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Cucuzzella, Michele; Ferrara, Antonella

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Practical Second Order Sliding Modes in Single-Loop Networked Control of Nonlinear Systems

Michele Cucuzzella, Antonella Ferrara

Dipartimento di Ingegneria Industriale e dell'Informazione, University of Pavia, via Ferrata 5, 27100 Pavia, Italy

Abstract

This paper presents a novel Second Order Sliding Mode (SOSM) control algorithm for a class of nonlinear systems subject to matched uncertainties. By virtue of its Event-Triggered nature, it can be used as a basis to construct robust networked control schemes. The algorithm objective is to reduce as much as possible the number of data transmissions over the network, in order not to incur in problems typically due to the network congestion such as packet loss, jitter and delays, while guaranteeing satisfactory performance in terms of stability and robustness. The proposed Event-Triggered SOSM control strategy is theoretically analyzed in the paper, showing its capability of enforcing the robust ultimately boundedness of the sliding variable and its first time derivative. As a consequence, it is also possible to prove the practical stability of the considered system, in spite of the reduction of transmissions with respect to a conventional SOSM control approach. Moreover, in order to guarantee the avoidance of the notorious Zeno behaviour, a lower bound for the time elapsed between two consecutive triggering events is provided.

Key words: Sliding mode control; robust control of nonlinear systems; networked control systems.

1 Introduction

Networked Control Systems (NCSs) are the obvious solution to control problems in several field implementations because of their advantages in terms of flexibility and reduction of modification and update costs. In NCSs, the presence of the network in the control loop can determine a deterioration of the performance because of critical issues such as packet loss and transmission delays [1]. Usually, the network malfunctions tend to increase with the network congestion. Thus, the design of control schemes able to reduce the transmissions over the network can be beneficial. In the literature, the so-called Event-Triggered (ET) control [2–5] has been proposed as an effective solution for NCSs. In contrast to conventional time-triggered implementation, which features periodic transmissions of the state measurements, ET control approach enables the state transmission only when some triggering condition is satisfied (or violated). For this reason, ET control can reduce the transmissions over the network avoiding the possible network congestion.

On the other hand, Sliding Mode (SM) control is a well-

known robust control approach, especially useful to control systems subject to matched uncertainties [6]. The same holds for higher order and, in particular, Second Order Sliding Mode (SOSM) control [7, 8], in which not only the sliding variable but also its time derivatives are steered to zero in a finite time. This is confirmed by the numerous applications described in the literature (see, for instance, [9–12]).

In this paper, SOSM control and ET control are coupled to design a novel robust control scheme with a reduced transmission requirement that can be appropriate for NCSs [13, 14]. The proposed control approach is based on two triggering conditions and two control laws that depend on the sliding variable and its first time derivative. Moreover, the proposed control strategy is very easy to implement, it does not require to transmit the state at any time instant, and by virtue of its low implementation complexity, it can be adequate also in case of NCSs. Moreover, the proposed algorithm provides the reduction of the control amplitude when the origin of the auxiliary system state space is approached, with a consequent reduction of the total control energy. The considered system controlled via the proposed strategy is theoretically analyzed in the paper, proving the ultimately boundedness, in a suitable convergence set, of the sliding variable and its first time derivative, even in presence of the uncertainties. In the paper it is also proved that in the convergence set an approximability property analogous to that of classical SM control holds. As a consequence, it is

* This is the final version of the accepted paper submitted to Automatica. Corresponding author Michele Cucuzzella.

Email addresses: michele.cucuzzella@gmail.com (Michele Cucuzzella), antonella.ferrara@unipv.it (Antonella Ferrara).

also possible to prove the practical stability of the considered uncertain nonlinear system. Finally, in order to guarantee the avoidance of the notorious Zeno behaviour, a lower bound for the time elapsed between two consecutive triggering events is provided.

2 Problem Formulation

Consider the uncertain nonlinear system

$$\dot{x} = a(x) + b(x)u + d_m(x), \quad (1)$$

where $x \in \Omega$ ($\Omega \subset \mathbb{R}^n$ bounded) is the state vector, the value of which at the initial time instant t_0 is $x(t_0) = x_0$, and $u \in [-U_{\max}, U_{\max}]$ is the input, while $a(x) : \Omega \rightarrow \mathbb{R}^n$ and $b(x) : \Omega \rightarrow \mathbb{R}^n$ are uncertain functions of class $C^1(\Omega)$. Moreover, the external disturbance d_m is assumed to be matched, i.e.,

$$d_m(x) = b(x)d, \quad d \in \mathcal{D} \subset \mathbb{R}, \quad (2)$$

$\mathcal{D}^{\text{sup}} := \sup_{d \in \mathcal{D}} \{|d|\}$ being a known positive constant. Define a suitable output function (the so-called ‘‘sliding variable’’) $\sigma : \Omega \rightarrow \mathbb{R}$ of class $C^2(\Omega)$, it being defined as follows.

