The Nullcone of the Lie Algebra of $G_2$

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Abstract

This paper investigates the nilpotent conjugacy classes of the Lie algebra of the simple algebraic group of type $G_2$. These classes are determined by first finding the stratification, and then finding the classes within the strata. Except for characteristic 3, the classes coincide with the strata. In characteristic 3, one stratum splits into two orbits. If the characteristic differs from 2 and 3, the classes are determined by the singularities of the nilpotent variety. In characteristic 3, the matter is undecided yet. In characteristic 2, different classes have the same singularities.

Key words: simple group; $G_2$; nilpotent; singularity; stratification.

1 Introduction

Let $G$ be a reductive group with Lie algebra $\mathfrak{g}$. Can the nilpotent conjugacy classes in $\mathfrak{g}$ be characterized by the singularities of the nilpotent variety? In the paper [7], a positive answer to this question was given for the cases that $G$ is $GL(n)$, or $Sp(n)$ and $\text{char}(K) \neq 2$. The primary aim of the present paper is to extend this result to the group $G_2$.

The first thing to do is to determine the nilpotent conjugacy classes in $\mathfrak{g}$. Traditionally, they are classified by means of the Theorem of Jacobson-Morozov. This leads, however, to unnatural assumptions on the characteristic of the field. Stuhler [24] was one of the first to determine the nilpotent conjugacy classes for $G_2$ in all characteristics. More recently, the book [13] determines these classes for all simple groups.

We propose a two-step approach for the determination of the nilpotent conjugacy classes for $G_2$. The first step is the determination of the stratification of the nullcone of $\mathfrak{g}$ in the sense of [9]. This can be done independently of the characteristic. In the second step, the orbits within the strata are determined. Here, the characteristics 2 and 3 need special attention. The stratification theory of [9] is presented here over a field of arbitrary characteristic, and is extended slightly for the sake of efficient computation.

There are two classical ways to construct $G_2$: either as the fixpoint set for the outer symmetry of the Dynkin diagram of $D_4$, or as the automorphism group of an octonion algebra. The group $D_4$ is most easily represented as $SO(8)$. We can therefore combine these ways in an eight dimensional representation of $G_2$, except for a minor gap in characteristic 2.

In general, there are five nilpotent conjugacy classes (orbits): a regular class of dimension 12, a subregular class of dimension 10, the subsubregular class (dimension 8), the class of the long root vector (dimension 6), and the origin (dimension 0). In characteristic 3, however, the subsubregular class splits into two orbits of dimensions 8 and 6.
The nilpotent variety or nullcone is the zero set of the homogeneous invariant polynomials. For $G_2$, the ring of the invariant polynomials is generated by two homogeneous elements, one of degree 2, and the other of degree 3 if $\text{char}(K) = 2$, and of degree 6 otherwise.

In [7], the singularities in the nilpotent varieties of $\mathbb{G}_L(n)$ and $\mathbb{S}p(n)$ are characterized by a numerical criterion $\text{ord}^*$. This criterion is used here as well. It separates the orbits in the nilpotent variety of $G_2$ in characteristics $\neq 2, 3$. In characteristic 3, the singularities in the two subsingular orbits seem to be different, but they are not separated by $\text{ord}^*$. In characteristic 2, the nilpotent variety is smooth in the regular orbit and in the subregular orbit, while the other two nonzero orbits have singularities that are smoothly equivalent.

Overview

Section 2 deals with the general theory of the stratification of the nullcone, for a reductive group over an algebraically closed field of arbitrary characteristic. In Section 3 an eight dimensional representation of $G_2$ is constructed, more precisely of the split version of $G_2$ over an arbitrary field. Section 4 presents the stratification of the nullcone of the Lie algebra of $G_2$, followed by the determination of the orbit structure. In Section 5 the nilpotent variety is defined, and its singularities are related to the orbits.

2 The Stratification of the Nullcone

The stratification of the nullcone is based on classical ideas of Hilbert and Mumford [17], presented in Sections 2.1 and 2.2, and in particular on the optimality theory of Kempf [11], presented in the Sections 2.3 and 2.4.

The stratification theory of [9, 10] is presented in Section 2.5. This theory is extended here slightly to make it easier to determine the stratification in concrete cases. In Section 2.6 it is shown that the nullcone of the Lie algebra $g$ of the group $G$ is the set of the nilpotent elements of $g$.

2.1 Concentration and the nullcone

Let $K$ be an algebraically closed field of arbitrary characteristic. Let $G$ be a linear algebraic group over $K$, cf. [1]. Let $X(G)$ denote the abelian group of the characters $\chi : G \to \mathbb{G}_L(1)$, and let $Y(G)$ be the set of the homomorphisms $\lambda : \mathbb{G}_L(1) \to G$. If $\chi \in X(G)$ and $\lambda \in Y(G)$ then $(\chi, \lambda) \in \mathbb{Z}$ is defined by $\chi(\lambda(t)) = t^{(\chi, \lambda)}$. If $\lambda \in Y(G)$ and $n \in \mathbb{Z}$ then $n\lambda \in Y(G)$ is defined by $(n\lambda)(t) = \lambda(t^n)$. If $G$ is a torus, $X(G)$ and $Y(G)$ are free $\mathbb{Z}$-modules of finite rank and $(,)$ defines a duality between them.

The set $M(G)$ is defined as the set of equivalence classes for the equivalence relation $\sim$ on $Y(G) \times \mathbb{N}_+$ where $(\mu, m) \sim (\nu, n)$ if and only if $n\mu = m\nu$. The elements of $M(G)$ are called coweights. If $G$ is a torus, $M(G) = Y(G) \otimes \mathbb{Q}$ is a vector space and $X(G) \otimes \mathbb{Q}$ is its dual.

Let $V$ be a pointed affine $G$-variety, i.e., $G$ acts on $V$ and $V$ has a $G$-invariant base point $\ast$. A point $v \in V$ is called concentrated if there is $\lambda \in Y(G)$ such that $\lim_{t \to 0} \lambda(t)(v) = \ast$. The assertion $\lim_{t \to 0} \lambda(t)(v) = \ast$ means that there is a morphism of algebraic varieties $f : \mathbb{A}^1 \to V$ with $f(0) = \ast$ and $f(t) = \lambda(t)v$ for $t \neq 0$. Mumford [17] defined $m(v, \lambda)$ to be the multiplicity of the fiber $f^{-1}(\ast)$. By convention $m(\ast, \lambda) = +\infty$. If $\lambda \in M(G)$ then $n\lambda \in Y(G)$ for some $n > 0$ and we can define $m(v, \lambda) = n^{-1}m(v, n\lambda)$. For rational $r$, the set $V(\lambda, r) = \{v \in V \mid r \leq m(v, \lambda)\}$ is a closed subset of $V$. 

The nullcone $Nc(V)$ is defined as the set of concentrated points of $V$. In general, the nullcone need not be closed, see [10, 1.3].

### 2.2 The Hilbert-Mumford theory

Two central results in this area must be mentioned. The first one is Theorem A.1.0 of [17, p. 192]:

**Theorem 1** The algebraic group $G$ is reductive (in the sense of [1]) if and only if, for every finitely generated $K$-algebra $R$ on which $G$ acts rationally by $K$-automorphisms, the ring of invariants $R^G$ is finitely generated.

The second result (given e.g. in [10, Section 1.2]) shows that concentration is closely related to invariant theory.

**Theorem 2** Assume that the algebraic group $G$ is reductive. Let $V$ be a pointed affine $G$-variety. For $v \in V$, the following three conditions are equivalent:

(i) $v$ is concentrated,

(ii) the point $*$ is in the closure of the orbit $Gv$,

(iii) $f(x) = f(*)$ for every $G$-invariant function $f$ on $V$.

Let $A(V)$ be the $K$-algebra of the polynomial functions on $V$. The group $G$ acts of $A(V)$ by $K$-automorphisms. If $G$ is reductive, Theorem 1 implies that $A(V)^G$ is finitely generated, and Theorem 2 implies that the nullcone $Nc(V)$ is the zero-set of the ideal in $A(V)$ generated by the functions in $A(V)^G$ that vanish in the point $*$. In particular, the nullcone is closed.

### 2.3 Optimality

A norm $q$ on $M(G)$ is a function $q : M(G) \to \mathbb{Q}$ such that

1. If $\lambda \in M(G)$ is nonzero then $q(\lambda) > 0$.

2. If $\lambda \in M(G)$ and $g \in G$ then $q(\text{int}(g)\lambda) = q(\lambda)$.

3. If $T$ is a subtorus of $G$ the restriction of $q$ to $M(T)$ is a quadratic form on the vector space $M(T)$, with an associated inner product such that $(\lambda, \lambda) = q(\lambda)$.

It is well known that a norm of $M(G)$ exists. More precisely, if $T$ is a maximal torus of $G$ and $W$ is the Weyl group, every $W$-invariant norm on $M(T)$ has a unique extension to a norm of $M(G)$, and every norm on $M(G)$ restricts to a $W$-invariant norm on $M(T)$, see [8, 17].

From now, a norm $q$ on $M(G)$ is fixed. If $X$ is a subset of $V$, the number $q^*(X)$ is defined by

$$ q^*(X) = \inf \{ q(\lambda) \mid \lambda \in M(G) : X \subset V(\lambda, 1) \} \, . $$

The set $X$ is said to be **concentrated** iff $q^*(X) < \infty$, i.e., if $X \subset V(\lambda, 1)$ for some $\lambda$. The **optimal class** $\Lambda(X)$ is defined by

$$ \Lambda(X) = \{ \lambda \in M(G) \mid X \subset V(\lambda, 1) \land q(\lambda) = q^*(X) \} \, . $$

For simplicity, we assume henceforward that $V$ is a $G$-module pointed by $0$. If $T$ is a torus in $G$, let $V = \sum_{\pi} V_\pi$ be the corresponding weight space decomposition where $\pi$ ranges over $X(T)$. For any subset $R$ of $X(T)$, let $V[R] = \sum_{\pi \in R} V_\pi$. The **Newton polytope** $R(X, T)$ of $X$ is defined as the smallest subset $R$ of $X(T)$ with $X \subset V[R]$. For any $\lambda \in M(T)$, let
Lemma 3 Let \( X \) be a subset of \( V \) and let \( T \) be a torus in \( G \). Then \( \Lambda(X) \cap M(T) \) contains at most one element.

Assume \( X \) is a concentrated set. Let \( T \) be a maximal torus of \( G \). As all maximal tori of \( G \) are conjugate, it holds that

\[
q^\ast(X) = \inf\{q^\ast(R(g^{-1}X,T)) \mid g \in G\}.
\]

As all Newton polytopes for \( T \) are contained in the finite set \( R(V,T) \), there exists \( h \in G \) with

\[
q^\ast(X) = q^\ast(R(h^{-1}X,T)) = q^\ast(R(X,\operatorname{int}(h)T)).
\]

Putting \( T_0 = \operatorname{int}(h)T \) and \( \delta_0 = \delta(R(X,T_0)) \), we have \( \delta_0 \in \Lambda(X) \cap M(T_0) \). As the other implication is trivial, this proves

Lemma 4 Let \( X \) be a subset of \( V \). The set \( \Lambda(X) \) is nonempty if and only if the set \( X \) is concentrated.

2.4 Kempf’s theorem

From now onward, the group \( G \) is assumed to be reductive and connected.

The interior action of \( G \) on itself, given by \( \operatorname{int}(g)h = ghg^{-1} \) and pointed by \( * = e \), is of particular importance. Because \( G \) is reductive, the corresponding subset \( P(\lambda) = G(\lambda,0) \) is a parabolic subgroup of \( G \), e.g., by [17, p. 55]. If \( \mu = \operatorname{int}(p)\lambda \) for some \( p \in P(\lambda) \), then \( V(\mu,r) = V(\lambda,r) \) for any pointed affine \( G \)-variety \( V \); in particular \( P(\mu) = P(\lambda) \). We therefore have an equivalence relation \( \sim \) on \( (M(G) \setminus \{0\}) \) defined by \( \lambda \sim \mu \) iff \( \mu = \operatorname{int}(p)\lambda \) for some \( p \in P(\lambda) \). The quotient set is called the \textit{vectorial building} \( \mathbf{Vb}(G) = (M(G) \setminus \{0\}) \). For a pointed affine \( G \)-variety \( V \) and \( \lambda \in \mathbf{Vb}(G) \), we can now define \( V(\lambda,r) = V(\lambda,r) \) for any \( \lambda \in \Lambda \). In particular \( P(\lambda) = P(\lambda) \).

Lemma 5 [11, 9]. Let \( \Lambda_1, \Lambda_2 \in \mathbf{Vb}(G) \) be such that \( (\Lambda_1 \cup \Lambda_2) \cap M(T) \) contains at most one element for every torus \( T \) in \( G \). Then \( \Lambda_1 = \Lambda_2 \).

This result is used to prove Kempf’s optimality theorem [11, 9]:

Theorem 6 Let \( X \) be a concentrated set. Then \( \Lambda(X) \) is a single equivalence class of \( M(G) \), i.e. an element of \( \mathbf{Vb}(G) \).

