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Asymptotic Stability Analysis of Lur'e Systems with Butterfly Hysteresis Nonlinearities

M. A. Vasquez Beltran, B. Jayawardhana, R. Peletier

***Abstract*—We study the asymptotic stability of Lur'e systems with butterfly hysteresis nonlinearities modeled by the Preisach operators with respect to a set of equilibrium points. We present the input-output rate property of the Preisach operators that exhibit butterfly hysteresis behavior. Based on this characterization, we present sufficient conditions on the linear systems that guarantee the asymptotic stability of the closed-loop system to a set of equilibrium points via circle criterion. Finally numerical simulations are presented to demonstrate these results.**

***Index Terms*—Stability of nonlinear systems, nonlinear output feedback, mechatronics**

I. Introduction

HYSTERESIS is a nonlinear phenomenon that affects many classes of physical systems, such as, electromechanical systems that use ferroelectric, piezoelectric, ferromagnetic and/or magnetostrictive materials and mechanical systems with friction. In literature, a number of models have been proposed and analyzed that can well describe the hysteresis behavior [1]–[4].

Some of the mathematical properties inherent in these models have accommodated the design of control systems which contain such hysteresis in the feedback loop. There are a variety of approaches for dealing with the control design problem of systems with hysteresis. For instance, a well-known control design strategy is to deploy an inverse hysteresis model for linearizing the hysteresis operator as pursued in [5], [6] where they assume either the Duhem or Preisach models, respectively. Such an approach has a number of challenges. Firstly, the domain discretization of an infinite dimensional operator, such as Preisach operator, introduces discretization error and the finite-dimensional approximation loses its predictive capability when it is used beyond the compact support of the approximator. Secondly, these models are phenomenological models that may not capture all essential physical elements in the physical systems, such as, the accommodation property in ferromagnetic materials is not captured in Preisach model despite its widespread use for modeling ferromagnetic systems. This affects strongly the exactitude.

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Another control design approach is pursued in [7]–[9] where, instead of using the inverse model to cancel the hysteresis behavior, a particular mathematical property of the hysteresis is taken into account in the control structure. The property that is of interest in these papers is the dissipativity property which is related to the energy dissipation inherent in physical systems, including, hysteretic systems. Particularly, such dissipativity property lends to the commonly observed counterclockwise or clockwise behavior in many hysteretic systems.

Despite all of the aforementioned state-of-the-art approaches in literature, the stability analysis of control systems, where a hysteresis element that exhibits a butterfly hysteresis behavior is in the feedback loop, remains an open problem. Butterfly hysteresis behavior is characterized by the presence of both clockwise (CW) and counter-clockwise (CCW) input-output dynamics. Roughly speaking, the input-output phase portrait with a periodic input of constant amplitude contains two loops in opposite directions which are connected at a crossing-point. In our recent work [10], a framework based on the use of Preisach operator is proposed that is capable of describing such butterfly hysteresis behavior and used to model an asymmetric butterfly hysteresis behavior in a new piezoelectric material developed for a novel hysteretic deformable mirror application [11].

Motivated by the results in [10], in this paper we study the asymptotic stability of the feedback interconnection of a linear system with a Preisach butterfly operator with respect to a set of equilibrium points. We follow a differential approach as in [12] where a time-varying relation between the input and output rates of a given Preisach operator that satisfies a sector boundary condition (under mild assumptions over the weighting function of the Preisach operator) is established. In this way, we can analyze the interconnection as a Lur'e type system so that classical circle criterion can directly be applicable.

This paper is organized as follows. In Section II we give the preliminaries regarding the Preisach operator and the recently introduced Preisach butterfly operator. In Section III a differential formulation of the Preisach operator as a time-varying relation between the input and output rates is introduced. Section IV presents the analysis of the feedback interconnection of a linear system with a Preisach butterfly operator. A numerical example is included in Section V and the conclusions are finally given in Section VI.

II. Preliminaries

Notation. We denote by $C(U,Y)$ and $C_{\mathrm{pw}}(U,Y)$ the space of all continuous and piecewise continuous functions $f: U \to Y$ and $AC(U,Y)$ the space of all absolute continuous functions $f: U \to Y$, respectively.