Definition 1 (Sliding variable) σ is a sliding variable for system (1) provided that the pair (σ, u) has the following property: if u in (1) is designed so that, in a finite time $t_r^* \geq t_0$, $\forall x_0 \in \Omega$, $\sigma = \dot{\sigma} = 0 \quad \forall t \geq t_r^*$, then $\forall t \geq t_r^*$ the origin is an asymptotically stable equilibrium point of (1) constrained to $\sigma = 0$.

Now, regarding the sliding variable σ as the controlled variable associated with system (1), assume that system (1) is complete in Ω and has a uniform relative degree equal to 2. The following definitions are introduced.

Definition 2 (Ideal SOSM) Given $t_r^* \geq t_0$ (ideal reaching time), if $\forall x_0 \in \Omega$, $\sigma = \dot{\sigma} = 0 \quad \forall t \geq t_r^*$, then an ‘‘ideal SOSM’’ of system (1) is enforced on the sliding manifold $\sigma = \dot{\sigma} = 0$.

Definition 3 (Practical SOSM) Given $t_r \geq t_0$ (practical reaching time), if $\forall x_0 \in \Omega$, $|\sigma| \leq \delta_1$, $|\dot{\sigma}| \leq \delta_2 \quad \forall t \geq t_r$, then a ‘‘practical SOSM’’ of system (1) is enforced in a vicinity of the sliding manifold $\sigma = \dot{\sigma} = 0$.

Moreover, assume that system (1) admits a global normal form in Ω , i.e., there exists a global diffeomorphism of the form $\Phi = [\Psi, \sigma, a \cdot \nabla \sigma]^T = [x_r, \xi]^T$, with $\Phi : \Omega \rightarrow \Phi_\Omega$ ($\Phi_\Omega \subset \mathbb{R}^n$ bounded), and $\Psi : \Omega \rightarrow \mathbb{R}^{n-2}$, $\nabla \sigma = (\partial \sigma / \partial x_1, \dots, \partial \sigma / \partial x_n)^T$, $x_r \in \mathbb{R}^{n-2}$, $\xi = [\sigma, \dot{\sigma}]^T \in \mathbb{R}^2$, such that

$$\begin{cases} \dot{x}_r = a_r(x_r, \xi) & (3a) \\ \dot{\xi}_1 = \xi_2 & (3b) \\ \dot{\xi}_2 = f(x_r, \xi) + g(x_r, \xi)(u + d), & (3c) \end{cases}$$

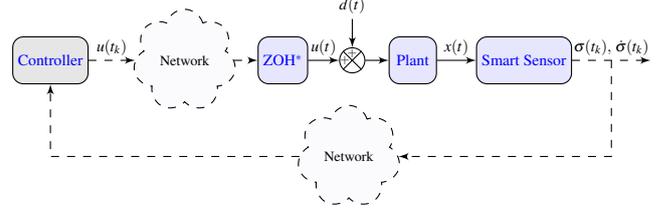


Figure 1. The proposed single-loop networked control scheme.

with $a_r = \frac{\partial \Psi}{\partial x} a$, $f = a \cdot \nabla(a \cdot \nabla \sigma)$, and $g = b \cdot \nabla(a \cdot \nabla \sigma)$. Note that, since a, b are functions of class $C^1(\Omega)$, and σ is a function of class $C^2(\Omega)$, with $\Omega \subset \mathbb{R}^n$ bounded, then functions f, g exist for all $(x_r, \xi) \in \Phi_\Omega$. Moreover, as a consequence of the uniform relative degree assumption, one has that $g \neq 0$. In the literature, see for instance [7], subsystem (3b)-(3c) is called ‘‘auxiliary system’’. Since a_r, f, g are continuous functions and Φ_Ω is a bounded set, one has that

$$\exists F > 0 : |f(x_r, \xi)| \leq F, \quad \exists G_{\max} > 0 : g(x_r, \xi) \leq G_{\max}. \quad (4)$$

In this paper we assume that F and G_{\max} are known. Moreover, we assume that

$$\exists G_{\min} > 0 : g(x_r, \xi) \geq G_{\min}, \quad (5)$$

G_{\min} being a priori known.

Relying on (3)-(5), a first control problem can be stated.

