Evolutionary Game Dynamics for Two Interacting Populations in a Co-evolving Environment

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Published in:
2018 IEEE Conference on Decision and Control (CDC)

DOI:
10.1109/CDC.2018.8619801

Evolutionary Game Dynamics for Two Interacting Populations under Environmental Feedback

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Abstract— **We study the evolutionary dynamics of games under environmental feedback using replicator equations for two interacting populations. One key feature is to consider jointly the co-evolution of the dynamic payoff matrices and the state of the environment: the payoff matrix varies with the changing environment and at the same time, the state of the environment is affected indirectly by the changing payoff matrix through the evolving population profiles. For such co-evolutionary dynamics, we investigate whether convergence will take place, and if so, how. In particular, we identify the scenarios where oscillation offers the best predictions of long-run behavior by using reversible system theory. The obtained results are useful to describe the evolution of multi-community societies in which individuals' payoffs and societal feedback interact.**

I. INTRODUCTION

Evolutionary game theory [1] is widely used to model population dynamics for social and ecological systems since it offers insightful results under meaningful simplification. There exist various classic models, among which the replicator dynamics play a prominent role [2]; in this model, the individuals in a well-mixed population play games with each other according to a mutually known payoff matrix. When the collective of individuals can be divided into several populations according to certain constraints, such as local interactions or genetic relationships, multi-population games take place. Then each population can be taken as a cohesive community whose members play games with players from other populations according to corresponding payoff matrices [3], [4]. The replicator dynamics for the multi-population evolutionary games have been developed in [5], [6], [7].

On the other hand, in classic evolutionary game theory, it is generally assumed that the related payoff matrix is constant and not affected by the evolution itself. That is to say, the players will receive pre-determined payoffs whenever every player has chosen a candidate strategy for the current game play. In practical scenarios from ecological systems or human society, however, the incentives or punishments for the individuals in a game may dynamically change because of changing environment [8], [9], [10], [11]. In public goods dilemmas where the use of common resource, such as water and pasture, is involving, the contest in real population not only modifies the social composition, but also may have a marked effect on the value of subsequent rewards [12]. If the shared resource is limited, "the tragedy of the commons" will be the inevitable fate for all players [13]. However, in some practical situations, the outcome of such evolutionary public goods games can be much more complicated when the shared resources are not only affected by the actions of individuals but also act back on the strategic choices of the populations through influencing the payoffs in the game process [14], [15], [16]. In fact, some actions may be in favor of enhancing the environment while the others weaken it. On the other hand, the environmental context can also in turn have influence on the individual actions by changing the current payoff matrices. Thus a feedback mechanism arises from the environmental state to the game dynamics.

So far, most of the existing works have been focusing on the dynamics of population profiles under the given fixed payoff matrices, but environmental feedback on game dynamics has not been taken into account adequately. Recently, the environment factors have received surging interests, and their influence is attracting more and more attention [17]. The co-evolution of strategies and games have been studied in [18], which shows the path to the collapse of cooperation; in particular, the evolvability of payoffs in a fixed environment is discussed, but the direct relation between payoffs and environment is not considered. It is studied in [19] how a single population evolves with a changing environmental resource using replicator equations. The corresponding system therein exhibits interesting oscillating behavior when the game is characterized by a payoff matrix modified from the prisoner dilemma. Indeed, the single population replicator equations combined with an environment factor take the form of an integrable system which admits constants of motion. Hence, it can be proved that the periodic orbits exist using level sets of the corresponding energy function.

When the structure of population is extended from a single population to multiple populations, it is of great interest to study whether convergence will take place in such situation, and if so, how. In addition, it will be more challenging to identify complicated dynamics in such a multi-dimensional system. The main contributions of the paper are as follows. The dynamic and environment-dependent payoff matrices are utilized to indicate the feedback of environment to game dynamics. Then we consider the multi-population game dynamics and the environment state simultaneously and obtain a model of a closed-loop and coupled system. For two interacting populations case, we derive sufficient conditions

The work of Gong and Gao was supported in part by China Scholarship Council (CSC). The work of Cao was supported in part by the European Research Council (ERC-CoG-771687) and the Netherlands Organization for Scientific Research (NWO-vidi-14134).

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for the convergence to boundary points. More importantly, we identify the scenarios where oscillation offers the best predictions of long-run behavior by using reversible system theory.