A. Preisach operator

The operator $\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}: AC(\mathbb{R}_+,\mathbb{R}) \times \{-1,1\} \to C_{\mathrm{pw}}(\mathbb{R}_+,\mathbb{R})$ is the (counterclockwise) relay operator parametrized by $\alpha > \beta$ and the initial condition r_0, and is defined by

$$\big(\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}(u,r_0)\big)(t) := \begin{cases} 1 & \text{if } u(t) > \alpha, \\ -1 & \text{if } u(t) < \beta, \\ \big(\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}(u,r_0)\big)(t_-) & \begin{array}{l}\text{if } \beta \le u(t) \le \alpha, \\ \quad \text{and } t > 0,\end{array} \\ r_0 & \begin{array}{l}\text{if } \beta \le u(t) \le \alpha, \\ \quad \text{and } t = 0,\end{array} \end{cases} \tag{1}$$

Note that the unconventional use of r_0 for initial condition of the relay, instead of using directly -1 or 1, is to accommodate the incorporation of initial condition in the Preisach operator using the notion of interface as defined below.

Let P be the Preisach plane which is a subset of $\mathbb{R}^2$ defined by $P := \{(\alpha,\beta) \,|\, \alpha \ge \beta\}$. We denote by $\mathscr{I}$ the set of all so-called interfaces $L \subset P$, which are monotonically decreasing staircase curves $\{\gamma(x) \in P \,|\, x \in \mathbb{R}_+\}$ where $\gamma: \mathbb{R}_+ \to P$ defines a curve in P and $\gamma(0) = (\beta,\beta)$ for some $\beta \in \mathbb{R}$. By monotonically decreasing we mean that for every pair of points $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in L$ with $L \in \mathscr{I}$, we have that $\alpha_1 \ge \alpha_2$ whenever $\beta_1 \le \beta_2$. Following the formulation in [3] and with an explicit description on the initial conditions of the relays, the Preisach hysteresis operator $\Phi: AC(\mathbb{R}_+,\mathbb{R}) \times \mathscr{I} \to AC(\mathbb{R}_+,\mathbb{R})$ is defined by

$$\big(\Phi(u,L_0)\big)(t) := \iint\limits_{(\alpha,\beta)\in P} \mu(\alpha,\beta)\ \Big(\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}\big(u,\hat{r}_{\alpha,\beta}(L_0)\big)\Big)(t)\ \mathrm{d}\alpha \mathrm{d}\beta \tag{2}$$

where $\mu \in C_{\mathrm{pw}}(P,\mathbb{R})$ is a weighting function that takes value from the Preisach plane P and $L_0 \in \mathscr{I}$ is the initial interface. The parametrized function $\hat{r}_{\alpha,\beta}: \mathscr{I} \to \{-1,1\}$ is defined as

$$\hat{r}_{\alpha,\beta}(L_0) := \begin{cases} 1, & \text{if } L_0 \cap \{(\alpha_1,\beta_1) \,|\, \alpha \le \alpha_1, \beta \le \beta_1\} \ne \emptyset, \\ -1 & \text{otherwise.} \end{cases}$$

In definition (2) of the Preisach hysteresis operator, $\hat{r}_{\alpha,\beta}$ is an auxiliary function that determines the initial condition r_0 of every relay $R^{\circlearrowleft}_{\alpha,\beta}$ according to its (α,β)-coordinates w.r.t. the initial interface L_0, such that $r_0 = -1$ for all relays above L_0 whereas $r_0 = +1$ for all relays below L_0.

One can observe from the definition in (1) that at time $t = 0$ the state of every relay $R^{\circlearrowleft}_{\alpha,\beta}(u,r_0)$ at $t = 0$ is consistent with the input $u(0)$, i.e., the initial state r_0 only plays a role when $\beta \le u(0) \le \alpha$ and otherwise $u(0)$ defines the state of the relay at $t = 0$. However, the use $\hat{r}_{\alpha,\beta}(L_0)$ in (2) may result in the inconsistency of the initial state of the relays $R^{\circlearrowleft}_{\alpha,\beta}(u,\hat{r}_{\alpha,\beta}(L_0))$ at $t = 0$ for some initial input $u(0)$ so that $\big(\Phi(u,L_0)\big)(0)$ is not well posed. Therefore, for well-posedness, we assume throughout this paper that every pair of input u and the initial interface L_0 in the Preisach operator (2) satisfy $\big(u(0),u(0)\big) \in L_0$. Another formulation of the Preisach operator where the Preisach plane P is rotated by $-3\pi/4$ can be found in [1].