Problem 1 Design a feedback control law

$$u^* = \kappa(\sigma, \dot{\sigma}), \quad (6)$$

with the following property: $\forall x_0 \in \Omega$, $\exists t_r^* \geq t_0$ such that $\sigma = \dot{\sigma} = 0, \forall t \geq t_r^*$, in spite of the uncertainties.

Note that the solution to Problem 1 is in fact a control law capable of robustly enforcing an ‘‘ideal SOSM’’ of system (1)-(5) in a finite time (see Definition 2). In other terms, any SOSM control law is an admissible solution to Problem 1. Note that, since σ is selected to be a sliding variable (see Definition 1), if Problem 1 is solved, one has that $\forall x_0 \in \Omega$, the origin of the state space is a robust asymptotically stable equilibrium point for (1)-(5).

Typically, the state is sampled at time instants $t_k, k \in \mathbb{N}$, and the control law is computed as $u(t) = u(t_k), \forall t \in [t_k, t_{k+1}[$, the sequence $\{t_k\}_{k \in \mathbb{N}}$ being periodic, with $T = t_{k+1} - t_k$ a priori fixed (‘‘time-triggered’’). In this paper, instead of relying on time-triggered executions, we will introduce two triggering conditions, transmitting the values of $\sigma, \dot{\sigma}$ and u only when such conditions are verified (‘‘event-triggered’’). Moreover, we assume that the plant is equipped with a particular zero-order-hold, indicated in Fig.1 with ZOH*, capable of holding constant $u, \forall t \in [t_k, t_{k+1}[$. Relying on (3)-(5), we can formulate the problem that will be solved in the paper.

Problem 2 Design a feedback control law

$$u = u(t_k) = \kappa(\sigma(t_k), \dot{\sigma}(t_k)) \quad \forall t \in [t_k, t_{k+1}[, \quad (7)$$

with the following property: $\forall x_0 \in \Omega, \exists t_r \geq t_0$ such that $|\sigma| \leq \delta_1$, and $|\dot{\sigma}| \leq \delta_2, \forall t \geq t_r$, in spite of the uncertainties, with δ_1 and δ_2 positive constants arbitrarily set.

Note that the solution to Problem 2 is an event-triggered control law capable of robustly enforcing a ‘‘practical SOSM’’ of system (1)-(5) in a finite time (see Definition 3) when a ZOH* is used to generate $u(t)$.

3 The Proposed Solution

The control scheme proposed to solve Problem 2 is reported in Fig. 1. The existence of a communication network is considered. Yet, we do not explicitly model the network, but we propose a control strategy such that the number of transmissions is reduced to avoid the network congestion. Under these considerations we assume that at the time instants when the triggering conditions are verified, the network is available (we refer to [14] for the case with delayed transmissions due to the unavailability of the network). The proposed control scheme contains two key blocks: the ‘‘Smart Sensor’’ and the ‘‘Controller’’.

3.1 The Smart Sensor

First, let us define the convergence set

$$\mathcal{B} := \mathbb{R}^2 \setminus \{S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5\}, \quad (8)$$

where

$$\begin{aligned} S_1 &:= \{(\sigma, \dot{\sigma}) : |\dot{\sigma}| \geq \delta_2\} \\ S_2 &:= \{(\sigma, \dot{\sigma}) : \sigma \geq \delta_1, -\delta_2 < \dot{\sigma} \leq 0\} \\ S_3 &:= \{(\sigma, \dot{\sigma}) : \sigma \leq -\delta_1, 0 \leq \dot{\sigma} < \delta_2\} \\ S_4 &:= \{(\sigma, \dot{\sigma}) : \sigma \geq -\frac{\dot{\sigma}|\dot{\sigma}|}{2U_r} + \delta_1, 0 < \dot{\sigma} < \delta_2\} \\ S_5 &:= \{(\sigma, \dot{\sigma}) : \sigma \leq -\frac{\dot{\sigma}|\dot{\sigma}|}{2U_r} - \delta_1, -\delta_2 < \dot{\sigma} < 0\}, \end{aligned}$$

with

$$U_r := G_{\min}(U_{\max} - \mathcal{D}^{\sup}) - F > 0, \quad (9)$$

where U_{\max} is the control amplitude. In the following, $\partial\mathcal{B}$ will denote the boundary of the convergence set \mathcal{B} (see Fig. 2), and $\sigma_k, \dot{\sigma}_k$ will denote the values of $\sigma(x(t_k))$ and $\dot{\sigma}(x(t_k))$. The considered sensor is smart in the sense that it is able to verify two different triggering conditions.