The rest of the paper is organized as follows. Section II introduces how the game and environment dynamics are modeled with necessary background information. Section III provides the main results: First, the conditions for convergence in the closed-loop system is analyzed with an illustrative example; then more complicated dynamics are analyzed thoroughly when the payoffs are in the form of a modified prisoner's dilemma; and lastly, a qualitative approach is used to prove the resulting periodic orbits by applying reversible system theory. In the last section, we conclude our contributions.

II. BACKGROUND AND PROBLEM STATEMENT

A. Replicator dynamics

As one of the most well-known models in evolutionary game theory, the *replicator dynamics* describe the evolution of the frequencies of strategies in a well-mixed population [2]. Consider a matrix game with a finite set of m pure strategies, $\{s_1,\ldots,s_m\}$, and the entries of the payoff matrix A are a_{ij} for $i,j=1,\ldots,m$. Let p_i be the proportion of the individuals who choose s_i, and denote $p=[p_1,\ldots,p_m]^T$. Then the single-population replicator dynamics are determined by setting the growth rate $\dot{p}_i$ proportional to the difference between the expected utility of s_i and the average utility in the whole population, namely

$$\dot{p}_i = p_i[U_i(p)-\bar{U}(p)], \tag{1}$$

where $U_i(p)=(Ap)_i$ is s_i's and $\bar{U}(p)=p^TAp$ is the average utility. Note that the replicator dynamics (1) are defined on the simplex $\Delta=\{p|p_i\geq 0,\ \sum_i p_i=1\}$. It has been proved in [2] that Δ is invariant under (1).

Now we extend the single-population replicator dynamics (1) to the multi-population case. Consider an n-population system and its replicator dynamics take

$$\dot{p}_i^k = p_i^k[U_i^k(\mathbf{p})-\bar{U}^k(\mathbf{p})], \tag{2}$$

where p_i^k is the proportion of individuals in population k, $k=1,\ldots,n$, who are currently using s_i and $\mathbf{p}=[p^1,\ldots,p^n]^T$, where p^k is determined by the proportions of the players of different strategies in population k. It can be easily verified that the state space of the whole population, which becomes a polyhedron, denoted by Θ, is invariant under (2), so do Θ's interior and boundary respectively.

B. Dynamic payoff matrices

To model the feedback of the environment to game dynamics, we introduce the dynamic payoff matrices. According to [11], [19], changes in the richness of biological environments, or the economic situation of governing bodies in social settings, sometimes can be captured by varying the payoff matrices by a multiplication factor. Therefore, we consider the matrix games where the payoff varies with a scalar environment variable r, which will be explained in detail in the coming subsection. In particular, we assume that the entries of each payoff matrix depend affinely on r, i.e.,

$$\begin{bmatrix} a_{11}r+b_{11} & \ldots & a_{1m}r+b_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1}r+b_{m1} & \ldots & a_{mm}r+b_{mm} \end{bmatrix}. \tag{3}$$

C. Environmental factor

To address in depth the environment's influence on game dynamics and vice versa, the environment in [19] is modeled by a scalar function coupled with game dynamics. We will use this mechanism to consider the multi-population games and environment dynamics. To be more specific, we represent the change of the environmental resource r by a continuous scalar function, i.e.,

$$\dot{r} = r(1-r)H(\mathbf{p}), \tag{4}$$

where r and initial state r_0 are confined in the range $[0,1]$; $H(\mathbf{p})$ denotes the feedback of the individuals' actions on the environment. The environmental state changes as a result of the states of the populations and the sign of $H(\mathbf{p})$ determines whether r will decrease or increase, corresponding to environmental degradation or enhancement, respectively.

D. Mathematical model

For the sake of simplicity, we start by considering the case of two interacting populations. In this case, each individual from population $k=1,2$ only interacts with an individual from the other population. Such games are usually referred to as *bi-matrix games*. In addition, we assume that each population k has only two strategies $\{s_1,s_2\}$. Then the population profiles can be defined by $p^k=[p_1^k \quad 1-p_1^k]^T$. In every population we only need to focus on the evolution of the proportion of one strategy. Hereafter we denote $x=p_1^1$ and $y=p_1^2$. We consider the strategy set in the context of the cooperation-defection game as a means to motivate our analysis, i.e., the proportion of the first strategy, cooperation, enriches the resource and the proportion of the second strategy, defection, consumes the resource.

