsilon,j)}_\varepsilon \right) \cup \left( \bigcup_{j=1}^{N} \omega^j_\varepsilon \right).$$ So, the tube structure $B_\varepsilon$ is a union of all thin cylinders having edges as the heights plus small smoothing domains $\omega^j_\varepsilon$ in the neighborhoods of the nodes. Their role is to avoid artificial corners in the boundary of intersecting cylinders, and we will assume that $B_\varepsilon$ is a bounded domain (connected open set) with a $C^2$-smooth boundary. However for the numerical tests we consider a domain with corners.

### 4.2.2 The full dimension fluid flow problem

Let us consider the stationary Stokes or the non-stationary Navier–Stokes equations in $B_\varepsilon$ with the no-slip conditions at the boundary $\partial B_\varepsilon$ except for some parts $\gamma^j_\varepsilon$ of the boundary where the velocity field is given as known inflows and outflows (for alternative boundary conditions on the inlet and outlet boundaries of the domain, the reader is referred to Bégue et al. [Bèg+87] and Bertoglio et al. [Ber+18a]).

The inflow and outflow boundaries are denoted $\gamma^j_\varepsilon = \partial \omega^j_\varepsilon \cap \partial B_\varepsilon$, $\gamma^j = \partial \omega^j \cap \partial B^j_1$ where $B^j_1 = \{y : y \varepsilon + O_j \in B_\varepsilon\}$ and $\gamma_\varepsilon = \cup_{j=N}^{N+1} \gamma^j_\varepsilon$.

#### Unsteady Navier–Stokes Problem

The initial boundary value problem for the non-stationary Navier–Stokes equations reads

\begin{equation}
\begin{aligned}
\nu &\frac{\partial u_\varepsilon}{\partial t} + (u_\varepsilon \cdot \nabla) u_\varepsilon - \nu \Delta u_\varepsilon + \nabla p_\varepsilon = 0, \\
\nabla \cdot u_\varepsilon &= 0, \\
\left. u_\varepsilon \right|_{\partial B_\varepsilon} &= g_\varepsilon, \\
\left. u_\varepsilon \right|_{\partial B_\varepsilon} &= 0,
\end{aligned}
\end{equation}

where $u_\varepsilon$ is the unknown velocity vector, $p_\varepsilon$ is the unknown pressure, $g_\varepsilon$ is a given vector-valued function with $g_\varepsilon(x, t) = g \left( \frac{x - O_j}{\varepsilon}, t \right)$ if $x \in \gamma^j_\varepsilon$, $j = N_1 + 1, \ldots, N$ and equal to zero for the remaining part of the boundary $\partial B^j_1 \setminus \gamma_\varepsilon$ and satisfying the additional conditions given in Bertoglio et al. [Ber+19].

Introduce the space $H^1_{\text{div}0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}(B_\varepsilon)$ as the subspace of vector valued functions from $H^1(B_\varepsilon)$ satisfying the conditions $\nabla \cdot v = 0$, $v|_{\partial B_\varepsilon \setminus \gamma_\varepsilon} = 0$, i.e.,

$$H^1_{\text{div}0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}(B_\varepsilon) = \{ v \in H^1(B_\varepsilon) | \nabla \cdot v = 0; v|_{\partial B_\varepsilon \setminus \gamma_\varepsilon} = 0\}.$$
4.3. MAPDD: THE NEW JUNCTION CONDITIONS

A weak formulation corresponding to the Navier–Stokes problem (4.1) is given by the following definition.

**Definition 1.** By a weak solution we understand the couple of the vector-field \( \mathbf{u}_\varepsilon \) and a scalar function \( p_\varepsilon \) such that \( \mathbf{u}_\varepsilon(x, 0) = 0 \), \( \mathbf{u}_\varepsilon \in L^2(0, T; H^1_{\text{div}}(\partial B_\varepsilon \setminus \gamma_\varepsilon)(B_\varepsilon)), \) \( \mathbf{u}_{\varepsilon, t} \in L^2(0, T; L^2(B_\varepsilon)), \) \( p_\varepsilon \in L^2(0, T; L^2(B_\varepsilon)), \) \( \mathbf{u}_\varepsilon = g_\varepsilon \) on \( \gamma_\varepsilon \) and \( (\mathbf{u}_\varepsilon, p_\varepsilon) \) satisfy the integral identity for every vector-field \( \phi \in H^1_0(B_\varepsilon) \) for all \( t \in (0, T) \),

\[
\int_{B_\varepsilon} (\mathbf{u}_{\varepsilon, t} \cdot \phi + \nu \nabla \mathbf{u}_\varepsilon : \nabla \phi + ((\mathbf{u}_\varepsilon, \nabla \mathbf{u}_\varepsilon) \cdot \phi)) = \int_{B_\varepsilon} p_\varepsilon \nabla \cdot \phi.
\]