Proof. If \( \lambda \sim \mu \) then \( V(\lambda,1) = V(\mu,1) \) and \( q(\lambda) = q(\mu) \). Therefore \( \Lambda(X) \) is a union of equivalence classes. Lemma 3 and Lemma 5 together imply that \( \Lambda(X) \) is contained in one equivalence class. Lemma 4 says that \( \Lambda(X) \) is nonempty. \( \Box \)

In view of the above, for concentrated set \( X \), the \textit{saturation} \( S(X) \) and the \textit{Kempf group} \( P(X) \) of \( X \) are defined by \( S(X) = V(\Lambda(X),1) \) and \( P(X) = P(\Lambda(X)) \). The set \( S(X) \) is a concentrated closed subset of \( V \) that contains \( X \). The group \( P(X) \) is a parabolic subgroup of \( G \), and it is the stabilizer of \( S(X) \), i.e. \( P(X) = \{g \in G \mid gS(X) \subset S(X)\} \), cf. [9].

The functions \( q^\ast, \lambda, S, P \) implicitly depend on the group \( G \). If useful, an index \( G \) is used to make the dependence explicit.

For a not-necessarily reductive subgroup \( H \) of \( G \), the set \( M(H) \) is a subset of \( M(G) \), and the restriction of \( q \) to \( M(H) \) is a norm on \( M(H) \). A subgroup \( H \) is called \textit{optimal} for \( X \) iff \( q^\ast_H(X) = q^\ast_G(X) \), in which case \( \Lambda_X(\lambda) = M(H) \cap \Lambda_G(\lambda) \).
2.5 The stratification

For the stratification and the subsequent orbit classification we are interested mainly in the case that \( X \) is a singleton set \( \{ v \} \), in which case we write \( q^*(v), \Lambda(v), P(v), S(v) \) instead of \( q^*(\{ v \}), \Lambda(\{ v \}), \) etc.

The stratification of the nullcone is defined in [9] by means of the equivalence relations \( \sim \) and \( \approx \) on \( \text{Nc}(V) \) given by

\[
\begin{align*}
    x \approx y & \iff \Lambda(x) = \Lambda(y) , \\
    x \sim y & \iff \Lambda(gx) = \Lambda(y) \text{ for some } g \in G.
\end{align*}
\]

An equivalence class \( [v] = \{ x \mid x \approx v \} \) is called a \textit{blade}. An equivalence class \( G[v] = \{ x \mid x \sim v \} \) is called a \textit{stratum}. In [9, 4.2], the following result is proved.

**Proposition 7** Let \( v \in \text{Nc}(V) \).

(a) \( [v] = \{ x \in S(v) \mid q^*(x) = q^*(v) \} \). It is open in \( S(v) \) and \( S(v) \) is its closure.

(b) \( \text{GS}(v) \) is an irreducible closed subset \( V \), contained in \( \text{Nc}(V) \).

(c) \( G[v] = \{ x \in \text{GS}(v) \mid q^*(x) = q^*(v) \} \). It is open and dense in \( \text{GS}(v) \).

A coweight \( \lambda \) is called \textit{optimal} for \( V \) iff the set \( b(V, \lambda) = \{ x \in V \mid \lambda \in \Lambda(x) \} \) is nonempty, or equivalently iff \( b(V, \lambda) \) is a blade.

In the next results, the theory of [9] is extended slightly to make it easier to determine the stratification. Fix a maximal torus \( T \) and a Borel group \( B \) of \( G \) with \( T \subset B \). Let \( B_u \) be the maximal unipotent subgroup of \( B \).

**Proposition 8** Let \( \lambda \in M(T) \) and \( v \in \text{Nc}(V) \). It holds that \( \lambda \in \Delta_G(v) \) if and only if \( v \in \Lambda(v,1) \) and \( q(\lambda) \leq q(\mu) \) for every optimal coweight \( \mu \) and every \( g \in B_u \) with \( gv \in V(\mu,1) \).

**Proof.** Assume \( \lambda \in \Delta_G(v) \). Then \( v \in V(\lambda,1) \) by definition. Moreover \( q(\lambda) = q^*(v) = q^*(gv) \leq q(\mu) \) whenever \( gv \in V(\mu,1) \) (and \( g \in B_u \) and \( \mu \) optimal).

Conversely, assume that \( v \in V(\lambda,1) \) and \( \lambda \notin \Delta_G(v) \). Then \( q^*(v) < q(\lambda) \). By [8, 5.4(b)], the parabolic group \( B \) is optimal for \( v \). Therefore, \( B \) has an optimal coweight \( \mu' \in M(B) \cap \Lambda(v) \) with \( q(\mu') = q^*(v) < q(\lambda) \). Moreover \( B \) has a maximal torus \( T' \) with \( \mu' \in M(T') \). As all maximal tori of \( B \) are conjugate under the unipotent group \( B_u \), there is \( g \in B_u \) with \( \text{int}(g)T' = T \). Then \( \mu = \text{int}(g)\mu' \in M(T) \cap \Lambda(gv) \) has \( q(\mu) < q(\lambda) \) and \( gv \in V(\mu,1) \). \( \Box \)

Although they are not equivalent, this proposition plays here the same role as the Kirwan-Ness criterion [12, 18] in e.g. [20] and [4].

Coming back to the concepts and notations of Section 2.3, a coweight \( \lambda \in M(T) \) is called \textit{preoptimal} for \( V \) iff \( \lambda = \delta(H(\lambda) \cap R(V, T)) \).

**Lemma 9** (a) \( \lambda \in M(T) \) be optimal for \( V \). Then it is a preoptimal for \( V \).

(b) In \( M(T) \), the number of preoptimal coweights for \( V \) is finite.

**Proof.** (a) There is \( v \in \text{Nc}(V) \) with \( \lambda \in \Lambda(v) \). Then \( \lambda = \delta(R(v, T)) \). The set \( R(v, T) \) is a subset of \( H(\lambda) \cap R(V, T) \), and that the latter set has the same minimal coweight as \( R(v, T) \). Therefore \( \lambda \) is a preoptimal.

(b) The set \( R(V, T) \) is finite and has therefore only finitely many subsets. \( \Box \)

Recall that a coweight \( \lambda \) is called \textit{dominant} iff \( B \subset P(\lambda) \). A coweight \( \lambda \) is called \textit{critical} if it is both optimal and dominant. It is called a \textit{candidate} iff it is both preoptimal and dominant.

It is easy to see that every blade \( U \) is a concentrated set and satisfies \( \Lambda(U) = \Lambda(v) \) for all \( v \in U \). A blade \( U \) is called \textit{dominant} iff \( B \subset P(\Lambda(U)) \), or equivalently iff there is a critical coweight \( \lambda \in \Lambda(U) \).
Lemma 10  (a) If $X$ is a stratum of $\text{Nc}(V)$ there is a unique dominant blade $U$ with $X = G \cdot U$.
(b) If $U$ is a dominant blade of $\text{Nc}(V)$ there is a unique critical coweight $\lambda \in M(T)$ with $U = b(V, \lambda)$.
(c) Conversely, if $\lambda$ is a critical coweight, then $b(V, \lambda)$ is a dominant blade and is open and dense in $V(\lambda, 1)$, the set $G \cdot b(V, \lambda)$ is a stratum and is open and dense in the closed set $G \cdot V(\lambda, 1)$.
(d) The strata of $\text{Nc}(V)$ are in bijective correspondence with the dominant blades, and with the critical coweights.
(e) $\text{Nc}(V)$ has finitely many strata.

Proof.  (a) One can choose $v \in X$ with $B \subset P(v)$. The blade $[v]$ is dominant and satisfies $X = G \cdot [v]$. In order to prove uniqueness, assume $X = G \cdot U$ for some other dominant blade $U$. Then there is $g \in G$ with $gv \in U$. Put $P = P(v)$. Then $\text{int}(g)P = P(gv)$. As both $[v]$ and $[gv]$ are dominant blades, $B$ is a subset of both parabolic groups $P$ and $\text{int}(g)P$. Therefore, $\text{int}(g)P = P$ by [1, 11.17]. It follows that $g \in P$ by [1, 11.16]. This proves that $\Lambda(gv) = \Lambda(v)$ and, hence, $[gv] = [v]$.

The parts (b), (c), (d) can be left to the reader.

(e) The number of critical coweights is finite because of Lemma 9. □

In view of the above, the determination of the strata of $V$ begins with the determination of the candidate coweights. It is often convenient instead of the candidates to determine the candidate weight sets: a set of weights $R \subset X(T)$ is called a candidate weight set iff $\lambda = \delta(R)$ is dominant and satisfies $R = H(\lambda) \cap R(V, T)$ (so that $\lambda$ is a candidate). A candidate weight set $R$ is called critical iff $\delta(R)$ is critical.

Stratification is a step towards orbit classification because of

Lemma 11  Let $v \in \text{Nc}(V)$. Then $Gv \cap [v] = P(v)v$.

Proof.  First let $w \in Gv \cap [v]$. Then there is $g \in G$ with $w = gv$. It follows that $g[v] = [w] = [v]$. Therefore $g \in P(v)$ and $w \in P(v)v$. The converse inclusion is trivial. □

In some cases, the stratum is a single orbit:

Lemma 12  Let $V$ be a $G$-module. Let $v \in \text{Nc}(V)$ have $\dim(S(v)) = 1$. Then $[v]$ is the $P(v)$-orbit of $v$, and $G[v]$ is the $G$-orbit of $v$.

Proof.  Choose $\lambda \in \Lambda(v)$. Then $\lambda(t)v \in S(v)$ for all $t$. It fills $S(v)$ because $S(v)$ has dimension 1. □

2.6 The nullcone and nilpotency

We now specialize to the case that $V$ is the Lie algebra $g$ of $G$, which is a $G$-module for the adjoint action of $G$. It is well-known that the nullcone $\text{Nc}(g)$ is the set $\text{Nilp}(G)$ of the nilpotent elements of $g$, which is irreducible [21, (5.4)]. For lack of a suitable reference that applies to arbitrary characteristic, we provide the main arguments.

Lemma 13  Let $B$ be a Borel subgroup of $G$, with Lie algebra $\mathfrak{b}$. Let $B_u$ be the maximal unipotent subgroup of $B$, with Lie algebra $\mathfrak{b}_u$. Then $\text{Nilp}(G) = \text{Nc}(g) = \text{Ad}(G)\mathfrak{b}_u$.

Proof.  The set $\text{Nilp}(G)$ is a subset of $\text{Ad}(G)\mathfrak{b}_u$ because, by [1, 14.25], every nilpotent element of $g$ is conjugate to a nilpotent element of $\mathfrak{b}$, i.e., to an element of $\mathfrak{b}_u$. The set $\text{Ad}(G)\mathfrak{b}_u$ is contained in $\text{Nc}(g)$ because $\mathfrak{b}_u$ is a concentrated set. The set $\text{Nc}(g)$ is a subset of $\text{Nilp}(G)$ because every concentrated element has all eigenvalues zero and is therefore nilpotent because of Cayley-Hamilton. □
3 The Construction of $G_2$

There are two classical ways to construct the simple group $G_2$ and its Lie algebra. One is as the fixed points of the outer automorphisms of the group $D_4$ induced by the symmetry of its Dynkin diagram. The other is as the automorphism group of an octonion algebra. Both constructions involve several choices. In order to enforce compatibility of the choices we begin with the approach via $D_4$ represented as $\text{SO}(8)$; subsequently an octonion algebra is constructed in the associated eight dimensional vector space.

In this section, the field $K$ can be arbitrary, it need not be algebraically closed. The quadratic form, however, is supposed to be split. For simplicity we do not use Clifford algebras and spinors, as in [23, Chapter 3]. Therefore, the argument has a small gap in characteristic 2.

3.1 The orthogonal group in eight dimensions

Let $V$ be an eight dimensional vector space with a basis $b_0, \ldots, b_7$, over an arbitrary field $K$. Let the norm $N : V \to K$ be the quadratic form given by $N(\sum_i \xi_i b_i) = \xi_0 \xi_7 + \xi_1 \xi_6 + \xi_2 \xi_5 + \xi_3 \xi_4$. The associated bilinear form is given by $(x, y) = N(x + y) - N(x) - N(y)$. Note that

$$\langle b_i, b_j \rangle \neq 0 \iff i + j = 7. $$

The special orthogonal group $\text{SO}(V)$ is the group of the linear transformations of $V$ that preserve the norm $N$ and have determinant 1. The Lie algebra $\mathfrak{so}(V)$ consists of the matrices $Y$ that satisfy $(Y x, x) = 0$ for all $x \in V$. With respect to the basis $b_0, \ldots, b_7$, the elements of $\mathfrak{so}(V)$ have matrices of the form

$$Y = \begin{pmatrix}
    c_{12} & c_1 & -c_4 & c_9 & -c_8 & -c_{10} & -c_{11} & 0 \\
    -c_{26} & c_{13} & c_0 & c_5 & -c_6 & -c_7 & 0 & c_{11} \\
    c_{23} & -c_{27} & c_{14} & c_2 & -c_3 & 0 & c_7 & c_{10} \\
    -c_{18} & -c_{22} & -c_{25} & c_{15} & 0 & c_3 & c_6 & c_8 \\
    c_{19} & c_{21} & c_{24} & 0 & -c_{15} & -c_2 & -c_3 & -c_9 \\
    c_{17} & c_{20} & 0 & -c_{24} & c_{25} & -c_{14} & -c_0 & c_4 \\
    c_{16} & 0 & -c_{20} & -c_{21} & c_{22} & c_{27} & -c_{13} & -c_1 \\
    0 & -c_{16} & -c_{17} & -c_{19} & c_{18} & -c_{23} & c_{26} & -c_{12}
\end{pmatrix}$$

Here the indices and the signs are chosen carefully, in a not very natural way.