The interactions between the two populations are described by the following dynamic payoff matrices $A(r)$ and $B(r)$:

$$A(r)=\begin{bmatrix} a(r) & b(r) \\ c(r) & d(r) \end{bmatrix} \quad \text{and} \quad B(r)=\begin{bmatrix} e(r) & f(r) \\ g(r) & h(r) \end{bmatrix}.$$

Note that generally $A(r)\neq B(r)$, and thus the games are asymmetric. More precisely, if a player in population 1 plays s_1 against a player in population 2 playing s_2, then the first player receives the payoff $b(r)$ and the second player obtains the payoff $g(r)$. Then the utility functions are given by

$$U_i^1=(A(r)\mathbf{y})_i, \quad U_i^2=(B(r)\mathbf{x})_i, \tag{5}$$

where $\mathbf{x}=\begin{bmatrix} x & 1-x \end{bmatrix}^T$ and $\mathbf{y}=\begin{bmatrix} y & 1-y \end{bmatrix}^T$.

Combine the game dynamics (2) and environment equation (4), then we have the corresponding 3-dimensional co-evolutionary system

$$\begin{cases} \dot{x} = x(1-x)F(y,r) \\ \dot{y} = y(1-y)G(x,r) \\ \dot{r} = r(1-r)H(x,y), \end{cases} \tag{6}$$

where

$$\begin{aligned} F(y,r) &= U_1^1 - U_1^2 \\ &= (a(r)-b(r)-c(r)+d(r))y+b(r)-d(r), \end{aligned} \tag{7}$$

$$\begin{aligned} G(x,r) &= U_2^1 - U_2^2 \\ &= (e(r)-f(r)-g(r)+h(r))x+f(r)-h(r). \end{aligned} \tag{8}$$

In addition, we assume $h(x,y)$ is of the following form:

$$\begin{aligned} H(x,y) &= \theta_1 x-(1-x)+\theta_2 y-(1-y) \\ &= (1+\theta_1)x+(1+\theta_2)y-2, \end{aligned} \tag{9}$$

in which $\theta_1, \theta_2 > 0$ represent the ratios of enhancement rate to degradation rate.

Compared with the conventional replicator dynamics (2), (6) is more complicated because the payoff matrices are environment dependent and the strategies and environment dynamics are deeply coupled. One immediately can get some basic properties of this new system. Since all the variables, x, y and r, just vary in the range $[0,1]$, the domain for the system (6) is exactly the unit cube $[0,1]^3$ in the Cartesian coordinates. And through the structure of equations (6) one can easily prove the following lemmas.

Lemma 1 (Fixed points): The system (6) has eight obvious fixed points which are exactly on the eight corners of the cubic domain, i.e., $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$.

Lemma 2 (Invariant sets): The following statements hold for (6)

- the whole domain $[0,1]^3$ is a positively invariant set;
- the 6 faces of this cube are positively invariant sets.

Because of the above invariance, the system dynamics on the 6 faces are reduced to planar cases. Therefore it is trivial to analyze them, and we omit discussing these 2-dimensional dynamics hereafter and mainly focus on the 3-dimensional dynamics.

III. MAIN RESULTS

A. Convergence

The coexistence of states has always been the main subject of bi-matrix game. For this reason, we are going to analyze the convergence and non-convergence of the dynamics of (6).

Theorem 3: When the dynamic payoff matrices (3) admit weakly dominating strategies, the corresponding co-evolutionary dynamics (6) will always converge to the boundary for the initial conditions $(x_0,y_0,r_0) \in (0,1)^3$.

Proof: Without loss of generality, let the first strategy be the weakly dominating strategy for the first population. Then the payoff matrix $A(r)$ takes the following form

$$\begin{bmatrix} a(r) & b(r) \\ c(r) & d(r) \end{bmatrix}, \quad a(r) \geq c(r), b(r) \geq d(r).$$

As all entries of $A(r)$ are affine functions of r, the equality signs of the above inequalities hold only when r is exactly at its maximum or minimum. Then the replicator equation corresponding to x becomes

$$\dot{x} = x(1-x)[(a(r)-c(r))y+(b(r)-d(r))(1-y)].$$

One can check that the term in the square bracket is always positive for any initial $r_0 \in (0,1)$. As a result, x will converge to 1. When $r_0 = 0$ or $r_0 = 1$, the dynamics will be restricted in a plane because of the invariance according to Lemma 2. Similarly one can prove convergence for y. When the states x and y reach their maximums or minimums, the sign of $\dot{r}$ will depend on the parameters θ_1, θ_2. Then r will arrive at some fixed point asymptotically. ■

We now use an example to illustrate Theorem 3.