**Stokes problem**

Consider the Dirichlet’s boundary value problem for the stationary Stokes equation:

\[
\begin{align*}
-\nu \Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon &= 0, \quad x \in B_\varepsilon, \\
\nabla \cdot \mathbf{u}_\varepsilon &= 0, \quad x \in B_\varepsilon, \\
\mathbf{u}_\varepsilon &= g_\varepsilon, \quad x \in \partial(B_\varepsilon),
\end{align*}
\tag{4.2}
\]

where \( \nu \) is a positive constant, \( g_\varepsilon \) is a given vector-valued function \( g_\varepsilon(x) = g_j \left( \frac{x - O_j}{\varepsilon} \right) \) if \( x \in \gamma_\varepsilon^j, \) \( j = N_1 + 1, \ldots, N, \) equal to zero for the remaining part of the boundary \( \partial B_\varepsilon \setminus \gamma_\varepsilon \) and satisfying the additional conditions in Bertoglio et al. [Ber+19].

A weak formulation of the Stokes problem (4.2) is given by the following definition:

**Definition 2.** By a weak solution we understand the couple of the vector-field \( \mathbf{u}_\varepsilon \) and a scalar function \( p_\varepsilon \) such that \( \mathbf{u}_\varepsilon \in H^1_{\text{div}}(\partial B_\varepsilon \setminus \gamma_\varepsilon)(B_\varepsilon)), \) \( p_\varepsilon \in L^2(B_\varepsilon), \) \( \mathbf{u}_\varepsilon = g_\varepsilon \) on \( \gamma_\varepsilon \) and \( (\mathbf{u}_\varepsilon, p_\varepsilon) \) satisfy the integral identity: for any test function \( \mathbf{v} \in H^1_0(B_\varepsilon) \)

\[
\nu \int_{B_\varepsilon} \nabla \mathbf{u}_\varepsilon(x) \cdot \nabla \mathbf{v}(x) = \int_{B_\varepsilon} p_\varepsilon \nabla \cdot \mathbf{v}.
\]

It is well known that there exists a unique solution to this problem (see [Lad69]).

4.3 MAPDD: the new junction conditions

The classical MAPDD method was previously described in Panasenko and Pileckas [PP15]. We propose a new, more general, formulation of the method involving new junction conditions, which assumes that the flow through the cylinders \( B_\varepsilon^{(e)} \) has the shape of a Womersley flow. The advantages are twofold: (1) it removes a restriction on the velocity boundary condition (see Bertoglio et al. [Ber+19]), therefore being applicable for arbitrary transient regimes, and (2) it considerably simplifies the numerical implementation in the context of finite elements since only additional, easy-to-build integral terms need to be added to a standard weak form.
4.3.1 Formulation

Let $\delta$ be a small positive number much greater than $\varepsilon$ but much smaller than 1. For any edge $e = O_iO_j$ of the graph introduce two hyperplanes orthogonal to this edge and crossing it at the distance $\delta$ from its ends, see Figure 4.2.1.

Denote the cross-sections of the cylinder $B_i^e$ by these two hyperplanes respectively, by $S_{ij}$ (the cross-section at the distance $\delta$ from $O_i$), and $S_{ji}$ (the cross-section at the distance $\delta$ from $O_j$), and denote the part of the cylinder between these two cross-sections by $B_{ij}^{dec,\varepsilon}$. Denote $B_i^{\varepsilon,\delta}$ the connected truncated by the cross-sections $S_{ij}$, part of $B_i^{\varepsilon}$ containing the vertex or the node $O_i$.

The MAPDD model invokes the assumption of Womersley-type flow conditions within each of the truncated cylinders $B_{ij}^{dec,\varepsilon}$, namely

- the velocity is parallel to the edge $e$, i.e., the perpendicular (tangential to $S_{ij}$) components are zero,
- the longitudinal derivative (normal to $S_{ij}$) of the velocity is zero (i.e., identical velocity profiles at every cross-section of the cylinders $B_{ij}^{dec,\varepsilon}$),
- implying a constant longitudinal pressure derivative.