Let the matrices $Y_0, \ldots, Y_{27}$ be given by $Y = \sum_{i=0}^{27} c_i Y_i$. The indices in the matrix are chosen in such a way that $Y_0, \ldots, Y_{11}$ are upper triangular matrices, that $Y_{12}, \ldots, Y_{15}$ are diagonal matrices and $Y_{16}, \ldots, Y_{27}$ are lower triangular. The Lie products $H_i = [Y_{27-i}, Y_i]$ are diagonal matrices with $[H_i, Y_i] = 2Y_i$ if $0 \leq i < 12$ or $16 \leq i < 28$.

The group $\text{SO}(V)$ has an adjoint action on its Lie algebra $\mathfrak{so}(V)$ given by $\text{Ad}(g)(Y) = g Y g^{-1}$ for $g \in \text{SO}(V)$ and $Y \in \mathfrak{so}(V)$. Let $T$ be the maximal torus of the group $\text{SO}(V)$ that consists of the invertible diagonal matrices

$$t = \text{diag}(t_0, t_1, t_2, t_3, t^{-1}, t_2^{-1}, t_1^{-1}, t_0^{-1}).$$

Let $X(T)$ be the character group of $T$, written additively. The characters $\lambda_i$ for $0 \leq i < 4$ are given by $\lambda_i(t) = t_i$, and the characters $\alpha_i$ for $0 \leq i < 12$ by $\alpha_i(t) Y_i = \text{Ad}(t) Y_i$. Then $\alpha_0, \ldots, \alpha_{11}$ are the positive roots with respect to the Borel group of the upper triangular matrices in $\text{SO}(V)$. Among these, the simple roots are $\alpha_0 = \lambda_1 - \lambda_2, \alpha_1 = \lambda_0 - \lambda_1, \alpha_2 = \lambda_2 - \lambda_3, \alpha_3 = \lambda_2 + \lambda_3$.

Let $\theta$ be the involutive linear transformation of $V$ that interchanges the basis vectors $b_i$ and $b_{7-i}$ for $0 \leq i < 4$. It is clear that $\theta \in \text{SO}(V)$. Its adjoint action
Ad(θ) is the automorphism of \( \mathfrak{so}(V) \) that interchanges \( Y_i \) and \( Y_{27-i} \) for \( 0 \leq i < 12 \) and multiplies the matrices \( Y_{12}, \ldots, Y_{15} \) with \(-1\).

Recall that, for a semisimple algebraic group \( G \) with maximal torus \( T \), and root system \( R \), if \( \text{char}(K) = 0 \), a Chevalley system \cite[Chap. 8, §2]{3} is a family \( (X_\alpha)_{\alpha \in R} \) of vectors in \( g \) such that \( \text{Ad}(t)X_\alpha = \alpha(t)X \) for all \( t \in T \), that the elements \( H_\alpha = [X_{-\alpha}, X_\alpha] \) satisfy \( [H_\alpha, X_\alpha] = 2X_\alpha \), and that \( g \) has an automorphism that interchanges \( X_\alpha \) and \( X_{-\alpha} \). If \( \text{char}(K) = 0 \), the basis of \( \mathfrak{o}(V) \) constructed above therefore induces a Chevalley system given by \( X_\alpha = Y_i \) for \( 0 \leq i < 12 \) and \( 16 \leq i < 28 \).

Figure 1 shows the Dynkin diagram \( D_4 \) of \( \mathfrak{so}(V) \). It consists of a central node 0 with three neighbours 1, 2, 3. The symmetry of the diagram allows for the permutations of the three neighbours. The fixed positive roots are \( \alpha_0, \alpha_{10} = \sum_{i=0}^3 \alpha_i \), and \( \alpha_{11} = \alpha_0 + \alpha_{10} \). The triples \((\alpha_1, \alpha_4, \alpha_7), (\alpha_2, \alpha_5, \alpha_8), (\alpha_3, \alpha_6, \alpha_9)\) are permuted as the nodes 1, 2, 3 of the Dynkin diagram.

This symmetry is extended to the root vectors \( Y_i \). The minus signs in the above matrix of \( Y \) are chosen such that, if \( 0 \leq i < j < k < 12 \) and \( \alpha_i + \alpha_j = \alpha_k \), then \( [Y_i, Y_j] = Y_k \). Let \( r \in \mathfrak{o}(V) \) be the reflexion with \( rb_3 = -b_4 \), \( rb_4 = -b_3 \), and \( rb_i = b_i \) for \( i \neq 3, 4 \). The adjoint action of \( r \) on \( \mathfrak{so}(V) \) interchanges the pairs \((Y_2, Y_3), (Y_5, Y_6), (Y_8, Y_9), (Y_{18}, Y_{19}), (Y_{21}, Y_{22}), (Y_{24}, Y_{25}), \) and \((Y_{15}, -Y_{15})\). It leaves the other basic matrices unchanged. In short, it interchanges the nodes 2 and 3 of the Dynkin diagram.

Now assume that \( \text{char}(K) \neq 2 \). This assumption is needed to interchange the nodes 1 and 2 of the Dynkin diagram in the present representation of \( D_4 \). The root vectors \((Y_1, Y_2), (Y_4, Y_5), (Y_7, Y_8), \) etc., are interchanged in the same way as in the previous case. To determine the transformation needed for the Lie algebra \( \mathfrak{t} \) of the torus \( T \), we now use the elements

\[
\begin{align*}
H_0 &= [Y_{27}, Y_0] = Y_{13} - Y_{14}, \\
H_1 &= [Y_{26}, Y_1] = Y_{12} - Y_{13}, \\
H_2 &= [Y_{25}, Y_2] = Y_{14} - Y_{15}, \\
H_3 &= [Y_{24}, Y_3] = Y_{14} + Y_{15}.
\end{align*}
\]

These vectors form a basis of \( \mathfrak{t} \) because of \( \text{char}(K) \neq 2 \). The interchange of the nodes 1 and 2 is completed by interchanging \( H_1 \) and \( H_2 \), and keeping \( H_0 \) and \( H_3 \) fixed.

We thus have constructed two automorphisms of the Lie algebra \( \mathfrak{so}(V) \), which generate a group \( \Gamma \) isomorphic to the symmetric group of \( \{1, 2, 3\} \). The fixed points of \( \Gamma \) in \( \mathfrak{so}(V) \) of these automorphisms form a Lie algebra consisting of the matrices
by this basis. It turns out that $L$ Section 3.1 in such a way that the matrices $X$ group of type above group $\Gamma$ acts on $H$ of the algebra, i.e., the linear transformations of the algebra, $i.e.$, the linear transformations.

The multiplication of such matrices is defined here by $e_3$ one gets a system of homogeneous linear equations in 8 $K$ case $G$ $x$ octonion dimension 8 is called an $N$ elements of composition algebras are 1, 2, 4, and 8. A composition algebra $G$ is algebraically closed, every composition algebra over it is split. The possible $K$ of the same dimension are isomorphic, cf. [23, Thm. 1.8.1]. If the base field $x$ nonzero vector $N$ $e$ element algebra $G$ We turn to the second classical construction of $3.2$ A split octonion algebra

We turn to the second classical construction of $G_2$. Recall that a composition algebra $C$ over a field $K$ is a not necessarily associative algebra over $K$ with unit element $e$, such that there is a quadratic function $N : C \to K$ that satisfies $N(xy) = N(x)N(y)$, and for which the associated bilinear form $\langle x, y \rangle = N(x + y) - N(x) - N(y)$ is nondegenerate.

A composition algebra $C$ is called split if it has an isotropic vector, i.e., a nonzero vector $x \in C$ with $N(x) = 0$. All split composition algebras over $K$ of the same dimension are isomorphic, cf. [23, Thm. 1.8.1]. If the base field $K$ is algebraically closed, every composition algebra over it is split. The possible dimensions of composition algebras are 1, 2, 4, and 8. A composition algebra of dimension 8 is called an octonion algebra.

If $G$ is the automorphism group of an octonion algebra $C$ over $K$, it is a simple group of type $G_2$, see [23, Section 2.3]. Its Lie algebra consists of the derivations of the algebra, i.e., the linear transformations $D : C \to C$ with $D(xy) = (Dx)y + x(Dy)$.

Conversely, if one wants to define a multiplication on the vector space $V$ of Section 3.1 in such a way that the matrices $X$ of that section are derivations, one gets a system of homogeneous linear equations in $8^3$ unknowns. This system has a five-dimensional solution space. Adding the requirement of a unit element $e = \sum_{i=0}^{7} e_i b_i$ with $ex = x$ and $xe = x$, one gets 8 more unknowns and 128 additional equations. The final requirement $N(xy) = N(x)N(y)$ leads to four solutions.

One solution is chosen arbitrarily (we come back to the choice in Lemma 14 below). The solution can be described by so-called vector matrices, see [23, p. 20]. The vector $x = \sum_{i=0}^{7} \xi_i b_i$ in $V$ is represented by the matrix

$$x = \begin{pmatrix} \xi_3 & \xi_7, \xi_1, \xi_2 \\ \xi_0, \xi_5, \xi_5 \\ \xi_4 \end{pmatrix}$$

The multiplication of such matrices is defined here by

$$\left( \begin{array}{cc} \xi & x \\ y & \eta \end{array} \right) \left( \begin{array}{cc} \xi' & x' \\ y' & \eta' \end{array} \right) = \left( \begin{array}{cc} \xi \xi' - \langle x, y' \rangle & \xi x' + \eta' x - y \times y' \\ \eta y' + \xi' y - x \times x' & \eta \eta' - \langle y, x' \rangle \end{array} \right)$$

where the three dimensional space $K^3$ has the inner product $\langle x, y \rangle$ given by

$$\langle (\xi_0, \xi_1, \xi_2), (\eta_0, \eta_1, \eta_2) \rangle = \sum_{i=0}^{2} \xi_i \eta_i$$
and the outer product \( x \times y \) given by
\[
(x \times y, z) = \det(x, y, z)
\]
for all \( x, y, z \in K^3 \).

The full table of the multiplication in \( V \) is

<table>
<thead>
<tr>
<th></th>
<th>( b_0 )</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( b_3 )</th>
<th>( b_4 )</th>
<th>( b_5 )</th>
<th>( b_6 )</th>
<th>( b_7 )</th>
</tr>
</thead>
<tbody>
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<td>( b_0 )</td>
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<td>0</td>
<td>( b_0 )</td>
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<td>( -b_2 )</td>
<td>( -b_4 )</td>
</tr>
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<td>( -b_0 )</td>
<td>0</td>
<td>( b_1 )</td>
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<td>( b_2 )</td>
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<td>( b_0 )</td>
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<td>0</td>
<td>( b_2 )</td>
<td>( -b_3 )</td>
<td>0</td>
<td>( -b_6 )</td>
</tr>
<tr>
<td>( b_3 )</td>
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<td>( b_5 )</td>
<td>( -b_1 )</td>
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<td>( -b_4 )</td>
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<td>( b_7 )</td>
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<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

By computer algebra one can verify that the vector \( e = b_3 + b_4 \) is the unit element of the algebra and that the norm is multiplicative, i.e., satisfies \( N(xy) = N(x)N(y) \). Therefore the multiplication makes \( V \) an octonion algebra. As it is split and all split octonion algebras are isomorphic, it is the split octonion algebra. The multiplication is not commutative and not associative.

Let \( G_2 \) be the group of automorphisms of the octonion algebra \( V \) thus constructed, and let \( \mathfrak{g}_2 \) be its Lie algebra. It is known that \( \text{dim} \mathfrak{g}_2 = 14 \). By computer algebra one verifies that the matrices \( X \) of the Section 3.1 are derivations of the octonion algebra \( V \), and therefore elements of \( \mathfrak{g}_2 \). As the dimensions are equal, it follows that \( \mathfrak{g}_2 \) is the space of the matrices \( X \). This argument applies even if the characteristic of the field \( K \) is 2.

The freedom of choosing a multiplication is explained by the following result.

**Lemma 14** Let \((V, \cdot, e, N)\) be a split octonion algebra and let \( \mathfrak{g}_2 \) be the Lie algebra of its derivations. Let \( \text{Op} \) be the set of the bilinear operators \( \odot : V \times V \to V \) such that \((V, \odot, e', N)\) is an octonion algebra for some \( e' \in V \), and that \( \text{Der}(V, \odot) = \mathfrak{g}_2 \). Then \( \text{Op} \) is the set of the operators \( \odot_g \) given by \( x \odot_g y = g^{-1}(gx \cdot gy) \) where \( g \) ranges over the centralizer \( C \) of \( \mathfrak{g}_2 \) in \( \mathfrak{O}(V) \), i.e., the group of the elements \( g \in \mathfrak{O}(V) \) with \( \text{Ad}(g)X = X \) for all \( X \in \mathfrak{g}_2 \). If \( \odot_g = \odot_h \) for \( g, h \in C \), then \( g = h \).