Triggering Condition 1

$$\sigma = -\frac{\dot{\sigma}|\dot{\sigma}|}{2U_r} \pm \delta_1 \quad \forall (\sigma, \dot{\sigma}) \notin \{\mathcal{B} \cup \partial\mathcal{B}\}. \quad (10)$$

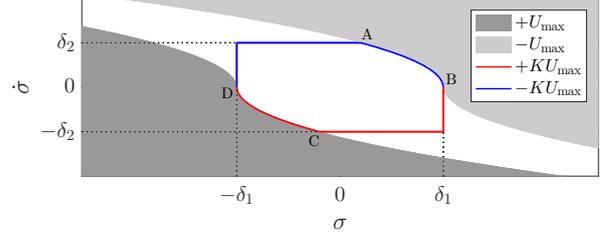


Figure 2. Representation of the convergence set \mathcal{B} .

Note that the Smart Sensor checks the Triggering Condition 1 only during the reaching of $\partial\mathcal{B}$. For the rest of the control interval a second triggering condition is adopted.

Triggering Condition 2

$$(\sigma, \dot{\sigma}) \in \partial\mathcal{B} \quad \forall (\sigma, \dot{\sigma}) \in \{\mathcal{B} \cup \partial\mathcal{B}\}. \quad (11)$$

Only when the triggering condition (10) or (11) is true, are σ and $\dot{\sigma}$ transmitted by the Smart Sensor to the Controller.

3.2 The Controller

The proposed control strategy is based on two different control laws.

Control Law 1

In analogy with [7], control law (7) can be expressed as

$$u(t_k) = -U_{\max} \text{sign} \left(\sigma_k + \frac{\dot{\sigma}_k |\dot{\sigma}_k|}{2U_r} \right) \quad \forall (\sigma_k, \dot{\sigma}_k) \notin \{\mathcal{B} \cup \partial\mathcal{B}\}, \quad (12)$$

with

$$U_{\max} > \frac{F}{G_{\min}} + \mathcal{D}^{\sup}. \quad (13)$$

When $(\sigma, \dot{\sigma})$ reaches $\partial\mathcal{B}$, a second control law is applied for the rest of the control interval. Before introducing the control law, $\forall (\sigma, \dot{\sigma}) \in \{\mathcal{B} \cup \partial\mathcal{B}\}$ we assume that

$$\exists \mathcal{F} : |f(x_r, \xi)| \leq \mathcal{F}, \quad \exists \mathcal{G}_{\min}, \mathcal{G}_{\max} : \mathcal{G}_{\min} \leq g(x_r, \xi) \leq \mathcal{G}_{\max}, \quad (14)$$

where $\mathcal{F} \leq F$, $\mathcal{G}_{\min} \geq G_{\min}$ and $\mathcal{G}_{\max} \leq G_{\max}$ are known positive constants.

Control Law 2

Control law (7) can be expressed as

$$u(t_k) = -KU_{\max} \text{sign}(\dot{\sigma}_k) \quad \forall (\sigma_k, \dot{\sigma}_k) \notin \{\mathcal{B} \cup \partial\mathcal{B}\}, \quad (15)$$

with

$$K = \frac{\frac{\mathcal{F} + U_r}{\mathcal{G}_{\min}} + \mathcal{D}^{\sup}}{\frac{\mathcal{F} + U_r}{\mathcal{G}_{\min}} + \mathcal{D}^{\sup}}, \quad (16)$$

where $K < 1$ is obtained by substituting (14) in (9).

Remark 1 (Control energy reduction) *The proposed algorithm provides the reduction of the control amplitude for any $(\sigma, \dot{\sigma}) \in \{\mathcal{B} \cup \partial\mathcal{B}\}$, with a consequent reduction of the total control energy [15].*

4 Stability Analysis

In this section, the stability properties of systems (1) and (3) controlled via the proposed strategy are analyzed. To this end, it is convenient to introduce the following definitions.

Definition 4 (Practical stability) *In analogy with [16], given the bounded sets $\Omega, \Omega_\delta \subset \Omega$ and \mathcal{D} as in (2), then, the origin of system (1) is said to be practically stable with respect to $(t_r^*, \tau_r, \Omega, \Omega_\delta, \mathcal{D})$ if there exists $\tau_r \geq t_r^*$ such that $\forall t_r^* \geq t_0, \forall d \in \mathcal{D}, \forall x_0 \in \Omega, x \in \Omega_\delta \forall t \geq \tau_r$.*

Definition 5 (Equivalent control) *Given system (1)-(5) controlled via (6), then $\forall t \geq t_r^*$, the so-called “equivalent control” in case of ideal SOSM can be defined by posing in (3c) $\ddot{\sigma} = \ddot{\xi}_2 = 0$, i.e.,*

$$u_{eq} := -\frac{f(x^*)}{g(x^*)} - d, \quad (17)$$

x^* denoting the ideal solution to system (1) when Problem 1 is solved, i.e., $\sigma = \dot{\sigma} = 0, \forall t \geq t_r^*$.