In particular, if the local variables $x^e$ for the edge $e$ coincide with the global ones, $x$, then the Womersley flow profile takes the form $W_p^e(x) = (v_1(x'/\varepsilon), 0, \ldots, 0)^T$, $v_1 \in H_0^1(\sigma^e)$. If $e$ has the direction cosines $k_{e1}, \ldots, k_{en}$ and the local variables $x^e$ are related to the global ones by equation $x^e = x^e(x)$ then the Womersley flow is given by

$$W_p^e(x) = \text{const} \left(k_{e1} v_1 \left(\left(x^e(x)\right)'/\varepsilon\right), \ldots, k_{en} v_1 \left(\left(x^e(x)\right)'/\varepsilon\right)\right)^T, x' = (x_2, \ldots, x_n).$$

Consider the Navier–Stokes problem, and furthermore the special case of geometries with $N = M + 1$, which leads to the following formulation:

find the vector-field $u^{\varepsilon,\delta}$ and the pressure $p^{\varepsilon,\delta}$ such that $u^{\varepsilon,\delta}(x, 0) = 0$, $u^{\varepsilon,\delta} \in L^\infty(0, T; H^1(B_i^{\varepsilon,\delta}))$, for all $i = 1, \ldots, N$, $u_{e,\varepsilon,t} \in L^2(0, T; L^2(B_i^{\varepsilon,\delta}))$, $u_e,\delta = g_\varepsilon$ at $\gamma_\varepsilon$, $u_{e,\delta} = 0$ at $(\partial B_i^{\varepsilon,\delta} \cap \partial B_i^\varepsilon)|\gamma_\varepsilon$, $p_{e,\delta} \in L^2(0, T; L^2(B_i^{\varepsilon,\delta}))$ for all $i = 1, \ldots, N$, $u_{e,\varepsilon} \cdot t = 0$ on $S_{ij} \cup S_{ji}$, $u_{e,\varepsilon} \cdot n|_{S_{ij}} + u_{e,\varepsilon} \cdot n|_{S_{ji}} = 0$, where $t$ is the unit tangent vector and $n$ the unit outward normal vector, and the couple $(u^{\varepsilon,\delta}, p^{\varepsilon,\delta})$ satisfies for all $t \in (0, T)$ the integral identity for every vector-field $\phi \in H^1(B_i^{\varepsilon,\delta})$, $q \in L^2(B_i^{\varepsilon,\delta})$, for all $i = 1, \ldots, N$, such that $\phi = 0$ at $\partial B_i^{\varepsilon,\delta} \cap \partial B_i^\varepsilon$, and for all edges $O_iO_j$, $\phi \cdot t = 0$ at $S_{ij} \cup S_{ji}$ and $\phi \cdot n|_{S_{ij}} + \phi \cdot n|_{S_{ji}} = 0$:

$$\sum_{i=1}^N \int_{B_i^{\varepsilon,\delta}} u_{e,\varepsilon,t} \cdot \phi + v^\varepsilon u_{e,\varepsilon} : \nabla \phi + (\nabla u_{e,\varepsilon}, \nabla u_{e,\varepsilon}) \cdot \phi - p_{e,\varepsilon} \nabla \cdot \phi + q^\varepsilon \cdot u_{e,\varepsilon}$$

$$+ \sum_{l=1}^M d_l \int_{\sigma_l^{(\varepsilon)}} u_{e,\varepsilon,t} \cdot \phi + v^\varepsilon \nabla x^{(\varepsilon)}_{\sigma_l} u_{e,\varepsilon} : \nabla x^{(\varepsilon)}_{\sigma_l} \phi = 0. \quad (4.3)$$
4.4. NUMERICAL EXAMPLES

For $e_l = O_iO_j$, $d_l$ is the distance between the cross sections $S_{ij}$ and $S_{ji}$.

The last sum of integrals in Eq. (4.3) is the contribution of the truncated tubes to the full-dimensional model describing the junctions. They enter the equation by substitution of the boundary integrals arising from integrating by parts the diffusion term. See Bertoglio et al. [Ber+19, Appendix A.3] for a complete derivation.

Finally, note that the last two terms in (4.3) are analogous to the ones obtained in the context the so called *Stokes-consistent* methods for backflow stabilization at open boundaries [BC16].