**Proof.** Let \( \odot \in \text{Op} \). Then \((V, \odot, e', N)\) is an octonion algebra for some unit \( e' \in V \). This algebra has the same norm \( N \) as the first algebra and is therefore split. As all split octonion algebras are isomorphic, there is an isomorphism \( g \) from \((V, \odot, e', N)\) to \((V, \cdot, e, N)\). This means that \( g(x \odot y) = gx \cdot gy \) for all \( x, y \in V \), that \( ge' = e \), and that \( N(gx) = N(x) \) for all \( x \in V \). This implies that \( g \in \mathfrak{O}(V) \), and that \( x \odot y = g^{-1}(gx \cdot gy) \) for all \( x, y \in V \).

Let \( X \in \mathfrak{g}_2 \). Then \( X \) is a derivation of \((V, \odot)\). So, for all \( u, v \in V \), it holds that \( X(u \odot v) = Xu \odot v + u \odot Xv \), or equivalently \( Xg^{-1}(gu \cdot gv) = g^{-1}(gxu \cdot gv) + g^{-1}(gu \cdot gXv) \). Substituting \( X' = gXg^{-1} \) and \( u' = g^{-1}u \) and \( v' = g^{-1}v \), we get that \( X'(u' \cdot v') = X'u' \cdot v' + u' \cdot Xv' \). This holds for all \( u' \) and \( v' \), showing that \( X' = \text{Ad}(g)X \) is a derivation of \((V, \cdot)\), i.e., \( X \in \mathfrak{g}_2 \). This holds for all \( X \in \mathfrak{g}_2 \). Therefore \( \text{Ad}(g) \) is an automorphism of \( \mathfrak{g}_2 \).

As every automorphism of the Lie algebra \( \mathfrak{g}_2 \) is an inner automorphism, it follows that the group \( G_2 \) has an element \( h \) with \( \text{Ad}(h)X = \text{Ad}(g)X \) for all \( X \in \mathfrak{g}_2 \). Then \( g_1 = h^{-1}g \) is an element of the centralizer and \( x \odot y = g_1^{-1}(g_1x \cdot g_1y) \) for all \( x, y \in V \).

Conversely, if \( g \) is in the centralizer, it is easy to see that \((V, \odot_g, g^{-1}e, N)\) is an octonion algebra with \( \text{Der}(V, \odot_g) = \mathfrak{g}_2 \).

Finally, assume \( \odot_g = \odot_h \) for \( g, h \in C \). If we put \( k = gh^{-1} \), it holds that \( x \odot_k y = kg^{-1}(gh^{-1}x \cdot gh^{-1}y) = h(h^{-1}x \odot_g h^{-1}y) = h(h^{-1}x \odot_h h^{-1}y) = x \cdot y \). This
implies that $k$ preserves the multiplication and hence that $k \in G_2$. As $k \in C$, it acts trivial on the Lie algebra $g_2$. As the adjoint action of $G_2$ on its Lie algebra is known to be faithful, this implies $k = 1$ and, hence, $g = h$. \hfill $\Box$

The centralizer of $g_2$ in $O(V)$ can be determined in the following way. One first determines the centralizer of $g_2$ in $\text{End}(V)$. This consists of the matrices

$$
g = \begin{pmatrix}
t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s & s-t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s-t & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t
\end{pmatrix}
$$

with $s, t \in K$. The centralizer in $O(V)$ is obtained by intersecting with $O(V)$. This boils down to the additional requirements $t^2 = 1$ and $s(s-t) = 0$. Therefore the centralizer of $g_2$ in $O(V)$ is generated by $-1$ and the reflexion $r$ mentioned in Section 3.1. It has four elements and is isomorphic to the multiplicative group $\{\pm 1\}^2$. By Lemma 14, it follows that there are four choices of an octonion algebra structure on $V$ compatible with the chosen Lie algebra $g_2$.

3.3 The root system of $G_2$

Let $T$ be the torus in $G_2$ of the diagonal matrices with respect to the basis $b_0, \ldots, b_7$. Let the characters $\chi_i \in X(T)$ be given by $t b_i = \chi_i(t) b_i$. Formula (0) implies that $\chi_i(t) \chi_j(t) = 1$ when $i + j = 7$. Writing the character group $X(T)$ additively, it follows that $\chi_i + \chi_j = 0$ when $i + j = 7$. The identity $b_1 b_2 = - b_0$ implies that $\chi_1 + \chi_2 = \chi_0$. The identities $b_3^2 = b_3$ and $b_4^2 = b_4$ imply that $\chi_3 = \chi_4 = 0$.

Conversely, one can use the multiplication table to verify that the diagonal matrices $\text{diag}(uv, u, v, 1, 1, v^{-1}, u^{-1}, u^{-1}v^{-1})$ with $u, v \neq 0$ are automorphisms of algebra $V$. This proves that $T$ is a two-dimensional torus. In fact, it is a maximal torus in $G_2$, because any element $g \in G_2$ that commutes with $T$ preserves the weight spaces, as well as the elements $b_3$ and $b_4$. The character group $X(T)$ is a free $\mathbb{Z}$-module with (e.g.) the basis $\chi_0, \chi_1$. The diagram of the weights $\chi_0, \ldots, \chi_7$ is drawn in the lefthand part of Figure 2. This diagram explains several zeroes in the multiplication table of $V$ because, if $b_i b_j = \pm b_k$, the corresponding weights adds up: $\chi_i + \chi_j = \chi_k$.

The group $G_2$ preserves the unit element $e = b_3 + b_4$ and the norm $N$. It therefore also preserves the bilinear form of $V$. 

Figure 2: The weights of $V$ and the root system of $g_2$. 

whh548 – 11
As the Lie algebra $g_2$ consists of the matrices $X$ of Section 3.1, it has the basis $X_0, \ldots , X_{13}$, defined by the condition $X = \sum_i a_i X_i$. Then, e.g., it holds that $X_0 = Y_1 + Y_2 + Y_3$, $X_1 = Y_0$, etc. As before, the indices are chosen in such a way that the matrices $X_0, \ldots , X_5$ are upper triangular, that $X_6$ and $X_7$ are diagonal, and that $X_8, \ldots , X_{13}$ are lower triangular. Again, the Lie products $H_i = [X_{13-i}, X_i]$ satisfy $[H_i, X_i] = 2X_i$ if $0 \leq i < 6$ or $8 \leq i < 14$.

The elements $X_0, \ldots , X_5$ are eigenvectors for the adjoint action of the torus $T$ on the Lie algebra, with the respective weights $\alpha_0, \ldots , \alpha_5$ given by $\alpha_0 = \chi_2$, $\alpha_1 = \chi_1 - \chi_2$, and $\alpha_2 = \alpha_0 + \alpha_1$, $\alpha_3 = 2\alpha_0 + \alpha_1$, $\alpha_4 = 3\alpha_0 + \alpha_1$, $\alpha_5 = 3\alpha_0 + 2\alpha_1$. These weights form the set $R_+$. Similarly, the elements $X_{13-i}$ ($0 \leq i < 6$) are eigenvectors for $T$ with weights $-\alpha_i$ for all $i$ with $0 \leq i < 6$. Then $R = \{\pm \beta \mid \beta \in R_+\}$ is a root system of type $G_2$, with positive system $R_+$, and simple roots $\alpha_0, \alpha_1$. It is drawn as a six-pointed star in Figure 2.

The transformation $\theta$ used in Section 3.1 is an automorphism of the octonion algebra $V$ and hence an element of the group $G_2$. The adjoint action $\text{Ad}(\theta)$ of $\theta$ is the automorphism of $g_2$ that interchanges $X_i$ and $X_{13-i}$ for $0 \leq i < 6$ and multiplies the diagonal matrices $X_6$, $X_7$ with $-1$. If $\text{char}(K) = 0$, the basis therefore induces a Chevalley system of $g_2$, just as in Section 3.1.

Now that we have the weight space decomposition of the Lie algebra $g_2$, we can also form the corresponding one-dimensional subgroups $U_\beta$ of the group $G_2$, cf. [1, Theorem (13.18)]. These are obtained by truncated exponential functions $g_i : K \to G_2$. For example, $g_1(u)$ is the transformation $1_C + uX_1$ of $V$, and a similar expression works for the other long roots. The function $g_1$ can be extended to a homomorphism $h_1 : \text{SL}(2) \to G_2$ given by

$$h_1\left(\begin{array}{cc} x & z \\ y & t \end{array}\right) = \text{diag} \left(1, \begin{array}{cc} x & z \\ y & t \end{array}\right), 1, 1, \begin{array}{cc} x & -z \\ -y & t \end{array}, 1\right)$$

The short roots need the quadratic term of the exponential function. For example, $g_0(u)$ is the transformation $1_C + uX_0 + \frac{1}{2}u^2X_0^2$. Strictly speaking, this expression requires $\text{char}(K) \neq 2$, but by evaluating the matrix $X_0^2$ one gets a factor 2 which can formally cancel the factor $\frac{1}{2}$. Therefore, with some care, the expression turns out to work in characteristic 2 as well. Indeed the function can be extended to a homomorphism $h_0 : \text{SL}(2) \to G_2$ given by

$$h_0\left(\begin{array}{cc} x & z \\ y & t \end{array}\right) = \left(\begin{array}{cccccc} x & z & 0 & 0 & 0 & 0 \\ y & t & 0 & 0 & 0 & 0 \\ 0 & 0 & x^2 & xz & -xz & z^2 \\ 0 & 0 & xy & xt & -yz & zt \\ 0 & 0 & -xy & -yz & xt & -zt \\ 0 & 0 & y^2 & yt & -yt & t^2 \end{array}\right)$$

It follows that the function $f : K^6 \to G_2$ given by $f(u) = \prod_{i=0}^5 g_i(u_i)$ is an isomorphism of varieties between $K^6$ and the unipotent radial $B_u$ of the upper-diagonal Borel group $B$ of $G_2$, see [22, 10.1.1]. In other words, every element $b \in B_u$ is in a unique way a product $b = \prod_i g_i(u_i)$, where $i$ ranges from 0 to 5 in some specified order. We use the clockwise order 1, 2, 5, 3, 4, 0 to fix the notation.

### 3.4 The nullcone of the octonions

As now the results of Section 2 are to be applied, assume that the field $K$ is algebraically closed. The nullcone $\text{Nc}(V)$ of the octonion algebra $V$ for the action of $G_2$ has a simple structure:
Theorem 15  (a) $\text{Nc}(V) = \{ x \in V \mid \langle x, e \rangle = \text{N}(x) = 0 \}$.
(b) $\dim(\text{Nc}(V)) = 6$.
(c) The nonzero elements of $\text{Nc}(V)$ form a single $G_2$-orbit.
(d) $\text{Nc}(V) = \{ x \in V \mid x^2 = 0 \}$.

Proof. It is easy to see that $\text{Nc}(V)$ is contained in the set $X = \{ x \in V \mid \langle x, e \rangle = \text{N}(x) = 0 \}$, and that $\dim(X) = 6$. It is clear that $b_0 \in \text{Nc}(V)$. Let $x$ be an arbitrary nonzero element of $\text{Nc}(V)$. We have $\dim G_2 - \dim(P(x)) + \dim(S(x)) = \dim G_2[x] \leq 6$ by [9, 4.5(c)], and hence $\dim(S(x)) \leq \dim(P(x)) - 8$. As all proper parabolic subgroups of $G_2$ have dimension 8 or 9, this proves that $\dim(S(x)) = 1$. Moreover, in this case $\dim(G_2[x]) = 6$. The set $X$ is irreducible, and therefore equals the closure of $G_2[x]$. This proves the parts (a) and (b). Part (c) follows from $\dim(S(x)) = 1$ and Lemma 12.

(d) By [23, Prop. 1.2.3], in any composition algebra, squaring satisfies
\[ x^2 = \langle x, e \rangle x - \text{N}(x)e. \]
By (a), this formula implies that every element $x \in \text{Nc}(V)$ satisfies $x^2 = 0$. Conversely, if $x^2 = 0$, the formula implies that $x \in \text{Nc}(V)$ unless $x$ is a multiple of $e$; the latter case is easily dealt with. $\square$

It is possible to give a direct proof of part (c) using the results of Section 3.3.

Lemma 16 The action of the unipotent radical $B_0$ of the Borel group on $\text{Nc}(V)$ restricts to a simply transitive action on the intersection of $\text{Nc}(V)$ with $b_7 + \sum_{i<7} K b_i$.