Lemma 1 (Finite time convergence to $\{\mathcal{B} \cup \partial\mathcal{B}\}$) *Given system (3b)-(3c) with $(\sigma_0, \dot{\sigma}_0) \notin \{\mathcal{B} \cup \partial\mathcal{B}\}$, controlled by (10), (12) and (13), then, the solution $(\sigma, \dot{\sigma})$ to (3b)-(3c) is steered to the convergence set $\{\mathcal{B} \cup \partial\mathcal{B}\}$ in a finite time.*

PROOF. For the proof of this Lemma we refer to [7, Theorem 2]. ■

Lemma 2 (Invariance of $\{\mathcal{B} \cup \partial\mathcal{B}\}$) *Given system (3b)-(3c) with $(\sigma_0, \dot{\sigma}_0) \in \{\mathcal{B} \cup \partial\mathcal{B}\}$, controlled by (11), (15) and (16), then, the convergence set $\{\mathcal{B} \cup \partial\mathcal{B}\}$ is a positively invariant set.*

PROOF. Since $\sigma, \dot{\sigma}$ and u are updated only when (11) holds, i.e., when $(\sigma, \dot{\sigma}) \in \partial\mathcal{B}$, the Lemma will be proved showing that for any $(\sigma_0, \dot{\sigma}_0) \in \partial\mathcal{B}$, the vector field $(\dot{\sigma}, \ddot{\sigma})$ never points outside \mathcal{B} . Let $\partial\mathcal{B}^+$ denote $(\sigma, \dot{\sigma}) \in \partial\mathcal{B} : \dot{\sigma} > 0$, and $\partial\mathcal{B}^-$ denote $(\sigma, \dot{\sigma}) \in \partial\mathcal{B} : \dot{\sigma} < 0$ (in Fig. 2, $\partial\mathcal{B}^+$ is blue and $\partial\mathcal{B}^-$ is red). Assume that $(\sigma_0, \dot{\sigma}_0) \in \partial\mathcal{B}^-$. The vector field is $(\dot{\sigma}, f + g(u + d))$ with $\dot{\sigma} < 0$ and, according to (15), $u = KU_{\max}$. Then, $\ddot{\sigma} \geq U_r > 0$, so that the vector field points up-left, that is inside \mathcal{B} . Note that, if $(\sigma_0, \dot{\sigma}_0) \in \overline{CD}$ (all the points on this curve verify $\sigma = -\frac{\dot{\sigma}}{2U_r} - \delta_1$), then the vector field can be, at most, tangent to \overline{CD} , never pointing outside \mathcal{B} . Analogous considerations can be done if $(\sigma_0, \dot{\sigma}_0) \in \partial\mathcal{B}^+$.

Theorem 1 (Ultimately boundedness of $(\sigma, \dot{\sigma})$) *Given system (3b)-(3c) controlled by (10), (12) and (13) when $(\sigma, \dot{\sigma}) \notin \{\mathcal{B} \cup \partial\mathcal{B}\}$, and by (11), (15) and (16) when $(\sigma, \dot{\sigma}) \in \{\mathcal{B} \cup \partial\mathcal{B}\}$, then, the solution $(\sigma, \dot{\sigma})$ to (3b)-(3c) is ultimately bounded with respect to the convergence set $\{\mathcal{B} \cup \partial\mathcal{B}\}$.*

PROOF. The proof is a straightforward consequence of Lemma 1 and Lemma 2. By virtue of Lemma 1, there exists a time instant t_r when the trajectory $(\sigma, \dot{\sigma})$ enters $\{\mathcal{B} \cup \partial\mathcal{B}\}$. Then, by virtue of Lemma 2, $\forall t \geq t_r, (\sigma, \dot{\sigma})$ is ultimately bounded with respect to the convergence set $\{\mathcal{B} \cup \partial\mathcal{B}\}$. ■

Theorem 2 (Approximability) *Given system (3b)-(3c) controlled by (10), (12) and (13) when $(\sigma, \dot{\sigma}) \notin \{\mathcal{B} \cup \partial\mathcal{B}\}$, and by (11), (15) and (16) when $(\sigma, \dot{\sigma}) \in \{\mathcal{B} \cup \partial\mathcal{B}\}$, then, the origin of system (1) is practically stable with respect to $(t_r^*, \tau_r, \Omega, \Omega_\delta, \mathcal{D})$ if*