4.3.2 Error estimate for the unsteady Navier–Stokes problem

The following estimate for the error of the MAPDD solution of the Navier–Stokes problem with respect to the exact solution can be obtained (see Bertoglio et al. [Ber+19]):

**Theorem 1.** Let $g_j \in C^{[\frac{1+4}{2}]}([0, T]; W^{3/2,2}(\partial \Omega_j))$. Given natural number $J$ there exists a constant $C$ (independent of $\epsilon$ and $J$) such that if $\delta = CJ\epsilon |\ln \epsilon|$, then

$$\sup_{t \in (0, T)} \left\| u_{\epsilon, \delta} - u_{\epsilon} \right\|_{L^2(B_\epsilon)} + \left\| \nabla(u_{\epsilon, \delta} - u_{\epsilon}) \right\|_{L^2((0, T); L^2(\Omega_j))} = O(\epsilon^J).$$  \hspace{1cm} (4.4)

4.3.3 Error estimate for the stationary Stokes problem

Applying the MAPDD method in a similar manner to the Stokes problem, the following error estimate can be proved with asymptotic analysis for the difference between the exact solution, $u_\epsilon$, and the MAPDD solution, $u_{\epsilon, \delta}$:

**Theorem 2.** Given natural number $J$ there exists a constant $C$ (independent of $\epsilon$ and $J$) such that if $\delta = CJ\epsilon |\ln \epsilon|$, then

$$\left\| u_\epsilon - u_{\epsilon, \delta} \right\|_{H^1(B_\epsilon)} = O(\epsilon^J).$$  \hspace{1cm} (4.5)

See Bertoglio et al. [Ber+19] for the full derivation and proofs.

4.4 Numerical examples

In this section, the previous analysis is complemented by numerical experiments for the new MAPDD formulation applied to the stationary Stokes problem and the transient Navier–Stokes problem, for a sequence of values of $\epsilon$. In the tests we used more natural Neumann’s condition for the outflow. The errors of the MAPDD solutions obtained in the truncated domain with respect to reference solutions computed in the full domain are evaluated in the norms given by Eqs. (4.4) and (4.5).
4.4.1 Problem setup

Consider the two-dimensional geometry illustrated in Fig. 4.2.1. Two junctions are connected by a straight tube. This straight tube (labeled $B_{d{e}{c},\epsilon}^{1,2}$) is included in the full reference model, or truncated when the reduced MAPDD model is used.

The radius of the tube is proportional to $\epsilon$ (we set $R = \epsilon$). For each value of $\epsilon$, the junction domains are contracted homothetically by a factor of $\epsilon$ with respect to the center points marked with plus signs in Fig. 4.2.1. The distance between these points, $L$, remains the same for all values of $\epsilon$. Straight tube extensions (blue areas, $B_{\epsilon,\delta}^{1,2}$) are added to the junction domains. Theorem 2 requires the associated distance, $\delta$, from the centers of the junction domains to the interfaces, to be

$$\delta = C\epsilon |\ln(\epsilon)|.$$  \hspace{1cm} (4.6)

$C$ is a user parameter. Pairs of full and reduced domains are created for a sequence of values $\epsilon = 2^{-k}$, $k = 1, \ldots, 6$. In the particular examples of the investigated geometry and our selection of $\epsilon$, $1/\ln(2) < C < 6/\ln(2)$ is necessary for $B_{1,2}^{d{e}{c},\epsilon} \neq \emptyset$ and for $B_{1,2}^{e,\delta} \neq \emptyset$, respectively. In what follows, we choose the values $C = K/\ln(2)$, $K = 2, 3$ and $4$. The factor $1/\ln(2)$ is added for convenience, to cancel with the $\ln(\epsilon)$ terms and leave rational numbers as the interface coordinates.

4.4.2 Stationary Stokes test case

Since one of our main motivations is the numerical simulation of blood flows, we choose for the viscosity and the density values that represent physiologically relevant conditions, assuming the fluid is incompressible and Newtonian. Typical parameters of blood are a dynamic viscosity of $\mu = 0.035$ cm$^2$/s and a density of $\rho = 1$ g/cm$^3$.

Remind the relation between the dynamic viscosity $\mu$ and the kinematic viscosity $v$: $v = \mu/\rho$. At the inlet $\Gamma_{in}$ of the upstream junction domain a Dirichlet boundary condition for the velocity is defined as

$$g_\epsilon = (0, 1.5U_0(1 - (x_1 - c_0)^2/\epsilon^2))^T,$$

where $c_0$ is the $x_1$ coordinate of the center of the boundary and $U_0$ is chosen such that $Re = 2\rho\epsilon U_0/\mu = 1$. A homogeneous Neumann boundary condition for the normal stress is applied on the outlet $\Gamma_{out}$ of the downstream junction domain.