Proof. Let $v = b_7 + \sum_{i<7} v_i b_i$ be an arbitrary element of the intersection. The claim is the unique existence of $g \in B_0$ with $gb_7 = v$. If one uses the parametrization of $B_0$ given at the end of Section 3.3, the image $\prod_{i<6} g_i(u_i) b_7$ equals
\[ (u_0 u_2 u_3 + u_0 u_5 - u_2 u_4 + u_0^2, -u_0 u_1 u_3 - u_0 u_2^2 + u_1 u_4 + 2u_2 u_3 + u_5, -u_0 u_3 + u_4, -u_0 u_2 + u_3, u_0 u_2 - u_3, u_0 u_1 + u_2, -u_0, 1). \]
This leads to unique values for $u_0, \ldots, u_5$. Finally, one uses the equalities of Theorem 15(a). $\square$

Theorem 17 The nonzero elements of $\text{Nc}(V)$ form a single orbit under the action of the group $G_2$.

Proof. Let $v = \sum_{i<8} v_i b_i$ in $\text{Nc}(V)$ be nonzero. The aim is to show that $v$ is in the orbit of $b_7$. Because of Theorem 15(a), there is an index $i \in \{1, 2, 3, 5, 6, 7\}$ with $v_i \neq 0$. The elements
\[ h_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad h_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
introduced in Section 3.3 can be used to interchange the coefficients $v_0, v_1, v_2$ and $v_7, v_6, v_5$. We may therefore assume that $v_i$ is nonzero for some index $i \in \{0, 7\}$. The element $\theta \in G_2$ can be used to interchange $v_0$ and $v_7$. We may therefore assume that $v_7 \neq 0$. The torus can be used to normalize $v_7$. Therefore, we may assume that $v_7 = 1$, i.e., that $v \in b_7 + \sum_{i<7} K b_i$. Finally, apply Lemma 16. $\square$

4 The Nilpotent Conjugacy Classes of $G_2$

In this section, we determine the stratification and the orbit structure of the nullcone of the Lie algebra of $G_2$ over an algebraically closed field of arbitrary characteristic.
In characteristic 0, the orbits (conjugacy classes) of the nilpotent elements of the Lie algebra of $G_2$ are known for more than 60 years. This is briefly explained in Section 4.1. The stratification of the nilcone is determined in Section 4.2. It turns out that, in characteristic 0, the strata are the nilpotent orbits. In Section 4.3, differential methods are used to obtain concrete information on the relations between strata and orbits. Section 4.4 treats the two remaining orbits, and draws the conclusion.

### 4.1 The orbits in characteristic 0

Let $G$ be a semisimple connected algebraic group over a field $K$ of characteristic 0, with Lie algebra $\mathfrak{g}$. An $\mathfrak{sl}_2$-triplet in a Lie algebra $\mathfrak{g}$, is a triple $(x, h, y)$ of elements of $\mathfrak{g}$ such that $[h, x] = 2x$, $[h, y] = -2y$, and $[y, x] = h$. The Theorem of Jacobson-Morozov [3, Chap. 8, §11] asserts that, if $x$ is a nilpotent element of $\mathfrak{g}$, then there exist elements $h$ and $y$, such that $(x, h, y)$ is an $\mathfrak{sl}_2$-triplet. Moreover, if $h'$ and $y'$ are such that $(x, h', y')$ is another $\mathfrak{sl}_2$-triplet, there is an element $g \in G$ such that $\text{Ad}(g)x = x$, $\text{Ad}(g)h' = h$ and $\text{Ad}(g)y' = y$. Strictly speaking, the book [3] disallows the $\mathfrak{sl}_2$-triplet $(0, 0, 0)$, but the extension to this case is trivial.

Following Dynkin [6], Springer and Steinberg [2, Part E, III, §4] apply $\mathfrak{sl}_2$-triplets to the classification of the nilpotent conjugacy classes in $\mathfrak{g}$. The result is as follows.

Assume that $\mathfrak{g}$ is split. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, let $R$ be the corresponding root system, and let $\Delta$ be a basis of $R$. An $\mathfrak{sl}_2$-triplet $(x, h, y)$ is called normalized iff $h \in \mathfrak{h}$ and that $\alpha(h) \in \{0, 1, 2\}$ for all $\alpha \in \Delta$. For every nilpotent conjugacy class $\mathcal{C}$, one can choose a normalized $\mathfrak{sl}_2$-triplet $(x, h, y)$ with $x \in \mathcal{C}$. The Dynkin diagram $D(\mathcal{C})$ of $\mathcal{C}$ is defined as the Dynkin diagram of $\mathfrak{g}$ with numbers $\alpha(h)$ attached to the nodes $\alpha \in \Delta$. It is proved that $D(\mathcal{C})$ is uniquely determined by $\mathcal{C}$, and that classes $\mathcal{C}$ and $\mathcal{C}'$ are equal if and only if $D(\mathcal{C}) = D(\mathcal{C}')$. In fact, Springer and Steinberg extend some of these results to some positive characteristics, but we do not persue this here.

Following Dynkin [6], we have the following table of normalized $\mathfrak{sl}_2$-triplets of the Lie algebra of $G_2$ over a field of characteristic 0, corresponding to the five nilpotent conjugacy classes.

<table>
<thead>
<tr>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>rep</th>
<th>co</th>
<th>description</th>
<th>dim</th>
<th>ord*</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>$X_0 + X_1$</td>
<td>$10X_{12} + 6X_{13}$</td>
<td>regular</td>
<td>12</td>
<td>( )</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>$X_1 + X_4$</td>
<td>$2X_9 + 2X_{12}$</td>
<td>subregular</td>
<td>10</td>
<td>(1)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$X_3$</td>
<td>$X_{10}$</td>
<td>short root</td>
<td>8</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>$X_5$</td>
<td>$X_8$</td>
<td>long root</td>
<td>6</td>
<td>(1, 1, 1)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>zero</td>
<td>0</td>
<td>(2, 1, 1, 1, 1)</td>
</tr>
</tbody>
</table>

The first two columns give the Dynkin diagram, the numbers $\alpha(h)$. The column “rep” gives a representative $x$ of the nilpotent class in terms of the basis of $\mathfrak{g}$ constructed in Section 3.3. The column “co” gives a Jacobson-Morozov companion $y$. The column “dim” gives the dimension of the class or its closure. The column “ord*” describes the singularity of $\text{Nilp}(G)$ at an element of the class as determined below in Section 5.4.

### 4.2 Stratification in arbitrary characteristic

The theory of Section 2 is applied to the adjoint action of the group $G_2$ on its Lie algebra $\mathfrak{g}_2$. All norms on $M(G_2)$ are equivalent because $G_2$ is simple. We use the representation and the coordinates of Section 3.3. Let $B$ be the Borel group of $G_2$ that corresponds to the positive system $R_+$, and let $T \subset B$ be the torus of the diagonal matrices. By inspection of the root system one easily obtains
Lemma 18  The candidate weight sets of $G_2$ are: $R_0 = \emptyset$, $R_1 = \{\alpha_5\}$, $R_2 = \{\alpha_3, \alpha_4, \alpha_5\}$, $R_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $R_4 = R_+$, $R_5 = \{\alpha_4, \alpha_5\}$, and $R_6 = \{\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

Lemma 19  The sets $R_5$ and $R_6$ of Lemma 18 are noncritical.

Proof. Let $v \in g_2$ with $\delta(R_5) \in \Lambda_T(v)$. Then $v = \xi X_4 + \eta X_5$ for some nonzero scalars $\xi, \eta$. Now the group element $g = g_1(-\eta/\xi)$, introduced at the end of Section 3.3, satisfies $Ad(g)v = \xi X_4$. This implies $q^*(Ad(g)v) < q^*(v)$, contradicting optimality of $\delta(R_5)$. This shows that $R_5$ is noncritical.

Let $v \in g_2$ with $\delta(R_6) \in \Lambda_T(v)$. Let $U_2$ be the span of $X_3, X_4, X_5$. Then $v \in \xi X_0 + \eta X_2 + U_3$ for some nonzero scalars $\xi, \eta$. Again the group element $g_1(-\eta/\xi)$ is used to eliminate $X_2$, giving a contradiction with optimality of $R_6$. This proves that $R_6$ is noncritical.$\Box$

Let the subspaces $U_i$ ($0 \leq i \leq 4$) be defined by $U_i = g_2[R_i]$. Putting $\delta_i = \delta(R_i)$, we have $U_i = g_2(\delta_i, 1)$. Note that the sets $R_i$ with $0 \leq i \leq 4$ are ordered in such a way that $R_{i-1} \subset R_i$ and $q(\delta_{i-1}) < q(\delta_i)$ for $0 < i \leq 4$. Put $U_0^i = b(2, \delta_i)$.

Theorem 20  The dominant blades of $\mathcal{Nc}(g_2)$ are the sets $U_0^i$ for $0 \leq i \leq 4$, with the elements $0 \in U_0^0, X_2 \in U_0^1, X_3 \in U_0^2, X_1 + X_4 \in U_0^3, X_0 + X_1 \in U_0^4$. For $0 \leq i \leq 4$, the set $U_0^i$ is open and dense in $U_i$.

Proof. As, by the Lemmas 18 and 19, all critical coweight is in the set $\{\delta_0, \delta_1, \delta_2, \delta_3, \delta_4\}$, it follows from Lemma 10 that the dominant blades of $\mathcal{Nc}(g)$ are the nonempty sets among $U_0^i$ for $0 \leq i \leq 4$. It therefore remains to verify that the sets contain the elements claimed.

As $\delta_0 = 0$, it holds that $0 \in b(2, \delta_0) = U_0^0$. In the other four cases, Proposition 8 is applied. We have $X_5 \in g_2(\delta_1, 1)$. As $Ad(b)X_5 = X_5$ for every $b \in B_u$, Proposition 8 gives $\delta_1 \in \Lambda_G(X_5)$. It follows that $X_5 \in U_0^1$.

It is clear that $X_3 \in U_2 = g_2(\delta_3, 1)$. For every $b \in B_u$, it holds that $Ad(b)X_3 \in X_3 + KX_4 + KX_5$. This implies that $q^+(Ad(b)X_3) = q(\delta_2)$. Proposition 8 gives $\delta_2 \in \Lambda_G(X_3)$. It follows that $X_3 \in U_0^2$.

Similarly, $X_1 + X_4 \in U_3 = g_2(\delta_3, 1)$. For every $b \in B_u$, it holds that $Ad(b)(X_1 + X_4) = X_1 - \xi X_2 - \xi^2 X_3 + (1 - \xi^3)X_4 + \eta X_5$

for some $\xi, \eta \in K$. This implies that $q^+(Ad(b)(X_1 + X_4)) = q(\delta_3)$ because the coefficient of $X_1$ and the coefficient of $X_3$ or $X_4$ is nonzero. Proposition 8 gives $\delta_3 \in \Lambda_G(X_1 + X_4)$. It follows that $X_1 + X_4 \in U_0^3$.

The proof of $X_0 + X_1 \in U_0^4$ is similar but simpler.$\Box$

The dimensions of the strata $G_2 \cdot U_0^i$ are determined with the formula $\dim(G_2 \cdot [v]) = \dim G_2 - \dim P(v) + \dim S(v)$. This gives the dimensions $0, 6, 8, 10, 12$, respectively. The table of Section 4.1 thus extends nicely to arbitrary characteristic if one replaces conjugacy class by stratum, and ignores the numbers $\alpha(h)$ and the companions.

In our view the above determination of the stratification of $\mathcal{Nc}(g_2)$ is simpler and more elementary than the methods of [4]. It shows that the stratification is independent of the characteristic of the field, confirming the results of [4]. In principle, our methods can be used for any simple group and pointed affine $G$-variety $V$ but, in every case, the calculational bottleneck is the action of the Borel group of $G$ on $V$.

In [14, 15], G. Lusztig proposed a definition of nilpotent pieces which leads to a stratification of the nullcone. According to [4, Remark 1 in Section 7.3], in the case of the classical groups, this stratification coincides with the stratification determined here. Our results may make it possible to see if the same idea works for the group $G_2$. This is a matter of future research.
4.3 Open orbits

Let the stabilizer $P_i$ of $U_i$ in $G_2$ have Lie algebra $\mathfrak{p}_i$. An element $v \in U_i$ has an open $P_i$-orbit in $U_i$ if the tangent mapping $dp$ of the action $\rho : P_i \to U_i$ is surjective. This tangent mapping satisfies $dp(X) = \text{ad}(X)(v) = -\text{ad}(v)(X)$. Surjectivity of $dp$ is therefore equivalent to the condition that $\text{ad}(v)$ has rank equal to $\dim(U_i)$. In each separate case the matrix of $\text{ad}(v)$ is a submatrix of the 8 by 10 matrix of $\text{ad}(v) : \mathfrak{b} + KX_{12} + KX_{13} \to \mathfrak{b}$, where $\mathfrak{b} = \sum_{i=0}^{7} KX_i$ is the Lie algebra of the Borel group. If $v = \sum_{i=0}^{7} v_iX_i$, the matrix is

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -v_0 & v_2 & 0 \\
v_1 & -v_0 & 0 & 0 & 0 & 0 & -v_2 & 0 & 0 & 2v_3 \\
-2v_2 & 0 & 2v_0 & 0 & 0 & -v_4 & -v_5 & 0 & 0 & 4 \\
-3v_3 & 0 & 0 & 3v_0 & 0 & 0 & -v_4 & -2v_4 & 0 & 0 \\
0 & -v_4 & -3v_3 & 3v_2 & v_1 & 0 & -2v_5 & -v_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -v_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1 & -2v_0 \\
\end{pmatrix}
$$

with respect to the basis $X_0, \ldots, X_7, X_{12}, X_{13}$.