Example 3.1: A version of game [10] takes the following environment-dependent form,

$$A(r) = \begin{bmatrix} 0.5r & 1 \\ 0 & 0.5 \end{bmatrix}; \quad B(r) = \begin{bmatrix} 0.5 & 0 \\ 1 & 0.5r \end{bmatrix}.$$

Obviously the cooperation strategy corresponds to the unique Nash Equilibrium (NE) for the first population and the defection strategy for the second population respectively, whenever $r \in (0,1)$. But when r reaches its maximum or minimum, these two strategies are not NE anymore. Therefore, they are only weakly dominating strategies in this sense.

The corresponding system now becomes

$$\begin{cases} \dot{x} = x(1-x)(0.5ry-0.5y+0.5) \\ \dot{y} = y(1-y)(0.5rx-0.5x-0.5r) \\ \dot{r} = r(1-r)[(1+\theta_1)x+(1+\theta_2)y-2]. \end{cases} \tag{10}$$

One can check the stability of the corner equilibria by the Jacobians. Obviously, five of the equilibria, $(0,0,0)$, $(0,0,1)$, $(1,1,0)$, $(0,1,1)$ and $(1,1,1)$, are unstable because all the Jacobians have at least one positive eigenvalue. It is, however, not clear whether the other equilibria are stable or not, since their Jocobians are dependent of parameters.

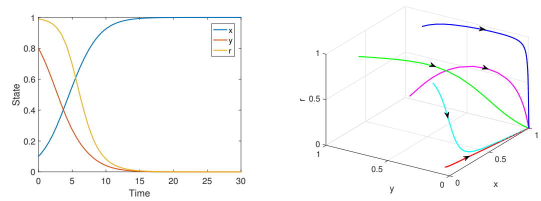
However, in the first equation $(0.5ry - 0.5y + 0.5) = 0.5ry + 0.5(1-y) \geq 0$ and the equality holds only when $y = 1, r = 0$. Thus, x will converge to 1 if $x_0 \neq 0$. In contrast, in the second equation $(0.5rx-0.5x-0.5r) = 0.5x(r-1) - 0.5r \leq 0$ and the equality holds only when $x = 0, r = 0$. So it always holds that y will converge to 0 as expected if $y_0 \neq 0$ initially. As a consequence, there are two more critical sets, i.e., lines $\{y = 1, r = 0\}$ and $\{x = 0, r = 0\}$.

For r, the open set $(0,1)^3$ contains no equilibrium point as $(0.5ry-0.5y+0.5) > 0$ and $(0.5rx-0.5x-0.5r) < 0$. Hence, every trajectory will go towards a point on the boundary $\{x = 1, y = 0\}$, when the starting point is in $(0,1)^3$. Further, on the line $\{x = 1, y = 0\}$, we have

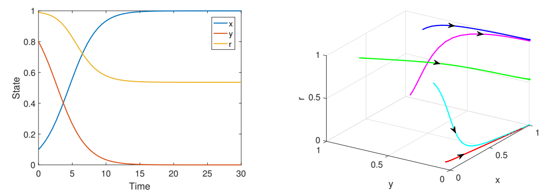
$$\dot{r} = r(1-r)[(1+\theta_1)x+(1+\theta_2)y-2] = r(1-r)(\theta_1 - 1).$$

Thus, the evolution of r depends on the parameter θ_1. To sum up, the trajectory will asymptotically converge to the fixed point $(x = 1, y = 0, r = 1)$ if $\theta_1 > 1$ or to the fixed point $(x = 1, y = 0, r = 0)$ if $\theta_1 < 1$. For the case $\theta_1 = 1$, the trajectory will stay there after it reaches the line $\{x=1, y=0\}$. The different situations are showed in Fig. 1.