The mathematical properties and dynamical behavior of Preisach operator have been widely studied in [1], [2]. Roughly speaking, the Preisach operator is an integral of infinitesimal relay operators (also known as hysterons) that are modulated by the weighting function μ. All infinitesimal relay operators react instantaneously and simultaneously to the input signal. Consequently, at any given time $t > 0$, the interface function $L_t \in \mathscr{I}$ that separates between the relays in 1 state and those in -1 state will dynamically be changed according to the history of the input from the initial time 0 until the current time t and we always have that $L_t \in \mathscr{I}$ for all $t \ge 0$ [3]. We denote by $P_+(t)$ the domain in Preisach plane P where the relay operators are in -1 state at time t and by $P_-(t)$ the domain in P where the relays are in $+1$ state at time t. Thus $P = L_t \cup P_+(t) \cup P_-(t)$. One can check that due to the particular property of relays in the Preisach operator, the interface L_t determines completely the state of all relay operators and output value of the Preisach operator. We remark that although L_t is dependent on the input signal u, we remove its dependence to u in its notation for conciseness.

B. Preisach butterfly operator

In literature, it is common to restrict the Preisach operator weighting function μ as a positive or negative semi-definite function which helps in the fitting of the Preisach model to the measured hysteresis input-output data. Moreover, it has recently been established that such sign definite function μ exhibits nice input-output behavior of the Preisach operators [13]. Positive semi-definite μ gives a counterclockwise (CCW) hysteresis operator while negative semi-definite μ yields a clockwise (CW) hysteresis operator.

On the other hand, the input-output property of Preisach operator with sign-indefinite function μ has not been well-studied. In our recent paper [10], we introduce the notion of *butterfly hysteresis operator* which is a class of hysteresis operator that exhibits both the clockwise (CW) and counter-clockwise (CCW) input-output behavior. The following formal definition of *butterfly hysteresis operator* is adapted from [10].

Definition 2.1: A Preisach hysteresis operator Φ is called a *Preisach butterfly hysteresis operator* if there exist an input-output pair of signals $y,u \in AC(\mathbb{R}_+,\mathbb{R})$ and $L_0 \in \mathscr{I}$, where u is periodic with period of $T > 0$ and $y = \Phi(u,L_0)$, and a constant $T_0 > 0$ such that

$$\frac{1}{2}\int_{T_1}^{T_2} \big(u(\tau)\dot{y}(\tau) - y(\tau)\dot{u}(\tau)\big)\mathrm{d}\tau = 0 \tag{3}$$

holds for all $T_1 > T_0$ and $T_2 = T_1 + T$. △

The left-hand side of (3) denotes the signed area that is enclosed by the input-output phase portrait $\{(y(\tau),u(\tau)) \,|\, \tau \in [T_1,T_2]\}$.

In the main results of [10], we show that a Preisach operator with a two-sided μ can exhibit the butterfly hysteresis behavior under some additional mild assumptions. The two-sided weighting functions are characterized by the existence of two disjoint regions in the Preisach plane P that are separated by a simple monotonically decreasing curve. The following result also from [10] shows a sufficient condition on μ such that the resulting Preisach operator is a Preisach butterfly hysteresis operator.

Theorem 2.2: Consider a Preisach operator Φ as in (2) with a two-sided piecewise continuous weighting function μ satisfying

$$\int_r^{\infty} \mu(\alpha,\beta)\beta \mathrm{d}\beta = \infty, \qquad \int_{-\infty}^{r} \mu(\alpha,\beta)\alpha \mathrm{d}\alpha = \infty, \quad (4)$$

and whose boundary curve $B := \{(\alpha,\beta) \,|\, \mu(\alpha,\beta) = 0\}$ is monotonically decreasing. Then Φ is a Preisach butterfly hysteresis operator.

One can observe from the condition on the first moment of μ with respect to the first and second arguments in (4) that μ cannot decay too rapidly to zero as its arguments go to infinity. This condition can, in fact, be relaxed by only imposing the two-sidedness of μ and μ can be a compactly-supported function. The proof for this claim follows a similar line as the proof of Theorem 2.2 in [10] with a modification on the way a sub-plane $\hat{P}$ within P is identified such that

$$\iint\limits_{(\alpha,\beta)\in\hat{P}} \mu(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta = 0,$$

in which case, the weighted area under the counterclockwise loop is equal to the weighted area under the clockwise loop. We will present this result in an extension paper of [10].