In the case of $U_4$ the stabilizer $P_4$ is the Borel group with Lie algebra $\mathfrak{b}$. The matrix of $\text{ad}(v) : \mathfrak{b} \to U_4$ is

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -v_0 & 0 & 0 \\
v_1 & -v_0 & 0 & 0 & 0 & 0 & -v_2 & 0 & 0 & 0 \\
-2v_2 & 0 & 2v_0 & 0 & 0 & -v_3 & -v_5 & 0 & 0 & 0 \\
-3v_3 & 0 & 0 & 3v_0 & 0 & 0 & -v_4 & -2v_4 & 0 & 0 \\
0 & -v_4 & -3v_3 & 3v_2 & v_1 & 0 & -2v_5 & -v_5 & 0 & 0 \\
\end{pmatrix}
$$

This matrix has rank 6 if and only if $6v_0v_1 \neq 0$.

In the case of $U_5$ the stabilizer $P_5$ has the Lie algebra $\mathfrak{p}_5 = \mathfrak{b} + KX_{13}$. The matrix of $\text{ad}(v) : \mathfrak{p}_5 \to U_5$ for $v = \sum_{i=1}^{5} v_iX_i \in U_5$ with respect to the appropriate basis vectors $X_i$ is

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -v_1 & -3v_2 \\
v_1 & 0 & 0 & 0 & 0 & 0 & -v_2 & 0 & 2v_3 \\
-2v_2 & 0 & 0 & 0 & 0 & -v_3 & -v_5 & 0 & 0 \\
-3v_3 & 0 & 0 & 0 & 0 & -v_4 & -2v_4 & 0 & 0 \\
0 & -v_4 & -3v_3 & 3v_2 & v_1 & 0 & -2v_5 & -v_5 & 0 \\
\end{pmatrix}
$$

This matrix has rank 5 if and only if $3(v_1^2v_2^2 + 6v_1v_2v_3v_4 - 4v_1v_3^3 + 4v_2^3v_4 - 3v_2^2v_3^2) \neq 0$.

In the case of $U_2$ the stabilizer $P_2$ has the Lie algebra $\mathfrak{p}_2 = \mathfrak{b} + KX_{12}$. The matrix of $\text{ad}(v) : \mathfrak{p}_2 \to U_2$ for $v = \sum_{i=3}^{7} v_iX_i \in U_2$ with respect to the appropriate basis vectors $X_i$ is

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -v_3 & v_5 \\
-3v_3 & 0 & 0 & 0 & 0 & -v_4 & v_5 \\
0 & -v_4 & -3v_3 & 0 & 0 & -2v_5 & v_5 \\
\end{pmatrix}
$$

This matrix has rank 3 if and only if $v_3 \neq 0$, and $3 \neq 0$ or $v_4 \neq 0$ or $v_5 \neq 0$.

In view of these rank computations, we define on each of the spaces $U_i$ for $0 \leq i \leq 4$ a polynomial $f_i$, viz. $f_0 = 1$, $f_1 = v_5$, $f_2 = v_3$, $f_4 = v_0v_1$, and

$$
f_3 = v_1^2v_2^2 + 6v_1v_2v_3v_4 - 4v_1v_3^3 + 4v_2^3v_4 - 3v_2^2v_3^2.
$$

Here, $v_0, \ldots$, are used as coordinates in the subspaces $U_i$. Let the zerosets be defined as $C_i = \{ u \in U_i \mid f_i(u) = 0 \}$.
Remark. The polynomial $f_3$ is the main invariant of the $\mathfrak{sl}(2)$-module dual to the module of the cubic forms. If $\text{char}(K) \neq 3$, the module of the cubic forms is self-dual so that $f_3$ is equivalent to the discriminant. This is not the case for $\text{char}(K) = 3$. Anyhow, $f_3$ can be called the codiscriminant.

Lemma 21 (a) If $\text{char}(K) \neq 2, 3$, then $U_4 \backslash C_4$ is a single orbit for $P_4$.
(b) If $\text{char}(K) \neq 3$, then $U_3 \backslash C_3$ is a single orbit for $P_3$.
(c) If $\text{char}(K) \neq 3$, then $U_2 \backslash C_2$ is a single orbit for $P_2$.
(d) Assume that $\text{char}(K) = 3$. Then $U_2 \backslash C_2$ is the union of the $P_2$-orbits $U_{2}\alpha = U_2 \backslash KX_3$ and $U_{2}\beta = \{tX_3 \mid t \neq 0\}$.
(e) $U_1 \backslash C_1$ is always a single $P_1$-orbit.

Proof. (a) As $\text{char}(K) \neq 2, 3$, the tangent map at every element $v \in U_4 \backslash C_4$ is surjective, so that $v$ is an interior point of its $P_4$-orbit in $U_4$. As $U_4$ is irreducible, it follows that all elements of $U_4 \backslash C_4$ are conjugate under $P_4$. At every point $v \in C_4$, the tangent map is not surjective. Therefore, these points are not conjugate to the points in $U_4 \backslash C_4$. The proofs for the cases (b) and (c) are similar.

(d) The argument used in the proofs of (a), (b), (c) also shows that $U_{2}\alpha$ is a conjugate of $\{X_3 \mid \text{char}(K) = 3\}$. These two sets have optimal coweights smaller than 3.

Remark. For $\text{char}(K) \neq 3$, the module of the cubic forms. If $\text{char}(K) \neq 3$, the module of the cubic forms is self-dual so that $f_3$ is equivalent to the discriminant. This is not the case for $\text{char}(K) = 3$. Anyhow, $f_3$ can be called the codiscriminant.

Lemma 22 (a) $U_1^0 = U_i \backslash C_i$ for $1 \leq i \leq 4$.
(b) $U_0^0 = U_0 = \{0\}$.

Proof. Let $i = 4$. Every element $v \in U_4 \backslash C_4$ satisfies $\delta(R(v, T)) = \delta_4$, and the set $U_4 \backslash C_4$ is invariant under conjugation by the group $B_u$. By Proposition 8 this implies that $U_4 \backslash C_4 \subset U_4^0$. On the other hand, the set $C_4$ is the union of the sets $U_3$ and $g[R_0]$. These two sets have optimal coweights smaller than $\delta_4$. Therefore $C_4$ is disjoint with $U_4^0$. Together, this proves $U_4 \backslash C_4 = U_4^0$.

Let $i = 2$. Every element $v \in U_2 \backslash C_2$ satisfies $\delta(R(v, T)) = \delta_2$, and the set $U_2 \backslash C_2$ is invariant under conjugation by the group $B_u$. By Proposition 8 this implies that $U_2 \backslash C_2 \subset U_2^0$. On the other hand, the set $C_2$ equals $g[R_3]$ and this set has an optimal coweight smaller than $\delta_2$. Therefore $C_2$ is disjoint with $U_2^0$. Together, this proves $U_2 \backslash C_2 = U_2^0$.

The treatment of the cases $i = 1$ and $i = 0$ is simpler and can be left to the reader.

It remains to treat $i = 3$. The function $f_3$ is invariant under the adjoint action of $B_u$. Therefore $C_3$ and its complement are invariant under $B_u$. If $v \in U_3 \backslash C_3$ then $a_1 \neq 0$ or $a_2 \neq 0$, and also $a_3 \neq 0$ or $a_4 \neq 0$. Therefore $R(v, T)$ contains $\alpha_1$ or $\alpha_2$, and also $\alpha_3$ or $\alpha_4$. It follows that $\delta(R(v, T)) = \delta_3$. As $U_3 \backslash C_3$ is invariant under $B_u$, Proposition 8 implies that $U_3 \backslash C_3 \subset U_3^0$.

It remains to prove that the set $C_3$ and $U_3^0$ are disjoint. This is quite complicated. Let $v \in C_3$ be arbitrary. We have to show that $v \notin U_3^0$. We use the coordinates $v_1, \ldots, v_5$ as above. If $v_1 = v_2 = 0$, then $v \in U_2$ and hence $v \notin U_3^0$. Therefore, assume $v_1 \neq 0$ or $v_2 \neq 0$. Let $U_2^0 \subset g_2$ be the span of the basis vectors $X_1, X_2, X_5$. The set $U_2$ is a conjugate of $U_2$, so that $g(\delta(U_2)) < g(\delta_3)$. By Proposition 8, it suffices to show that $\text{Ad}(g)v \in U_2^0$ for some $g \in B_u$. The one-dimensional subgroup $g_0$ is used for this purpose. The action of $g_0$ on $U_3$ is given by

$$g_0(\xi) \left( \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \right) = \left( \begin{array}{c} v_1 \\ v_2 - v_1 \xi \\ v_3 + 2v_2 \xi - v_1 \xi^2 \\ v_4 + 3v_3 \xi + 3v_2 \xi^2 - v_1 \xi^3 \\ v_5 \end{array} \right)$$
First assume \( v_1 \neq 0 \). Then we can solve the quadratic equation \( v_3 + 2v_2\xi - v_1\xi^2 = 0 \). Then \( y = g_0(\xi)/(v) \) in \( U_3 \) has the coordinates \((y_1, \ldots, y_5)\) with \( y_1 = v_1 \neq 0 \) and \( y_3 = 0 \). Note that \( f_3(y) = 0 \) because \( f_3 \) is invariant under \( B_u \). If \( y_4 = 0 \), then \( y \in U_2' \) as required. Therefore assume \( y_4 \neq 0 \). We then calculate \( z = g_0(\eta) \) with \( \eta = 2y_2/y_1 \). Let \((z_1, \ldots, z_5)\) be the coordinates of \( z \). Then \( z_3 = 0 \) by construction and \( z_4 \) satisfies

\[
z_4 = y_4 + 3y_3\eta + 3y_2\eta^2 - y_1\eta^3 = y_4^2(y_1^2y_4 + 4y_2^2) = y_4^{-2}f_3(y) = 0.
\]

This proves that \( z \in U_2' \).

Otherwise \( v_1 = 0 \) and \( v_2 \neq 0 \). First assume \( \text{char}(K) \neq 2 \). For \( \xi = -v_3/2v_2 \), the vector \( y = g_0(\xi)v \) in \( U_3 \) has the coordinates \((y_1, \ldots, y_5)\) with \( y_3 = 0 \) and

\[
y_4 = v_4 + 3v_3\xi + 3v_2\xi^2 = (4v_2^3)^{-1}(4v_2^2v_4 - 3v_2v_3^2) = (4v_2^3)^{-1}f_3(v) = 0.
\]

This proves that \( y \in U_2' \).

It remains the case that \( \text{char}(K) = 2 \) and \( v_2 \neq 0 = v_1 \). We then observe that \( 0 = f_3(v) = v_2v_3^2 \). This implies that \( v_3 = 0 \). If \( \xi \) solves the quadratic equation \( v_4 + 3v_2\xi^2 = 0 \), then \( g_0(\xi)v \in U_2' \). □

**Remark.** Alternatively, one can prove that \( C_3 \) and \( U_3^0 \) are disjoint by showing that \( C_3 \) is irreducible of dimension 4, and that \( U_3 \setminus U_3^0 \) is closed and has dimension \( \geq 4 \). The above proof is more explicit and illustrates Proposition 8. □

### 4.4 Almost all strata are orbits

The Lemmas 21 and 22 show that the each of the dominant blades of \( N_c(g) \) is an orbit under its associated parabolic group, except for some cases in characteristic 2 and 3. The remaining cases are treated here, as well as the conclusions.

**Lemma 23** The Borel group \( B \) has a transitive action on \( U_4^1 \).

**Proof.** For \( v \in U_4^1 \), say \( v = \sum_{i=0}^5 v_iX_i \), we claim that there is an element \( g \in B \) with \( \text{Ad}(g)v = X_0 + X_1 \). Lemma 22 implies \( v_0 \neq 0 \neq v_1 \). We now use that \( B = T \cdot B_u \) where \( T \) is a maximal torus of \( B \) and \( B_u \) is the unipotent subgroup of \( B \). It is easy to see that there is \( t \in T \) such that \( \text{Ad}(t)v \in X_0 + X_1 + \sum_{i=2}^5 KX_i \). It therefore suffices to show that \( B_u \) has a transitive action on \( X_0 + X_1 + \sum_{i=2}^5 KX_i \).

We may therefore assume \( v_0 = v_1 = 1 \). In terms of the parametrization of \( B_u \) of Section 3.3, the equation \( \text{Ad}(b)(X_0 + X_1) = v \) with \( b \in B_u \) is equivalent to the system of equations

\[
\begin{align*}
-u_0 + u_1 &= v_2 \\
-u_0^2 - 2u_2 &= v_3 \\
-u_0^3 - 3u_3 &= v_4 \\
-u_0^4u_1 - 3u_0^2u_2 + 3u_0u_3 - 3u_1u_3 - 3u_2^2 - u_4 &= v_5 .
\end{align*}
\]

If the field \( K \) has characteristic \( \neq 2, 3 \), one can take \( u_0 = 0 \) and solve \( u_1, \ldots, u_4 \) in a unique way.