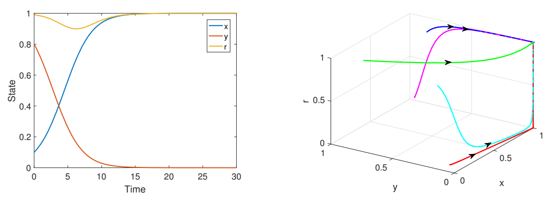
In this example the states of both of the two populations cannot co-exist no matter what values the parameters θ_1, θ_2 take. In other words, the environment factor does not make a big difference to the outcome of game dynamics even when the payoff matrices admit only weakly dominating strategies.



(a) $\theta_1 = 0.5$



(b) $\theta_1 = 1$



(c) $\theta_1 = 1.5$

Fig. 1: Convergence of the trajectories when $\theta_2 = 0.5$ and $\theta_1 = 0.5$ (a), $\theta_1 = 1$ (b), $\theta_1 = 1.5$ (c). The left panels show the time evolution of (x,y,r) corresponding to the same initial condition $(0.1, 0.8, 0.99)$. The right panels show the distinct trajectories corresponding to initial conditions $(0.1, 0.1, 0.01)$ (red), $(0.1, 0.8, 0.99)$ (green), $(0.1, 0.2, 0.9)$ (cyan), $(0.9, 0.9, 0.1)$ (magenta) and $(0.9, 0.9, 0.1)$ (blue).

Theorem 3 only gives a sufficient condition for convergence to the boundary point. It would be very difficult to construct the general conditions for non-convergence of the co-evolutionary system. In the following section, we will provide a specific type of payoff structure which results in oscillating behaviors.

B. Periodic Orbits

We adopt the following payoff matrices which have been used in [19]. They are modified from the general Prisoner Dilemma (PD) game:

$$A(r) = (1-r)\begin{bmatrix} T_1 & P_1 \\ R_1 & S_1 \end{bmatrix} + r\begin{bmatrix} R_1 & S_1 \\ T_1 & P_1 \end{bmatrix}, \tag{11}$$

$$B(r) = (1-r)\begin{bmatrix} T_2 & P_2 \\ R_2 & S_2 \end{bmatrix} + r\begin{bmatrix} R_2 & S_2 \\ T_2 & P_2 \end{bmatrix}, \tag{12}$$

where $P_1 > S_1$, $T_1 > R_1$ and $P_2 > S_2$, $T_2 > R_2$ such that mutual cooperation is a NE when $r \to 0$ and mutual defection is a NE when $r \to 1$. Intuitively, more players prefer to cooperate in the period of scarce resources and incline to defect in the period of ample resources.

Using (6), we arrive at

$$\begin{cases} \dot{x} = x(1-x)[\delta_{PS_1} + (\delta_{TR_1} - \delta_{PS_1})y](1-2r) \\ \dot{y} = y(1-y)[\delta_{PS_2} + (\delta_{TR_2} - \delta_{PS_2})x](1-2r) \\ \dot{r} = r(1-r)[(1+\theta_1)x + (1+\theta_2)y - 2], \end{cases} \tag{13}$$

where $\delta_{PS_1} = P_1 - S_1 > 0$, $\delta_{TR_1} = T_1 - R_1 > 0$ and $\delta_{PS_2} = P_2 - S_2 > 0$, $\delta_{TR_2} = T_2 - R_2 > 0$.

1) Equilibria and stability: Except for the eight isolated fixed points identified in Lemma 1, there is also a set of fixed points in the interior of the cube, i.e., $\left\{(x,y,r) : (1+\theta_1)x + (1+\theta_2)y = 2, r = \frac{1}{2}\right\}$. To study the local stability at these fixed points, one needs to calculate their Jacobians. It is easy to check that all the eight corner fixed points are unstable because each of their Jacobian matrices has at least one positive eigenvalue.

In addition, the Jacobian matrix of an interior fixed point (x^*, y^*, r^*) is of the following form:

$$J^* = \begin{bmatrix} 0 & 0 & -2x^*(1-x^*)[\delta_{PS_1} + (\delta_{TR_1} - \delta_{PS_1})y^*] \\ 0 & 0 & -2y^*(1-y^*)[\delta_{PS_2} + (\delta_{TR_2} - \delta_{PS_2})x^*] \\ \dfrac{1+\theta_1}{4} & \dfrac{1+\theta_2}{4} & 0 \end{bmatrix}, \tag{14}$$

where $(1+\theta_1)x^* + (1+\theta_2)y^* = 2$.