Although it can be checked that several of the known properties that are derived for Preisach operator with sign-definite μ, such as the Lipschitz continuity (see, for instance, [14]), hold also for the Preisach butterfly operator as in the Theorem 2.2, the monotonicity property of the output due to the piecewise monotone input functions does not hold any longer because of the sign-indefiniteness of μ. Therefore, it restricts the use of inversion methods as presented in [6], [15] for the control design when the Preisach butterfly operator exists in the feedback loop.

III. Input-output rate property of the Preisach butterfly hysteresis

We present in this section a differential formulation of the Preisach operator that allows us to express explicitly the time-varying relation between the output rate and the input rate. When we consider an infinitesimal change to the input, the infinitesimal change to the output will be proportional to the infinitesimal state change of the hysterons that are located on the interface L_t and modulated by the associated weight. In other words, in our differential formulation of the Preisach operator, we show that the output rate is proportional to input rate where the proportionality factor is given by a weighted line integral over the last horizontal or vertical linear segment of the interface L_t, corresponding to the instantaneous change of the relays' state on L_t. To properly state this, let us denote by L_t^+ and L_t^- the last horizontal and vertical linear segments of the interface L_t respectively, defined by

$$L_t^+ = \{(\alpha,\beta) \in L_t \,|\, \alpha = u(t)\},$$
$$L_t^- = \{(\alpha,\beta) \in L_t \,|\, \beta = u(t)\}.$$

Then we define two functions $m_t : \mathscr{I} \to \mathbb{R}$ and $M_t : \mathscr{I} \to \mathbb{R}$, which will be used later in our main results, by

$$m_t = \inf_{\beta}\{\beta \,|\, (\alpha,\beta) \in L_t^+\},$$

$$M_t = \sup_{\alpha}\{\alpha \,|\, (\alpha,\beta) \in L_t^-\},$$

where we remove the dependence on L_t in the above notations for conciseness.

A simple interpretation of the scalar value of m_t and M_t can be made using the Preisach operator memory behavior as studied in [1], [3]. In these books, the corners of L_t are given by the points in (α,β)-coordinates determined by the subset of extrema of the truncated input $u_t = \{u(\tau) \,|\, 0 \le \tau \le t\}$. In this regards, M_t and m_t correspond to the last maximum and last minimum of the truncated input u_t that are stored in the Preisach memory and coincide with the corner whose coordinates are given by $(\alpha,\beta) = \big(M_t, m_t\big)$.

Proposition 3.1: Consider a Preisach operator Φ as in (2). Let $\dot{u} \in C(I,\mathbb{R})$ for some nonempty open interval $I \subset \mathbb{R}_+$. Then the time derivative of the Preisach operator output $y \in AC(\mathbb{R}_+,\mathbb{R})$ at every time instant $t \in I$ is given by

$$\dot{y}(t) = \psi(t)\dot{u}(t) \quad (5)$$

with

$$\psi(t) := \begin{cases} 2\displaystyle\int_{m_t}^{u(t)} \mu(u(t),\beta)\mathrm{d}\beta & \text{if } \dot{u}(t) > 0, \\ 2\displaystyle\int_{u(t)}^{M_t} \mu(\alpha,u(t))\mathrm{d}\alpha & \text{if } \dot{u}(t) < 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Proof Proposition 3.1. Let us first prove the case when $\dot{u}(t) > 0$ at some time instant $t \in I$. Since $\dot{u} \in C(I,\mathbb{R})$, there exists a constant $\Delta t > 0$ such that $\dot{u}(\tau) > 0$ for all $\tau \in [t, t+\Delta t]$. Assume without loss of generality that Δt is small enough to guarantee that $m_t = m_\tau$ for all $\tau \in [t, t+\Delta t]$. Let us define three subsets of Preisach domain given by

$$\begin{aligned} \Omega_1 &:= \{(\alpha,\beta) \mid \alpha \ge \beta,\ u(t) \le \alpha < u(t+\Delta t), \\ &\qquad\qquad u(t) \le \beta < u(t+\Delta t)\} \\ \Omega_2 &:= \{(\alpha,\beta) \mid u(t) \le \alpha < u(t+\Delta t),\ m_t \le \beta < u(t)\} \\ \Omega_3 &:= \{(\alpha,\beta) \mid \alpha \ge \beta,\ m_t \le \alpha < u(t),\ m_t \le \beta < u(t)\}, \end{aligned}$$

and use them to partition P (see an illustration of such partition in Figure 1) such that the Preisach operator output at t is given by

