In this thesis we study of the contact process – a particular type of interacting particle system – and different percolation models. Both the contact process and percolation are models of propagation of some material in an environment and have been the topic of intensive and fruitful research in the last decades due to their simplicity, rich behavior and mathematical tractability. Moreover, results often transfer from one model to the other, as a specific type of oriented percolation model can be viewed as a discrete-time version of the contact process.

We have studied how the introduction of inhomogeneities in the environment affects the behavior of the models. In general, random processes on infinite volume do not depend too much on local changes in the environment. In percolation models we can study how changing a small portion of the environment affects the occurrence of percolation. In the case of the contact process, since a single site or edge can affect the dynamics infinitely many times, one can ask whether its presence has an influence on the critical parameter of the process.

The contact process is usually taken as a model of epidemics on a graph: vertices are individuals, which can be healthy or infected. In the continuous-time Markov dynamics, infected individuals recover with rate 1 and transmit the infection to each neighbor with rate $\lambda > 0$ (“infection rate”). The “all healthy” configuration is a trap state for the dynamics; starting from a finite set of initially infected vertices the probability that the contact process ever reaches this state is either equal to 1 for any finite set or strictly less than 1 for any finite set. The process is said to “die out” in the first case and to “survive” in the latter. Whether it survives or dies out will depend on both the underlying graph $G$ and $\lambda$, so one defines the critical rate $\lambda_c$ as the supremum of the infection parameter values for which the contact process dies out on $G$.

The aim of Chapter 1 is to provide an understanding of how local modifications of the graph on which the contact process takes place can cause significant changes in relevant probabilities associated to the process. It is natural to expect that the critical rate $\lambda_c$ of the contact process is not affected by the addition or deletion of finitely many edges of $G$ as long as it remains connected. We consider a slightly different line of inquiry. We give a construction of a tree in which the contact process with any positive infection rate survives but, if a certain privileged edge $e^*$ is removed, one obtains two subtrees in which the contact process dies out for
small $\lambda$.

Percolation theory was introduced as a model of fluid flow through porous material. Consider a connected graph $G = (V, E)$. In a percolation configuration, each edge in $E$ can be “open” or “closed”. An “open path” in $G$ is a sequence of distinct vertices $v_0, v_1, \ldots, v_m \in V$ such that there is an open edge between each consecutive pair of vertices. We say that $v$ can be reached from $u$ either if they are equal or if there is an open path from $u$ to $v$. The set of vertices that can be reached from $v$ is called the “cluster” of $v$. The main object of the study is the probability of the presence of an infinite cluster in the graph.

The general inhomogeneous percolation framework treated in this thesis is as follows. Assume that $E$ is split into two disjoint sets, $E_1$ and $E_2$ and let each edge in $E_1$ be open with probability $p$ and each edge in $E_2$ with probability $q$. Whether or not there is an infinite cluster with positive probability depends on the parameters $p$ and $q$, so we can define $p_c(q)$ as the supremum of the values of $p$ for which there is almost surely no infinite cluster at parameters $p, q$. What can we say about $q \mapsto p_c(q)$?

In Chapter 2 we consider a “ladder graph”: starting with an arbitrary (unoriented, connected) graph $G = (V, E)$ we construct $\bar{G} = (\bar{V}, \bar{E})$ by placing layers of $G$ one on top of the other and adding extra edges to connect the consecutive layers. More precisely, let $V = \mathbb{V} \times \mathbb{Z}$ and

$$E = \{(u, n), (v, n)\} : (u, v) \in E, n \in \mathbb{Z}\} \cup \{(u, n), (u, n + 1)\} : u \in V, n \in \mathbb{Z}\}.$$

Fix an edge $e = \{u, v\} \in E$ in $G$ and let $E_2 := \{(u, n), (v, n)\} : n \in \mathbb{Z}\}. Let each edge of $E_2$ be open with probability $q$, and each edge in $E_1 = E \setminus E_2$ be open with probability $p$. One would expect the aforementioned function $p_c(q)$ to be constant in $(0, 1)$; our main result is that it is a continuous function.

Furthermore, we show that if we fix a finite number of edges on $G$ and simultaneously change the percolation parameter on the corresponding edge sets on $G$ to some $q_1, \ldots, q_n$, then $(q_1, \ldots q_n) \mapsto p_c(q_1, \ldots q_n)$ is continuous in $(0, 1)^n$. We also construct an oriented graph $\bar{G} = (\bar{V}, \bar{E})$ for each $G$ in a similar way, and prove an analogous theorem.

In Chapter 3 we consider an oriented graph whose vertex set is that of the $d$-regular, rooted tree, and containing “short edges” (with which each vertex points to its $d$ children) and “long edges” (with which each vertex points to its $d^k$ descendants $k$ generations below, for fixed $k \in \mathbb{N}$). Percolation is defined on this graph by letting short edges be open with probability $p$ and long edges with probability $q$.

The parameter space $[0, 1]^2$ can be decomposed in two regions: $\mathcal{N}$ is the set of $(p, q)$ for which there is almost surely no infinite cluster at parameters $p$ and $q$, and $\mathcal{P} = [0, 1]^2 \setminus \mathcal{N}$. The curve $p_c(q)$ separating $\mathcal{N}$ and $\mathcal{P}$ is continuous and strictly decreasing in the region where it is positive.

We first show that $p_c(q)$ is strictly above the line $dp + d^kq = 1$. Then we show that the cluster of the root has the same distribution as the family tree of a certain multi-type branching process, which allows us to state some limit
theorems. In the supercritical region we give an asymptotic limit for the number of vertices in the infinite cluster at distance $n$ from the root, denoted by $X_n$. In the subcritical region we show that the distribution of $X_n$ conditioned on being positive converges to a proper distribution. Furthermore, we show that along the critical curve, conditioned on having an infinite cluster, the limiting object can be described as a multi-type Kesten’s tree.