4.4.3 Transient Navier–Stokes test case

In the transient Navier–Stokes test case, the physical constants are set to the same values as for the Stokes problem, i.e., $\mu = 0.035$ cm$^2$/s and $\rho = 1$ g/cm$^3$. A pulsating inflow velocity is defined on $\Gamma_{in}^T$ via Dirichlet boundary conditions as

$$g_\epsilon = (0, 1.5U_0(1 - (x_1 - c_0)^2/\epsilon^2)\sin(\pi t/T))^T,$$

where $t$ is the actual time and $T = 0.8$ s is the duration of a cycle. $U_0$ is computed from the Reynolds number, $Re = 2\rho\epsilon U_0/\mu$. As for the Stokes problem, a homogeneous Neumann boundary condition defines the outflow on $\Gamma_{out}$. For the convergence study, Reynolds numbers $Re = 1, 25, 50, 80$ and
100 are considered. In addition, we analyze the MAPDD model for a high Reynolds number of $Re = 2500$.

### 4.4.4 Numerical discretization

A mixed finite element method is adopted for discretizing the Stokes and Navier–Stokes equations. We use monolithic velocity–pressure coupling with inf-sup stable second order Taylor-Hood elements on unstructured, uniform triangle meshes. The transient Navier–Stokes problem is discretized in time with the implicit Euler method. The convection term, written in skew-symmetric form, is treated semi-implicitly. The time step size is $\Delta t = 0.01 \, s$. The time interval of the simulations is a half cycle, i.e., $0 \leq t \leq T/2$. The numerical meshes of the domains are created such that the number of elements along the tube diameter is approximately 20 for each value of $\varepsilon$. The average grid size at the boundaries is therefore $h = \varepsilon/10$. This results in 170592 elements in the full domain for the smallest value of $\varepsilon = 2^{-6}$ and $C = 2/\ln(2)$, which corresponds to 784037 degrees of freedom in the Navier–Stokes system. The triangulation of the corresponding reduced domain consists of 15366 elements and the solution space contains 70741 degrees of freedoms. The problem is implemented and solved using the FEniCS finite element library [Aln+15]. The numerical meshes are generated with Gmsh [GR09].

### 4.4.5 Results

#### Stationary Stokes test case

The velocity and pressure field of the stationary Stokes problem, computed with the full model and with the MAPDD method, are illustrated in Fig. 4.4.1 for the largest value of $\varepsilon = 0.5$. No visible differences exist between the full and the MAPDD results.

The velocity error of the MAPDD model with respect to the full reference solution is analyzed quantitatively in Fig. 4.4.2 for the full range of values of $\varepsilon$. The error is computed in the $\mathcal{H}^1(B_\varepsilon)$ norm, cf. (4.5) in Theorem 2. Note that the error estimate depends on the solutions in the full domain, $B_\varepsilon$. The mesh nodes of the MAPDD and the full domains match for the junctions. In the truncated tube, the MAPDD solution was interpolated from one of the interfaces, $S_{12}$, to the mesh nodes of the full mesh. The rate of convergence can be estimated from the numerical results as

$$J_k = \frac{\log e_k/e_{k-1}}{\log e_k/e_{k-1}}$$

where $e_k = \|u_{\varepsilon_k} - u_{\varepsilon_k,\delta}\|_{\mathcal{H}^1(B_{\delta_2})}$, $e_k = 2^{-k}$, $k = 2, \ldots, 6$. While not constant, for $C = 2/\ln(2)$, $J_k$ is in the range $3 \leq J_k \leq 8$. The error drops at least with cubic convergence (in the investigated cases). For $C = 3/\ln(2)$ the convergence rate is greatly improved, and even more so using $C = 4/\ln(2)$, namely we obtain $J \approx 8$ and $J \approx 11$, respectively, discarding the points where the error stagnates. The stagnation of both cases for $\varepsilon <
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Figure 4.4.1: Pressure fields and velocity magnitude and vectors at the outflow boundaries obtained for the stationary Stokes problem using \( \epsilon = 0.5 \) with the full model (top row) and with the MAPDD model (bottom row).