$$\begin{aligned} y(t) = &-\iint\limits_{(\alpha,\beta)\in\Omega_1} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta - \iint\limits_{(\alpha,\beta)\in\Omega_2} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta \\ &+ \iint\limits_{(\alpha,\beta)\in\Omega_3} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta \\ &+ \iint\limits_{\substack{(\alpha,\beta)\in \\ P/(\Omega_1\cup\Omega_2\cup\Omega_3)}} \mu(\alpha,\beta)\ \Big(\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}\big(u, \hat{r}_{\alpha,\beta}(L_0)\big)\Big)(t)\ \mathrm{d}\alpha\mathrm{d}\beta. \end{aligned}$$

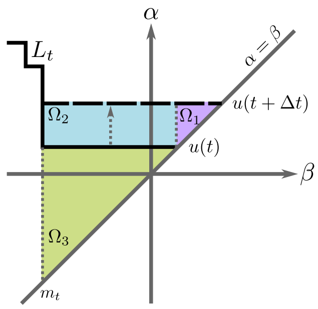

Figure 1: An illustration of a partitioning of the Preisach plane P that is used in the first part of the proof of Proposition 3.1. The domains Ω_1 (in purple), Ω_2 (in blue), and Ω_3 (in green) are defined according to the changes of the hysterons due to an increasing input (i.e., $\dot{u}(t) > 0$) in an infinitesimal time interval $[t, t+\Delta t]$.

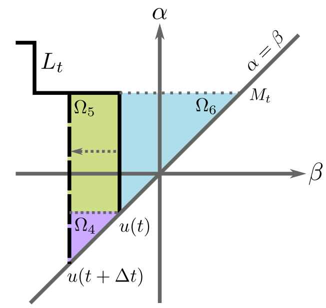

Figure 2: An illustration of a partitioning of the Preisach plane P that is used in the second part of the proof of Proposition 3.1. The domains Ω_4 (in purple), Ω_5 (in green), and Ω_6 (in blue) are defined according to the changes of the hysterons due to a decreasing input (i.e., $\dot{u}(t) < 0$) in an infinitesimal time interval $[t, t+\Delta t]$.

Since the relays $R^{\circlearrowleft}_{\alpha,\beta}$ with $(\alpha,\beta) \in P/(\Omega_1 \cup \Omega_2 \cup \Omega_3)$ have a constant output in the time interval $[t, t+\Delta t]$, the Preisach operator output at a time instance $t+\Delta t$ is given by

$$\begin{aligned} &y(t+\Delta t) = \\ &\quad \iint\limits_{(\alpha,\beta)\in\Omega_1} \mu(\alpha,\beta)\ \mathrm{d}\alpha \mathrm{d}\beta \ + \iint\limits_{(\alpha,\beta)\in\Omega_2} \mu(\alpha,\beta)\ \mathrm{d}\alpha \mathrm{d}\beta \\ &\quad + \iint\limits_{(\alpha,\beta)\in\Omega_3} \mu(\alpha,\beta)\ \mathrm{d}\alpha \mathrm{d}\beta \\ &+ \iint\limits_{\substack{(\alpha,\beta)\in \\ P/(\Omega_1\cup\Omega_2\cup\Omega_3)}} \mu(\alpha,\beta)\ \left(\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}\big(u, \hat{r}_{\alpha,\beta}(L_0)\big)\right)(t)\ \mathrm{d}\alpha \mathrm{d}\beta. \end{aligned}$$

Therefore the rate of change of the output can be computed by

$$\begin{aligned} &\frac{y(t+\Delta t)-y(t)}{\Delta t} = \frac{y(t+\Delta t)-y(t)}{u(t+\Delta t)-u(t)} \frac{u(t+\Delta t)-u(t)}{\Delta t} \\ &= 2\left[\frac{\iint\limits_{(\alpha,\beta)\in\Omega_1} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta}{u(t+\Delta t)-u(t)} + \frac{\iint\limits_{(\alpha,\beta)\in\Omega_2} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta}{u(t+\Delta t)-u(t)}\right] \frac{\Delta u}{\Delta t} \\ &= 2\left[\frac{\int\limits_{u(t)}^{u(t+\Delta t)} \int\limits_{u(t)}^{\beta} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta}{u(t+\Delta t)-u(t)} + \frac{\int\limits_{m_t}^{u(t)} \int\limits_{u(t)}^{u(t+\Delta t)} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta}{u(t+\Delta t)-u(t)}\right] \frac{\Delta u}{\Delta t}. \end{aligned}$$