- (1) *exists a Lipschitz constant L for the right-hand side of (1) obtained with respect to x^* by using the equivalent control (17), i.e.,*

$$\dot{x}^* = a(x^*) - b(x^*) \frac{f(x^*)}{g(x^*)}; \quad (18)$$

- (2) *the partial derivatives of the function $g(x)^{-1}b(x)$, exist and they are bounded in any bounded domain;*
- (3) *exist positive constants M and N such that*

$$\|a(x) + b(x)(u + d)\| \leq M + N\|x\|. \quad (19)$$

PROOF. In analogy with the Regularization Theorem in book [6, Chapter 2], we prove that for any pair of solutions x^*, x under the initial conditions $\|x(t_r^*) - x^*(t_r^*)\| \leq P\delta_2, P > 0$, there exists a positive number H such that $\|x - x^*\| \leq H\delta_2$ on a finite time interval $[t_r^*, T]$, T being the control interval. More precisely, when a practical SOSM is generated, the control u in (3c) differs from the equivalent control (17) and can be expressed as follows

$$u = -\frac{f(x)}{g(x)} - d + \frac{\ddot{\sigma}(x)}{g(x)}. \quad (20)$$

Then, by substituting (20) in (1), the dynamics of the system becomes

$$\dot{x} = a(x) - b(x) \frac{f(x)}{g(x)} + b(x) \frac{\ddot{\sigma}(x)}{g(x)}. \quad (21)$$

Now, relying on (18) and (21), one can compute the integral equations of x^* and x , respectively, i.e.,

$$x^* = x^*(t_r^*) + \int_{t_r^*}^t \left(a(x^*(\zeta)) - b(x^*(\zeta)) \frac{f(x^*(\zeta))}{g(x^*(\zeta))} \right) d\zeta, \quad (22)$$

$$\begin{aligned}
x &= x(t_r^*) + \int_{t_r^*}^t \left(a(x(\zeta)) - b(x(\zeta)) \frac{f(x(\zeta))}{g(x(\zeta))} \right) d\zeta \\
&+ \int_{t_r^*}^t \left(b(x(\zeta)) \frac{\dot{\sigma}(x(\zeta))}{g(x(\zeta))} \right) d\zeta. \tag{23}
\end{aligned}$$

Integrating the last term in (23) by parts and subtracting (22) to (23), it yields

$$\begin{aligned}
\|x - x^*\| &\leq \|x(t_r^*) - x^*(t_r^*)\| \\
&+ \left\| \int_{t_r^*}^t \left(a(x(\zeta)) - b(x(\zeta)) \frac{f(x(\zeta))}{g(x(\zeta))} \right) d\zeta \right. \\
&- \int_{t_r^*}^t \left(a(x^*(\zeta)) - b(x^*(\zeta)) \frac{f(x^*(\zeta))}{g(x^*(\zeta))} \right) d\zeta \left. \right\| \\
&+ \left[\|b(x(\zeta))\| \left\| \frac{\dot{\sigma}(x(\zeta))}{g(x(\zeta))} \right\| \right]_{t_r^*}^t \\
&+ \int_{t_r^*}^t \left\| \frac{d}{d\zeta} \frac{b(x(\zeta))}{g(x(\zeta))} \right\| |\dot{\sigma}(x(\zeta))| d\zeta. \tag{24}
\end{aligned}$$

Taking into account assumption (3) in the theorem statement, and according to the Bellman-Gronwall lemma, the solution x is bounded on the finite time interval $[t_r^*, T]$, i.e.,

$$\|x\| \leq (\|x(t_r^*)\| + M(T - t_r^*)) e^{N(T - t_r^*)}, \quad \forall t \in [t_r^*, T]. \tag{25}$$

Then, by virtue of Theorem 1 and (25), taking into account assumptions (1), (2) in the theorem statement, the inequality (24) can be expressed as

$$\|x - x^*\| \leq S\delta_2 + L \int_{t_r^*}^t \|x(\zeta) - x^*(\zeta)\| d\zeta \tag{26}$$