Otherwise, the characteristic of \( K \) is 2 or 3. As \( K \) is algebraically closed, it is perfect. If \( K \) has characteristic 2, one first solves the equation \( u_0^2 = -v_3 \), puts \( u_2 = 0 \), and subsequently solves \( u_1, u_3, \) and \( u_4 \). If \( K \) has characteristic 3, one first solves the equation \( u_0^3 = -v_4 \), puts \( u_3 = 0 \), and solves \( u_1, u_2, \) and \( u_4 \). □
It follows that the elements of $U_3^0$ are regular in the sense of [2, p. 227].

**Lemma 24** The blade $U_3^0$ is a single $P_3$-orbit.

**Proof.** By Lemma 21(b), it remains to treat the case of $\text{char}(K) = 3$. For this purpose, we use the subgroup $H$ of $G$, the image of the homomorphism $h_0 : \mathfrak{sl}(2) \to G$ considered in Section 3.3. This group $H$ is a subgroup of the parabolic group $P_3$. Therefore, $U_3$ is an $H$-module. Indeed, as an $H$-module, it is a direct sum of the $H$-modules $Q$, the span of $X_1, X_2, X_3, X_4$, and the trivial $H$-module $KX_5$.

We first determine the $H$-orbit of the point $X_1 + X_3$ of $Q$, using $\text{char}(K) = 3$. An element $a = \sum_{i=1}^{4} a_i X_i$ satisfies

$$a = h_0 \left( \begin{array}{ccc} x & z & \cdot \\ y & t & \cdot \\
\end{array} \right)(X_1 + X_3)$$

if and only if there exist numbers $x, y, z, t$ with

$$xt - yz = 1$$

$$a_1 = t^3$$

$$a_2 = -t^2y - z$$

$$a_3 = ty^2 + x$$

$$a_4 = -y^3.$$ 

As $K$ is perfect, the Frobenius mapping $x \mapsto x^3$ is an automorphism of the field $K$. The system of equations is therefore equivalent to $x^3t^3 - y^3z^3 = 1$ where $t^3 = a_1$, $y^3 = -a_4$, $x^3 = a_3^2 + a_4a_2$, $z^3 = -a_3^2 + a_4a_2$, or equivalently

$$-a_1^2a_4^2 + a_1a_3^2 - a_2^2a_4 = 1.$$ 

This is the equation $f_3(a) = -1$ because $\text{char}(K) = 3$. This proves that the $H$-orbit of $X_1 + X_3$ is the subset of $Q$ where $f_3 = -1$.

Let $T_0$ be the kernel of $a_0$ in torus $T$. Then $L_3 = T_0H$ is a Levi subgroup of the parabolic group $P_3$. The adjoint action of $T_0$ multiplies all elements of $Q$ with the same nonzero constant. Therefore the $L_3$-orbit of $X_1 + X_3$ is the subset of $Q$ where $f_3 \neq 0$.

Finally, let $w$ be an arbitrary element of $U_3^0$, say $w = q + \xi X_0$ with $q \in Q$ and $\xi \in K$. Then $f_3(q) = f_3(w) \neq 0$. Therefore $w$ has a conjugate under $L_3$ of the form $X_1 + X_3 + \eta X_0$ with $\eta \in K$. The one-parameter subgroup $g_0$ conjugates this to $X_1 + X_3$. $\square$

Using Lemma 11, Theorem 20, and the Lemmas 21, 23, 24, we obtain

**Theorem 25** (a) If $\text{char}(K) \neq 3$, each of the strata of $\text{Ne}(\mathfrak{g}_2)$ is a single $G_2$-orbit.

(b) Assume $\text{char}(K) = 3$. Each of the strata $G_2U_4^0$ with $i \neq 2$ is a single $G_2$-orbit; the stratum $G_2U_2^0$ is the union of two orbits: $G_2U_{2a}$, $G_2U_{2b}$ with $\dim(G_2U_{2a}) = 8$ and $\dim(G_2U_{2b}) = 6$.

These five nilpotent orbits (or six if $\text{char}(K) = 3$) correspond to the classes given in Table 22.1.5 of [13].

Moreover, as is easily verified, in each case, the adjacency structure of the orbits is the trivial one: orbit $O'$ is contained in the closure of a different orbit $O$ if and only if $\dim(O') < \dim(O)$. This orbit structure corresponds with the results of [19].

The sizes of the Jordan blocks of representatives of the orbits (as matrices in $\mathfrak{sl}(C)$) are most easily obtained by calculating the ranks of the powers of the representative. If $\text{char}(K) \neq 2$, the regular orbit has the sequence of sizes $(7,1)$, the subregular orbit has $(3,3,1,1)$. The next orbit has $(3,2,2,1)$, followed by $(2,2,1^4)$, and finally $(1^8)$. In characteristic 3, both orbits $GU_{2a}$ and $GU_{2b}$ have the same sequence $(3,2,2,1)$. For characteristic 2, the sequences are $(4,4)$, $(3,3,1,1)$, $(2^4)$, $(2,2,1^4)$, and $(1^8)$.
5 The Nilpotent Variety and its Singularities

Let $G$ be a reductive algebraic group with Lie algebra $\mathfrak{g}$. The starting point of the paper [7] was the question whether the $G$-orbits in $\text{Nilp}(G)$ can be classified using only the local structure of $\text{Nilp}(G)$. The paper gave a positive answer for the cases that $G$ is $\text{GL}(n)$ or $\text{Sp}(n)$ and $\text{char}(K) \neq 2$. This result is extended here to the group $G_2$ in characteristic $\neq 2, 3$.

In order to deal with its local structure, we need to know $\text{Nilp}(G)$ as a subvariety of $\mathfrak{g}$. The next step is to introduce cross sections to investigate the local structure of $\text{Nilp}(G)$ at specific points. Smooth equivalence is introduced to formalize the idea of local structure, the criterion $\text{ord}^*$ serves to quantify it. In Section 5.4, cross sections are used to determine $\text{ord}^*$ of $\text{Nilp}(G)$ at the points of the five orbits in characteristic $\neq 2$ and $3$. Characteristics 3 and 2 are treated in Sections 5.5 and 5.6, respectively.

5.1 The definition of the nilpotent variety

Let $G$ be a reductive algebraic group over a field $K$ with Lie algebra $\mathfrak{g}$. The affine quotient $[\mathfrak{g}/G]$ is the spectrum of the ring $A(\mathfrak{g})^G$ of the polynomial functions on $\mathfrak{g}$ that are invariant under the adjoint action of $G$. Let $p : \mathfrak{g} \to [\mathfrak{g}/G]$ be the canonical projection. The nilpotent variety $\text{Nilp}(G)$ is defined as the fiber $p^{-1}(p(0))$, see [7, (2.4)] (note that the affine quotient is universal because $K$ is a field). This means that the defining equations of $\text{Nilp}(G)$ in $\mathfrak{g}$ are the homogeneous invariant polynomials of positive degree. In fact, by the argument at the end of Section 2.2, we do the same for the nullcone of any affine $G$-variety.

Let $T$ be a maximal torus of $G$, with Lie algebra $\mathfrak{t}$ and Weyl group $W$. The restriction function $r : A(\mathfrak{g})^G \to A(\mathfrak{t})^W$ is injective and, under weak assumptions, it is an isomorphism [2, p. 200]. This holds in particular for the group $G_2$ in all characteristics. In Section 5.6 below it is shown for $\text{char}(K) = 2$.

If $\text{char}(K) \neq 2$ then $r$ is an isomorphism for $G_2$ because of [7, (2.6)]. In this case, the ring $A(\mathfrak{g})^G$ is a graded polynomial algebra generated by algebraically independent homogeneous polynomials $f_2, f_6$ of degrees 2 and 6 [5, p. 296].

Now recall that the characteristic polynomial of an endomorphism $x$ of a vector space $V$ is defined as the polynomial in the indeterminate $T$ given by $\chi(x) = \det(T - \text{id} - x)$. If $\dim V = n$, the symmetrical polynomials are the coefficients $\sigma_i$ given by $\chi(x) = T^n + \sum_{i=1}^n \sigma_i(x) \cdot T^{n-i}$. Each coefficient $\sigma_i$ is a homogeneous polynomial of degree $i$ in the matrix coefficients of $x$, and is invariant under conjugation by elements of $\text{GL}(V)$. The endomorphism $x$ is nilpotent if and only if $\sigma_i(x) = 0$ for all odd indices $i$ and $\sigma_1 = 0$. The only nonzero symmetrical polynomials are $\sigma_2, \sigma_4$, and $\sigma_6$. It is convenient to introduce the polynomial

$$\tau_2 = v_0v_1 + v_1v_2 + v_2v_3 + v_3v_4 + v_5v_4 + v_5v_6 + v_6v_7 + v_7^2,$$

which satisfies $\sigma_2 = 2\tau_2$ and $\sigma_4 = \tau_2^2$.

**Theorem 26** Assume that $\text{char}(K) \neq 2$. The ring of $G$-invariant polynomials $A(\mathfrak{g})^G$ on $\mathfrak{g}$ is generated by $\tau_2$ and $\sigma_6$, and these two generators are algebraically independent.

**Proof.** Above we saw that $A(\mathfrak{g})^G$ is generated by homogeneous algebraically independent elements $f_2$ and $f_6$ of degrees 2 and 6. As $\tau_2$ and $\sigma_6$ are homogeneous of degrees 2 and 6, there are scalars $s_1, s_2, s_3$ with $\tau_2 = s_1f_2$ and $\sigma_6 = s_2f_6^2 + s_3f_6$. 


It remains to show that \( s_1 \) and \( s_3 \) are nonzero. Well, \( s_1 \neq 0 \) because \( x = X_1 + X_{12} \) satisfies \( \tau_2(x) = 1 \). Similarly, \( s_3 \neq 0 \) because \( x = X_0 + X_1 + X_8 \) satisfies \( \tau_2(x) = 0 \) and \( \sigma_6(x) = 4 \neq 0 \). □

5.2 Smooth equivalence and cross sections

If \( V \) is an algebraic variety over a field \( K \), and \( v \in V \), then the pair \((V, v)\) is called a pointed variety. Pointed varieties \((X, x)\) and \((Y, y)\) are said to be smoothly equivalent, notation \((X, x) \sim (Y, y)\), iff there is a pointed variety \((Z, z)\) with smooth morphisms \( f : Z \to X \) and \( g : Z \to Y \) such that \( f(z) = x \) and \( g(z) = y \). This is an equivalence relation between pointed varieties.

Let the group \( G \) act on a variety \( V \) by means of a morphism \( h : G \times V \to V \). Let \( X \) be a subvariety of \( V \), and \( x \in X \). Then \( X \) is called a cross section at \( x \) if the restriction \( h : G \times X \to V \) is smooth in the point \((1, x)\), see e.g. [7, Section 2]. As the group \( G \) is a smooth variety, it follows that \((X, x) \sim (V, x)\).

Now assume that \( V \) is a \( G \)-module. The action of \( G \) on \( V \) induces an action of the Lie algebra \( g \) of \( G \) on \( V \). Let \( L \) be a linear subspace of \( V \). The restriction \( h : G \times (x + L) \to V \) induces a tangent morphism \( dh : g \times L \to V \) given by \( dh(u, v) = u \cdot x + v \). Therefore \( x + L \) is a cross section at \( x \) if and only if \( g \cdot x + L = V \).

If \( \text{char}(K) \neq 2 \) and 3, a cross section can be used in the following alternative proof of Theorem 26. Let \( x = X_0 + X_1 \) in \( g_2 \). The subspace \([g_2, x] \) is spanned by the vectors \( X_0, X_1, X_2, 2X_3, 3X_4, X_5, X_6, X_7, X_9, X_{10}, 2X_{11}, 3X_{12} - X_{13} \). As \( \text{char}(K) \neq 2, 3 \), the subspace \( L \) spanned by \( X_9 \) and \( X_{12} \) satisfies \([g_2, x] \oplus L = g_2 \). Therefore \( x + L \) is a cross section. By [7, Prop 2.2], the natural mapping \( A(g)^G \to A(x + L) \) is injective. Use the obvious coordinates \( v_8 \) and \( v_{12} \) on \( x + L \). Then the invariant polynomials satisfy \( (\tau_2 | x + L) = v_{12} \) and \( (\sigma_6 | x + L) = 4v_8 \). It follows that the subring \( K[\tau_2, \sigma_6] \) generated by the two invariant polynomials is mapped bijectively to \( A(x + L) \), that \( K[\tau_2, \sigma_6] \) equals \( A(g)^G \), and that \( \tau_2 \) and \( \sigma_6 \) are algebraically independent.

5.3 The sequence \( \text{ord} \)

The singularity of \((V, v)\) can be characterised by a partition \( \text{ord}(V, v) \) defined as follows. Assume that \( A \) is the local ring of \( V \) at the point \( v \). Assume that \( A \) is isomorphic to a quotient \( R/J \) where \( R \) is a regular local ring and \( J \) is an ideal of \( R \). Let \( M \) be the maximal ideal of \( R \). Then \( \text{ord}(V, v) = \text{ord}(A) \) is the sequence of numbers \( \text{ord}^i(A) = \text{rg}_{R/M}(J \cap (M^{i+1} + MJ))/MJ \) for \( i \geq 1 \). It is proved in [7] that this definition does not depend on the choices made, and that \( \text{ord}(X, x) = \text{ord}(Y, y) \) if \((X, x) \sim (Y, y)\). The sequence \( \text{ord}^* \) is a descending sequence of natural numbers, almost all of them 0. It is often represented by the finite sequence of its positive elements.