By taking the limit $\Delta t \to 0$,

$$\dot{y}(t) = \left(2\int\limits_{m_t}^{u(t)} \mu(u(t),\beta)\ \mathrm{d}\beta\right) \dot{u}(t)$$

holds when $\dot{u}(t) > 0$. Consider now the case when $\dot{u}(t) < 0$ for some time instance $t \in I$. By similar arguments we can take $\Delta t > 0$ such that $\dot{u}(\tau) < 0$ for all $\tau \in [t, t+\Delta t]$ and assume without loss of generality that Δt is small enough to guarantee that $M_t = M_\tau$ for all $\tau \in [t, t+\Delta t]$. As before, let us define three subsets of Preisach domain as follows

$$\begin{aligned} \Omega_4 &:= \{(\alpha,\beta) \mid \alpha \geq \beta,\ u(t+\Delta t) \leq \alpha < u(t), \\ &\qquad\qquad u(t+\Delta t) \leq \beta < u(t)\} \\ \Omega_5 &:= \{(\alpha,\beta) \mid u(t) \leq \alpha < M_t,\ u(t+\Delta t) \leq \beta < u(t)\} \\ \Omega_6 &:= \{(\alpha,\beta) \mid a \geq b,\ u(t) \leq \alpha < M_t,\ u(t) \leq \beta < M_t\}, \end{aligned}$$

so that the Preisach plane P can be partitioned as illustrated in Figure 2. Accordingly, we can compute the rate of change of the output as follows

$$\begin{aligned} &\frac{y(t+\Delta t)-y(t)}{\Delta t} = \frac{y(t+\Delta t)-y(t)}{u(t+\Delta t)-u(t)} \frac{u(t+\Delta t)-u(t)}{\Delta t} \\ &= -2\left[\frac{\iint\limits_{(\alpha,\beta)\in\Omega_4} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta}{u(t+\Delta t)-u(t)} + \frac{\iint\limits_{(\alpha,\beta)\in\Omega_5} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta}{u(t+\Delta t)-u(t)}\right] \frac{\Delta u}{\Delta t} \\ &= -2\left[\frac{\int\limits_{u(t+\Delta t)}^{u(t)} \int\limits_{\beta}^{u(t)} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta}{u(t+\Delta t)-u(t)} + \frac{\int\limits_{u(t+\Delta t)}^{u(t)} \int\limits_{u(t)}^{M_t} \mu(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta}{u(t+\Delta t)-u(t)}\right] \frac{\Delta u}{\Delta t}. \end{aligned}$$

By taking the limit $\Delta t \to 0$, we obtain

$$\dot{y}(t) = \left(2\int\limits_{u(t)}^{M_t} \mu(\alpha,u(t))\ \mathrm{d}\alpha\right) \dot{u}(t),$$

which holds for the case when $\dot{u}(t) < 0$. Finally, when $\dot{u}(t) = 0$ we have $\dot{y}(t) = 0$ and (5) holds. □

An immediate consequence of Proposition 3.1 is that when μ has a compact support, it is possible to find a sector bound for (6) which can be useful for the feedback loop analysis. We formalize this in the following proposition.

Proposition 3.2: Suppose that $\mu \in C_{\mathrm{pw}}(P,\mathbb{R})$ has a compact support. Then there exist $\lambda_m, \lambda_M \in \mathbb{R}$ with $\lambda_m \leq \lambda_M$ such that

$$\lambda_m \leq \psi(t) \leq \lambda_M \tag{7}$$

PROOF PROPOSITION 3.2. Let $P_1 \subset P$ be the compact support of μ. Then λ_m and λ_M are the extrema of (6) on P_1. In other words,

$$\lambda_m = 2 \ \min\left\{ \inf_{(\gamma,\kappa)\in P_1} \int_\kappa^\gamma \mu(\gamma,\beta)\mathrm{d}\beta, \ \inf_{(\gamma,\kappa)\in P_1} \int_\kappa^\gamma \mu(\alpha,\kappa)\mathrm{d}\alpha \right\} \tag{8}$$

and

$$\lambda_M = 2 \ \max\left\{ \sup_{(\gamma,\kappa)\in P_1} \int_\kappa^\gamma \mu(\gamma,\beta)\mathrm{d}\beta, \ \sup_{(\gamma,\kappa)\in P_1} \int_\kappa^\gamma \mu(\alpha,\kappa)\mathrm{d}\alpha \right\}. \tag{9}$$

□

In practice, when Preisach operator is used to model a physical phenomenon, it is commonly assumed that μ has a compact support which is partly due to the limited range of measurement data. We refer, for instance, to the works in [13], [15], [16]. We remark that a weaker condition on μ such that ψ lies in a sector bound is by requiring that the zero upper and lower partial moments of μ with respect to each of its arguments are finite.