S being a positive constant that depends on the right-hand side of (21), $x(t_r^*)$, $x^*(t_r^*)$, t_r^* , T and P . Now, applying again the Bellman-Gronwall lemma to (26), one has that $\|x - x^*\| \leq H\delta_2$, with $H = Se^{L(T - t_r^*)}$. Finally, since by Definition 1, $\forall t \geq t_r^*$, the origin is an asymptotically stable equilibrium point of (1) constrained to $\sigma(x^*) = 0$, then there exists $\tau_r \geq t_r^*$ such that $x \in \Omega_\delta$, $\forall t \geq \tau_r$, which proves the theorem. ■

Now, since the triggering time instants are known only at the execution times, we prove the existence of lower bounds for the inter-event times [2]. Let $\tau_{\min,1}$ and $\tau_{\min,2}$ be the minimum inter-event time when $(\sigma, \dot{\sigma}) \notin \{\mathcal{B} \cup \partial\mathcal{B}\}$ and when $(\sigma, \dot{\sigma}) \in \{\mathcal{B} \cup \partial\mathcal{B}\}$, respectively.

Theorem 3 (Minimum inter-event time $\tau_{\min,1}$) *Given system (3b)-(3c) with $(\sigma_0, \dot{\sigma}_0) \notin \{\mathcal{B} \cup \partial\mathcal{B}\}$, controlled by (10), (12) and (13), then, $\forall (\sigma, \dot{\sigma}) \notin \{\mathcal{B} \cup \partial\mathcal{B}\}$, the inter-event times are lower bounded.*

PROOF. Assume $\sigma_0 > 0$ and $\dot{\sigma}_0 > 0$. Let t_1 be the first triggering time instant when $\sigma = -\frac{\dot{\sigma}|\dot{\sigma}|}{2U_r} + \delta_1$ in (10) is verified. In order to compute the lower bound, we assume that the trajectory evolves with acceleration $-U_r$ from $(\sigma_0, \dot{\sigma}_0)$ to $(\sigma(t'_0), \dot{\sigma}(t'_0))$ that lies on the $\dot{\sigma} = 0$ axis, i.e.,

$$\sigma(t'_0) = \sigma_0 + \frac{\dot{\sigma}_0^2}{2U_r}, \quad \dot{\sigma}(t'_0) = 0. \tag{27}$$

Assume now that the trajectory evolves with acceleration $-U_R := -(G_{\max}(U_{\max} + \mathcal{D}^{\text{sup}}) + F)$ from (27) to $(\sigma(t_1), \dot{\sigma}(t_1))$, i.e.,

$$\begin{aligned}
\sigma(t_1) &= \frac{\dot{\sigma}^2(t_1)}{2U_r} + \delta_1 \\
\dot{\sigma}(t_1) &= -\sqrt{\frac{2U_R U_r}{U_R + U_r} \left(\sigma(t'_0) + \frac{\dot{\sigma}^2(t'_0)}{2U_r} - \delta_1 \right)}. \tag{28}
\end{aligned}$$

Finally, one can compute the time interval $\tau_{\min,1} = t_2 - t_1$ that the trajectory takes to evolve with acceleration $-U_R$ from (28) to $(\sigma(t_2), \dot{\sigma}(t_2))$ on the curve $\sigma = -\frac{\dot{\sigma}|\dot{\sigma}|}{2U_r} - \delta_1$, i.e.,

$$\tau_{\min,1} = \frac{\gamma \dot{\sigma}(t_1) + \sqrt{\gamma^2 \dot{\sigma}^2(t_1) - 2\gamma U_R \left(\frac{\dot{\sigma}^2(t_1)}{2U_r} - \sigma(t_1) - \delta_1 \right)}}{\gamma U_R} \tag{29}$$

with $\gamma = \frac{U_R}{U_r} + 1$. Analogous considerations can be done starting from different initial condition $(\sigma_0, \dot{\sigma}_0)$. ■

Theorem 4 (Minimum inter-event time $\tau_{\min,2}$) *Given system (3b)-(3c) with $(\sigma_0, \dot{\sigma}_0) \in \{\mathcal{B} \cup \partial\mathcal{B}\}$, controlled by (11), (15) and (16), then, $\forall (\sigma, \dot{\sigma}) \in \{\mathcal{B} \cup \partial\mathcal{B}\}$, the inter-event times are lower bounded by*

$$\tau_{\min,2} = \frac{\delta_2}{\mathcal{F} + \mathcal{G}_{\max}(KU_{\max} + \mathcal{D}^{\text{sup}})}.$$