Figure 4.4.2: Stationary Stokes test case: convergence of the error with respect to \( \epsilon \) for different values of \( C \) (see legend).

\[ 2^{-4} \text{ or } 2^{-3} \text{ is due to the precision of the numerical method being reached. Rounding errors gain importance for very small values of } \epsilon. \]
4.4. NUMERICAL EXAMPLES

Transient Navier–Stokes test case

The asymptotic behavior of the error of the MAPDD method with respect to the full model is shown for different Reynolds numbers in Fig. 4.4.3(a), for \( C = 2/\ln(2) \). The error is evaluated in the norm (4.4). For the lowest investigated Reynolds number \( Re = 1 \), the rate of convergence \( J \) was computed (omitting the two largest values of \( \varepsilon \)). The line \( \varepsilon J \) is included in the figure for comparison. With increasing Reynolds numbers the rate of convergence decreases. Exponential increase of the error was observed for \( Re = 100 \). Using \( C = 3/\ln(2) \) (see Fig. 4.4.3(b)), the rate of convergence obtained for Reynolds numbers \( Re > 1 \) is improved. In particular, for \( Re = 100 \) the error now decreases with \( \varepsilon \), at least for small values of \( \varepsilon \). The errors of the case \( Re = 100 \) obtained for \( C = K/\ln(2) \), \( K = 2, 3, 4 \), are shown in Fig. 4.4.4. Indeed, for higher \( K \), the errors are lower and convergence is improved for \( \varepsilon \leq 2^{-4} \). While the error estimate assumes a low Reynolds number, the MAPDD method can still be applied to these cases. Figures 4.4.5 and 4.4.6 show velocity streamlines and the pressure field obtained with the full reference model and the MAPDD method applied to the case \( \varepsilon = 1/4 \) and for a Reynolds number of \( Re = 2500 \), as an example. The boundary mesh size was set to \( h = \varepsilon/20 \), furthermore \( C = 2/\ln(2) \). The results match very well visually. The MAPDD model is able to recover the recirculation zones in both junctions accurately (Fig. 4.4.5(a) and (b)). For a more detailed comparison, the axial velocity profiles at the interfaces for the MAPDD solution and for the full solution in the corresponding location are shown in Figs. 4.4.7. At the left interface, the velocity interface conditions produce a pressure overshoot near the upper corner, since the Womersley hypothesis is in disagreement with the high Reynolds number flow conditions. This can be seen more clearly in Fig. 4.4.8(a), where the pressure profile at the interface is shown for both the MAPDD and the full solution. However, analyzing the pressure distribution along the cross-section the tube in a slightly more upstream position (shifted upstream by \( 2\varepsilon \)), the MAPDD recovers the behav-
Figure 4.4.4: Comparison of the Navier–Stokes error with different values of $C$ for $Re = 100$.

Figure 4.4.5: Velocity stream lines of the transient Navier–Stokes test case at peak time $t = 0.2$ s, for $Re = 2500$, $\varepsilon = 0.25$. Full model (a) versus MAPDD model (b).

ior observed for the full solution with an error of $< 8\%$ (Fig. 4.4.9). The pressure on the right interface does not suffer any nonphysical oscillations, as can be seen in Fig. 4.4.8(b), and the discrepancy between both models is within $2\%$.

4.5 Conclusion

The MAPDD was shown to be an efficient and accurate method for the steady Stokes problem and for the low Reynolds number Navier–Stokes problem. In these cases,
4.5. CONCLUSION

Figure 4.4.6: Pressure fields of the transient Navier–Stokes test case at peak time $t = 0.2$ s, for $Re = 2500$, $\epsilon = 0.25$. Full model (a) versus MAPDD model (b).

Figure 4.4.7: Axial velocity component $u_0$ at the interfaces for the MAPDD and the full solutions computed for $Re = 2500$, $\epsilon = 1/4$.

the error of the MAPDD method was in agreement with theoretical error estimates, (4.5) and (4.4), respectively. For slightly larger Reynolds numbers, the convergence can be improved by modifying the computational domain and adjusting the constant in Eq. (4.6). Although the theory is only valid for small Reynolds numbers, the method yields very good results also for high Reynolds numbers. For the (arbitrary) example of $Re = 2500$, $\epsilon = 1/4$, the MAPDD velocity and pressure solutions were in good agreement with the full solution, except for pressure oscillations that occur near the upstream interface.
Figure 4.4.8: Pressure along the interfaces for the MAPDD and the full solutions computed for $Re = 2500, \epsilon = 1/4$.

Figure 4.4.9: Pressure along the tube cross-section, at $2\epsilon$ upstream of $S_{12}$, for the MAPDD and the full solutions computed for $Re = 2500, \epsilon = 1/4$. 