Note that both Propositions 3.1 and 3.2 are valid for any Preisach operator (2) regardless whether it has a sign-definite or sign-indefinite weighting function. In particular, it follows from Proposition 3.2 that when μ is positive semi-definite (resp., negative semi-definite) then $0 \le \lambda_m \le \lambda_M$ (resp. $\lambda_m \le \lambda_M \le 0$). On the other hand, when μ is sign-indefinite as in the case of the Preisach butterfly operator (c.f. Theorem 2.2), the sector bound of $\psi(t)$ satisfies $\lambda_m < 0 < \lambda_M$.

IV. SET STABILITY WITH PREISACH BUTTERFLY HYSTERESIS IN THE FEEDBACK LOOP

Let us now analyze the interconnection of a linear system with a Preisach butterfly operator in the feedback loop. We will focus on the set stability and convergence analysis. Consider the following feedback interconnection of linear system Σ_1 and a nonlinear operator Σ_2 as follows

$$\begin{aligned} \Sigma_1 : &\begin{array}{ll} \dot{x}(t) = Ax(t) + Bv(t), & x(0) = x_0, \\ z(t) = Cx(t) & \end{array} \\ \Sigma_2 : &\, y(t) = \big(\Phi(u,L_0)\big)(t), \qquad L_0 \in \mathscr{I}, \\ & \qquad\qquad \text{with } v(t) = -y(t), \ u(t) = z(t), \end{aligned} \tag{10}$$

where $x(t) \in \mathbb{R}^n, z(t), v(t), y(t) \in \mathbb{R}$ and A, B, C are system's matrices with suitable dimension and transfer function of Σ_1 is given by $G(s) = C(sI - A)^{-1}B$. The set of equilibria of the combined state of systems in the interconnection (10) is given by

$$\mathscr{E} = \{(x_{\mathrm{ss}}, L_{\mathrm{ss}}) \in \mathbb{R}^n \times \mathscr{I} \mid Ax_{ss} - B\Phi(Cx_{\mathrm{ss}}, L_{\mathrm{ss}}) = 0\}.$$

Proposition 4.1: Let Φ be the Preisach hysteresis operator (2) with a compactly supported μ. Assume that (A,C) is observable and (A,B) is controllable. Suppose that $\overline{G}(j\omega)$ given by

$$\overline{G}(j\omega) := (1 + \lambda_M G(j\omega))\,(1 + \lambda_m G(j\omega))^{-1}, \tag{11}$$

with λ_M and λ_m be as in (9) and (8) is strictly positive real. Then $(x(t), L_t) \to \mathscr{E}$ as $t \to \infty$. Moreover, if A is invertible then

$$\frac{\big(\Phi(z,L_0)\big)(t)}{z(t)} \to \frac{1}{CA^{-1}B} \qquad \text{as } t \to \infty. \tag{12}$$

PROOF PROPOSITION 4.1. Using the differential form of Φ as given in (5) in Proposition 3.1, the output of Preisach operator Φ at any given time t can be expressed as

$$\begin{aligned} y(t) &= y(0) + \int_0^t \psi(\tau)\dot{u}(\tau)d\tau \\ &= \big(\Phi(u,L_0)\big)(0) + \int_0^t \psi(\tau)\dot{u}(\tau)d\tau, \end{aligned} \tag{13}$$

where $\psi(t) \in [\lambda_m, \ \lambda_M]$ is as in (6). Using (13), an equivalent representation to (10) is given by the following piecewise time differentiable state equations

$$\begin{aligned} \ddot{x}(t) &= A\dot{x}(t) + B\dot{v}(t) \\ \dot{z}(t) &= C\dot{x}(t) \\ \dot{v}(t) &= -\psi(t)\dot{z}(t), \qquad \text{a.a. } t \in \mathbb{R}_+. \end{aligned} \tag{14}$$