Lemma 27 Let \( R \) be a regular local ring with maximal ideal \( M \). Let \( A = R/J \) for some ideal \( J \subseteq M \). Let \( r, s \) be natural numbers with \( 1 \leq r < s \).

(a) Let \( J \) be generated by some element \( f \in M^r \setminus M^{r+1} \). Then \( \text{ord}^i(A) = 1 \) for \( 1 \leq i < r \), otherwise 0.

(b) Let \( J \) be generated by elements \( f, g \) with \( f \in M^r \setminus M^{r+1} \) and \( g \in M^s \setminus (fM \cup M^{s+1}) \). Then \( \text{ord}^i(A) = 2 \) for \( 1 \leq i < r \), and \( \text{ord}^i(A) = 1 \) for \( r \leq i < s \), and \( \text{ord}^i(A) = 0 \) otherwise.

Proof. Part (a) can be left to the reader. (b) The vector space \( J/MJ \) over the field \( R/M \) is generated by the residue classes of \( f \) and \( g \). The subspace \((J \cap (M^{i+1} + MJ))/MJ \) contains the class of \( f \) iff \( i < r \); it contains the class of \( g \) iff \( i < s \). Therefore, the classes are linearly independent, and the ranks are as described. □
5.4 Cross sections at the nilpotent elements for $G_2$

Cross sections are used to investigate the local structure of the nilpotent variety. Assume char($K$) $\neq 2, 3$.

For every conjugacy class of nilpotent elements, we need to consider only one representative. We use the representatives given in the table in Section 4.1.

The cross section $x + L$ for the regular element $x = X_0 + X_1$ is constructed in Section 5.2 in the alternative proof of Theorem 26. It shows that $x$ is a smooth point of $\text{Nilp}(G)$ with ord* = (1), i.e., ord* = 0 for all $i$.

Next the subregular element $x = X_1 + X_4$. The subspace $[g, x]$ is spanned by the vectors $X_i (0 \leq i < 8)$, and $X_{13}, X_9 - X_{12}$. A minimal cross section $L$ is spanned by $X_8, X_9, X_{10}, X_{11}$. The restrictions of the symmetrical polynomials to $x + L$ are $\tau_2 = v_9$ and $\sigma_6 = v_8^2 - 4v_9v_{10}v_{11} + 4v_{10}^3 - 4v_{11}^3$. After elimination of $v_9$, it results in the Kleinian singularity with the equation $v_9^2 + 4v_{10}^3 - 4v_{11}^3 = 0$. The singularity is at the origin, so the maximal ideal $M$ is generated by $v_8, v_{10}, v_{11}$. As the lowest order term is quadratic, Lemma 27(a) gives ord* = (1).

For the short root vector $X_3$, the subspace $[g, X_3]$ is spanned by $X_0, X_2, X_3, X_4, X_5, X_6 + X_2, X_{11}, X_{13}$. We can take $L$ spanned by $X_1, X_7, X_8, X_9, X_{12}$. The restriction of the symmetrical polynomial $\tau_2$ is $(\tau_2 | x + L) = v_1v_{12} - v_7^2 + 3v_3$. The restriction $(\sigma_6 | x + L)$ is rather messy. The variable $v_{10}$ is eliminated by putting $(\tau_2 | x + L) = 0$. Then $(\sigma_6 | x + L)$ modulo $M^4$ equals

$$4(v_1v_8^2 + v_7v_8v_9 + v_9^2v_{12}).$$

It follows that ord* = (1, 1). In fact, one can verify that this is the singularity $CC_3$ described in [7, Section (4.5)] which also occurs the the nilpotent variety of the group of type $G_2$.

The long root vector $X_5$ gives the subspace $[g, X_5]$ spanned by $X_1, X_2, X_3, X_4, X_5, X_6$. Therefore the linear subspace spanned by the vectors $X_0$ and $X_i$ with $7 \leq i \leq 13$ is a cross section. In this case $(\tau_2 | x + L) = v_9v_{13} - v_7^2 + 3v_8$. After elimination of $v_8$ by means of $\tau_2$, the restriction $(\sigma_6 | x + L)$ modulo $M^5$ equals

$$-v_9^2v_{12} - 6v_9v_{10}v_{11}v_{12} - 4v_9v_{11}^3 + 4v_{10}^3v_{12} + 3v_{10}^2v_{11}.$$  

It follows that ord* = (1, 1, 1). It can hardly be a coincidence that the function $f_3$ of Section 4.3 appears in this singularity.

In the case $x = 0$, the space $g$ itself is a minimal cross section. The singularity at the origin has ord* = (2, 1, 1, 1) by Lemma 27(b).

To summarize, this justifies the values of ord* of the singularities of $\text{Nilp}(G)$ in the five orbits given in the table of Section 4.1 when the characteristic of the field differs from 2 and 3. In particular, the singularities in the five orbits are different.

5.5 The local structure in characteristic 3

Now assume char($K$) = 3. For the regular element $x = X_0 + X_1$, take the cross section $x + L$ where $L$ is spanned by $X_4, X_8, X_{12}$. The intersection $(x + L) \cap \text{Nilp}(G)$ is given by the equations $v_{12} = 0$ and $v_8 - v_7^2v_9^2 = 0$. This is smooth at the origin, which corresponds to the point $x$. This shows that, also in this case, $x$ is a smooth point of $\text{Nilp}(G)$ with ord* = (1).

For the subregular element $X_1 + X_4$, take the cross section $x + L$ where $L$ is spanned by $X_4, X_8, X_9, X_{10}, X_{11}$. The intersection $(x + L) \cap \text{Nilp}(G)$ is given by the equations $v_4v_{10} = 0$ and $-v_7^2v_9^2 + v_7^2v_{10}^3 - v_4v_{11}^3 = 0$, where the point $x$ corresponds to $v_4 = 1, v_8 = v_9 = v_{10} = v_{11} = 0$. In particular, $v_4$ is invertible and the singularity has ord* = (1), just as the subregular elements in Section 5.4.

Take $X_3 + X_5$ as a representative of the orbits $G_2U_{20}$ of Theorem 25(b). The image $[g_2, x]$ is spanned by $X_0, X_2, X_3, X_4, X_5, X_1 - X_{13}, X_6 + X_7, X_6 - X_{11}$. It
follows that $x + L$ is a cross section at $x$ if $L$ is spanned by $X_i$ with $i = 1, 6, 8, 9, 10, 12$. The intersection $(x + L) \cap \text{Nilp}(G)$ is given by the equations $v_6^2 = v_1 v_{12}$ and $v_1 v_3 v_7 v_8 v_{11} + v_1 v_3 + v_1 v_8 v_{10} v_{12} - v_6 v_8 v_9 + v_8 v_7 v_{10} - v_6^2 v_{12} + v_5^2 v_{12} + v_3^2 v_{12} + v_1^2 v_{12} + v_{10}^2 = 0$. It follows that $\text{ord}^* = (2, 1)$ by Lemma 27(b).

One can take $X_3$ as a representative of the orbits $G_2 U_{26}$ of Theorem 25(b). The formulas get more complicated. It is therefore unlikely that the point $X_3$ is smoothly equivalent to $X_3 + X_5$. Yet ord* = (2, 1), so ord* does not separate the orbits.

One can determine the singularity of Nilp($G$) at the point $X_5$ by the same method. It turns out that ord* = (2, 1, 1). The singularity at the origin has ord* = (2, 1, 1, 1), just as in Section 5.4. Summarizing, the singularities in the five strata are separated by ord*, but the singularities in the orbits are not.

### 5.6 Invariant polynomials in characteristic 2

Assume that char($K$) = 2. In this case, the polynomial $\sigma_2$ vanishes but $\tau_2$ is $G_2$-invariant because $\tau_2^2 = \sigma_4$ and, in characteristic 2, squaring is injective. Similarly, the polynomial

$$\tau_2 = (v_6 + v_7)(v_6 v_7 + v_1 v_{12} + v_3 v_{10}) + v_6 (v_2 v_{11} + v_4 v_9) + v_7 (v_4 v_{13} + v_5 v_8) + v_0 v_1 v_{11} + v_5 v_{10} v_{11} + v_4 v_{10} v_{13} + v_2 v_{12} v_{13} + v_2 v_3 v_8 + v_0 v_3 v_9 + v_1 v_4 v_8 + v_5 v_9 v_{12}$$

satisfies $\tau_2^2 = \sigma_6$ and is therefore $G_2$-invariant. It is easy to see that $x \in \mathfrak{g}$ is nilpotent if and only if $\tau_2(x) = \tau_3(x) = 0$. Therefore, if $f(x) = 0$ for all nilpotent $x$, then some power of $f$ is in the ideal generated by $\tau_2$ and $\tau_3$, by Hilbert’s Nullstellensatz.

For the regular element $x = X_0 + X_1$, take the cross section $x + L$ where $L$ is spanned by $X_3, X_4, X_{11}, X_{12}$. Now $(\tau_2 \mid x + L) = v_{12}$ and $(\tau_3 \mid x + L) = v_{11}$. The intersection $(x + L) \cap \text{Nilp}(G)$ is therefore given by the equation $v_{11} = v_{12} = 0$. This implies that $x$ is a smooth point of $\text{Nilp}(G)$, as expected. It also follows that $\tau_2$ and $\tau_3$ are algebraically independent in $A(\mathfrak{g})^G$.

We now come back to the injective restriction $r : A(\mathfrak{g})^G \to A(t)^W$ of Section 5.1. In characteristic 2, the Lie algebra $\mathfrak{t}$ consists of the matrices

$$\text{diag}(v_{67}, v_6, v_7, 0, 0, v_7, v_6, v_{67}) \text{ with } v_{67} + v_6 + v_7 = 0.$$ 

The Weyl group is generated by the reflexions $s_0$ and $s_1$ in the simple roots $\alpha_0$ and $\alpha_1$. These reflexions act on $\mathfrak{t}$ as the adjoint actions of the group elements

$$w_0 = h_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } w_1 = h_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here $h_0$ and $h_1$ are the homomorphisms $\text{SL}(2) \to G_2$ introduced in Section 3.3. The element $w_0$ interchanges $v_6$ and $v_{67}$, while $w_1$ interchanges $v_6$ and $v_7$. This implies that the Weyl group acts as the permutation group of the symbols $v_6, v_7, v_{67}$, i.e., as the Weyl group of $A_2$ on the weight lattice. Therefore, the ring $A(t)^W$ is generated by homogeneous polynomials of degrees 2 and 3 by [5, p. 296]. As the functions $\tau_2$ and $\tau_3$ in $A(\mathfrak{g})^G$ restrict to $W$-invariant homogeneous polynomials of degrees 2 and 3, it follows that

**Theorem 28** Assume that char($K$) = 2. The restriction function $r : A(\mathfrak{g})^G \to A(t)^W$ is an isomorphism. The ring of $G$-invariant polynomials $A(\mathfrak{g})^G$ on $\mathfrak{g}$ is generated by $\tau_2$ and $\tau_3$, and these two generators are algebraically independent.
Above we saw that the regular elements are smooth points of $\text{Nilp}(G)$. It turns out that the subregular elements are smooth points of $\text{Nilp}(G)$ as well. Indeed, for the subregular element $X_1 + X_4$, take the cross section $x + L$ where $L$ is spanned by $X_8$, $X_9$, $X_{10}$, $X_{11}$. The restrictions of $\tau_2$ and $\tau_3$ to $x + L$ are $\tau_2 = \tau_9$ and $\tau_3 = \tau_8$.

For the short root vector $x = X_3$, take the cross section $x + L$ where $L$ is spanned by $X_i$ with $i = 0, 1, 2, 7, 8, 9, 10, 12$. Then $(\tau_2 \mid x + L) = v_9 v_1 + v_7 v_{13} + v_7 v_1$ and $(\tau_3 \mid x + L) = v_0 v_9 + v_7 v_{12} + v_7 v_8 + v_7 v_{10}$. After elimination of $v_{10}$, this leads to the Kleinian singularity with the equation $v_0 v_9 + v_2 v_3 = 0$ and $\text{ord}^* = (1)$.

For the long root vector $x = X_5$, take the cross section $x + L$ where $L$ is spanned by $X_i$ with $i = 0$ or $7 \leq i < 14$. Then $(\tau_2 \mid x + L) = v_8 + v_9 v_{13} + v_7$ and $(\tau_3 \mid x + L) = v_0 v_7 v_{12} + v_7 v_8 + v_9 v_{12} + v_{10} v_{11}$. After elimination of $v_8$, this leads to the equation $v_9 v_{12} + v_{10} v_{11} + v_7^2 = 0$. This gives the same singularity as at the short root vector.

The singularity at the origin has $\text{ord}^* = (2, 1)$.

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References