PROOF. Since $(\sigma, \dot{\sigma})$ and u are transmitted over the network only when the triggering condition (11) is verified, the theorem will be proved by computing the time interval $t_{k+1} - t_k$ that $\dot{\sigma}$ takes to evolve from 0 to δ_2 with acceleration $U_{\mathcal{R}} := \mathcal{F} + \mathcal{G}_{\max}(KU_{\max} + \mathcal{D}^{\text{sup}})$. Then, it yields

$$\begin{aligned}
\dot{\sigma}(t_{k+1}) - \dot{\sigma}(t_k) &= \int_{t_k}^{t_{k+1}} \ddot{\sigma}_{\max} d\tau \\
\delta_2 - 0 &= U_{\mathcal{R}} (t_{k+1} - t_k) \\
\delta_2 &= (\mathcal{F} + \mathcal{G}_{\max}(KU_{\max} + \mathcal{D}^{\text{sup}})) \tau_{\min,2}. \tag{30}
\end{aligned}$$

Analogous considerations can be done if we consider the evolution of $\dot{\sigma}$ from 0 to $-\delta_2$. ■

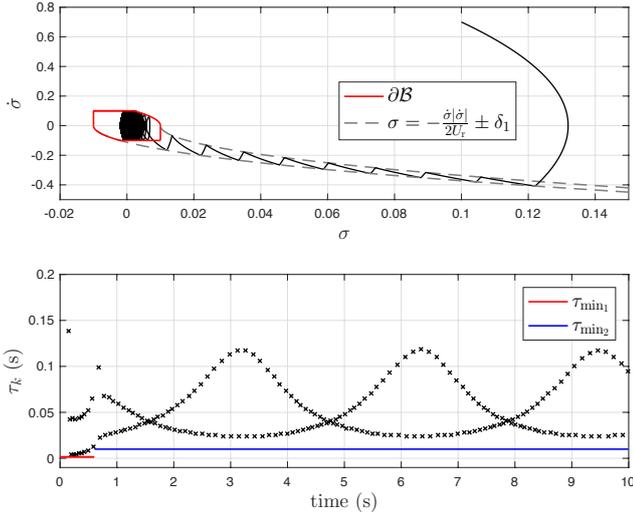


Figure 3. From the top: trajectory of the auxiliary system state; inter-event times.

Remark 2 (Zeno behaviour) Note that Theorems 3 and 4 guarantee that the time elapsed between two consecutive triggering events does not become arbitrarily small, avoiding the notorious Zeno behaviour [17]. The minimum inter-event times reasonably depend on the sizes of the desired convergence set \mathcal{B} , and on the bounds of the uncertainties. In practical cases, this result is very useful to assess the feasibility of the proposed control approach.

5 Illustrative Example

Consider the nonlinear uncertain system,

$$\begin{cases} \dot{x}_1 = x_2 + x_3 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = 0.1 \sin(x_2) + (1 + 0.1 \sin(x_2))(u + d) \end{cases}, \quad (31)$$

with $d = -\mathcal{D}^{\text{sup}} \cos(t)$, $\mathcal{D}^{\text{sup}} = 3.3$. Let $t_0 = 0$ and $x_0 = [0.1 \ -0.1 \ 0.8]^T$. Then, there exists $\Phi(x) = [x_2, x_1, x_2 + x_3]^T = [x_r, \xi_1, \xi_2]^T$. Relying on systems (31) it is possible to set the bounds in (4), (5) and (14) equal to $F = \mathcal{F} = 0.9$, $G_{\max} = \mathcal{G}_{\max} = 1.1$, and $G_{\min} = \mathcal{G}_{\min} = 0.9$. According to (9), (13) and (16) we choose $U_r = 0.63$, $U_{\max} = 5.0$ and $K = 1$. Moreover, we select in (8) $\delta_1 = 0.01$ and $\delta_2 = 0.1$. The trajectory of the auxiliary system state is shown in the top of Fig. 3. At the bottom of Fig. 3 the inter-event times $\tau_k = t_{k+1} - t_k$, which are greater than lower bounds $\tau_{\min,1} = 0.0016$ s and $\tau_{\min,2} = 0.01$ s are shown. Finally, considering a sampling time $T_s = 10^{-4}$ s, and a control interval $T = 10$ s, the number of transmissions is 99.8 % less then the number required by the conventional time-driven implementation.

6 Conclusions

In this paper a novel Second Order Sliding Mode control strategy of event-triggered type for uncertain nonlinear sys-

tems is presented. The proposed control scheme requires the transmission of the sliding variable and its first time derivative only when some suitably defined triggering condition is verified. In the paper we prove that the solution to the auxiliary system is ultimately bounded in a prescribed convergence set, implying the practical stability of the considered system in spite of the reduction of the transmissions, which makes the proposed control strategy suitable for networked implementations. Moreover, the avoidance of the notorious Zeno behaviour is guaranteed.

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