This is illustrated in Figure 3. Consequently, by the circle criterion results [17], [18], the above interconnected systems with time-varying sector-bounded ψ satisfying (11) is absolutely stable and $\dot{x}(t) \to 0$ as $t \to \infty$. It follows immediately from (10) that

$$Ax(t) - B\big(\Phi(z,L_0)\big)(t) \to 0 \quad \text{as } t \to \infty,$$

and therefore the combined state $(x(t), L_t)$ of the closed-loop system approaches $\mathscr{E}$ as $t \to \infty$. Furthermore, when A is invertible, a simple algebraic computation to the above limit shows that (12) holds. □

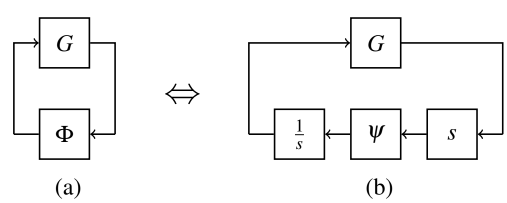

Figure 3: Feedback equivalence of a linear system with Preisach hysteresis in the feedback loop: (a). The original feedback loop as in (10); (b). The equivalent loop as in (14) which is based on the differential formulation as in (5).

From Proposition 4.1 we can see that the value of the ratio between the input and output of the Preisach operator converges to the negative inverse of the zero frequency gain of the linear system. This means that the input-output phase portrait of the Preisach operator will approach a line crossing the origin with slope given by (12). We note that the conditions of observability, controllability and strict positive realness of $\overline{G}$ can be relaxed to stabilizability, detectability and positive realness conditions when the version of the circle criterion in [17, Corollary 9] is considered.

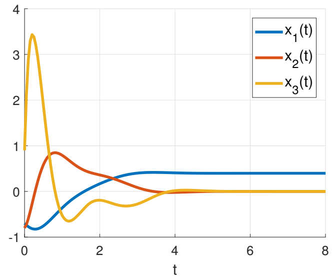

Figure 4: The plot of state trajectories $x(t)$ from the numerical simulation in Section V.

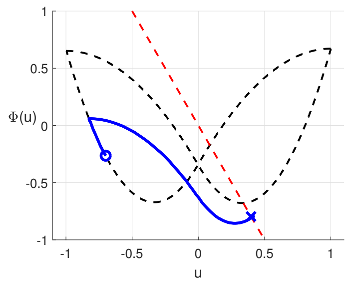

Figure 5: Input-output phase portrait of the closed-loop system from the numerical example. The dashed black line is the major hysteresis loop, the dashed red line is a linear line with the slope of $\frac{1}{CA^{-1}B} = -2$, and the solid blue line is the simulation data with initial values $(u(0), y(0))$ marked by a circle and final values $(u(\infty), y(\infty))$ marked by a cross.

V. Numerical example

Let us consider a numerical example to illustrate the results of the previous sections. Consider a feedback interconnection as in (10) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -20 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

and the weighting function μ of Φ be given by

$$\mu(\alpha,\beta) = \begin{cases} -1 & \text{if } \alpha \le -\beta, \ (\alpha,\beta) \in P_1, \\ 1 & \text{if } \alpha > -\beta, \ (\alpha,\beta) \in P_1, \\ 0 & \text{otherwise,} \end{cases}$$

where P_1 is defined by $P_1 := \{(\alpha,\beta) \in P \mid \alpha \ge \beta, \alpha \le 1, \beta \ge -1\}$. It is clear that P_1 is the compact support of μ and it can be checked that the sector bound defined in (8) and (9) satisfies $\lambda_m = -2$, $\lambda_M = 2$ and $\overline{G}_b(j\omega)$ defined as in (11) is strictly positive real. Therefore, following the result in Proposition 4.1, this feedback interconnection is stable and the steady state gain of the Preisach operator converges to $\frac{1}{CA^{-1}B} = -2$. Figures 4 and 5 show the simulation results of this feedback interconnection with the initial states of the linear system given by $x_0 = [-0.7, -0.8, 0.9]^\top$, and the initial interface given by $L_0 = \{(\alpha,\beta) \in P \mid \beta = -0.7, \ \alpha \ge \beta\}$.

VI. Conclusions

In this work we study the absolute stability property of a linear system with a Preisach butterfly operator in the feedback loop. Using the differential formulation of the operator, we provide sufficient conditions that guarantee the stability of the closed-loop and show the property of its asymptotic behavior.

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