In general the characteristic polynomial for a three dimensional system takes the form

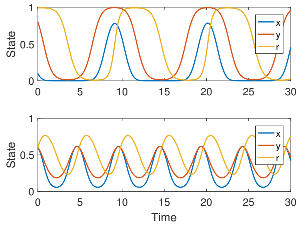
$$\lambda^3 - T\lambda^2 - K\lambda - D = 0, \tag{15}$$

where T, D indicate the trace and determinant of the Jacobian respectively. We calculate from the Jacobian (14) that $T = \text{trace}(J^*) = 0$, $D = \det(J^*) = 0$ and

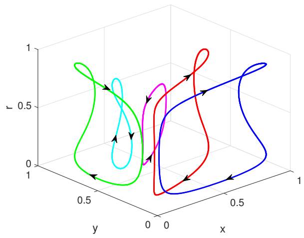
$$\begin{aligned} K = &-\frac{1+\theta_1}{2}y^*(1-y^*)[\delta_{PS_2} + (\delta_{TR_2} - \delta_{PS_2})x^*] \\ &-\frac{1+\theta_2}{2}x^*(1-x^*)[\delta_{PS_1} + (\delta_{TR_1} - \delta_{PS_1})y^*] \\ &< 0. \end{aligned}$$

Thus, one can conclude the eigenvalues are $\lambda_1 = 0$ and $\lambda_{2,3} = \pm\sqrt{-K}i$, in which $\lambda_{2,3}$ are conjugate pure imaginary numbers. When the real parts of all the eigenvalues are zero, one can only say that the interior equilibria are neutrally stable for the linearized system. As a consequence, to analyze the stability of the original nonlinear system through linearization is not effective. Moreover, the method in [19] utilizing the Hamiltonian system theory is not directly applicable anymore because the system's dimension is odd.

Through simulation we find the trajectories initialized with (x_0, y_0, r_0) in the interior of the cube $[0,1]^3$ and not at the fixed points exhibit closed periodic orbits as shown in Fig. 2. Therefore, it is natural to ask whether all the solutions for system (13) in $(0,1)^3$ are either fixed points or periodic orbits.



(a) Time Evolution



(b) Periodic Orbits

Fig. 2: Periodic Orbits of system (16) when $R_1 = 2$, $S_1 = -2$, $T_1 = 4$ and $P_1 = 3$; $R_2 = 3$, $S_2 = 0$, $T_2 = 5$ and $P_2 = 1$; $\theta_1, \theta_2 = 2$. (a) Time evolution of states with initial conditions $(0.1, 0.8, 0.99)$ and $(0.6, 0.6, 0.6)$. (b) Periodic trajectories with initial conditions $(0.1, 0.1, 0.1)$ (red), $(0.1, 0.8, 0.9)$ (green), $(0.6, 0.6, 0.6)$ (magenta), $(0.2, 0.6, 0.6)$ (cyan) and $(0.9, 0.1, 0.1)$ (blue).

In the following subsection, we are going to show that the neutral stability of linearized system at fixed points is preserved in the original nonlinear system. We will first analyze the trajectories around the equilibrium points, and then apply an inverse derivation to prove this stability.

2) Reversible system: Before proving the claim on periodic orbits, we need to introduce the *reversible system theory*. We say a transformation, denoted by G, is an *involution* if the the composition of itself is a identity mapping, i.e. $G \circ G = Identity$.

Definition 1: A dynamical system is said to be *reversible* if there is an involution in phase space which reverses the direction of time. For a general system of coupled ordinary differential equations,

$$\frac{d\mathbf{x}}{dt} = F(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^N, \tag{16}$$

it is reversible if there is an involution G which reverses the direction of time, i.e.,

$$\frac{dG(\mathbf{x})}{dt} = -F(G(\mathbf{x})). \tag{17}$$

The above definition implies that under the transformation of the N-dimensional phase space by G, the system (16) is transformed to that obtained by just putting $t \to -t$, so that under the combined action of involution and time reversal the equations are invariant.

By the transformation of $t \to -t$, $x \to x$, $y \to y$ and $r \to 1-r$, one can easily verify the resulting system is the same as the original one. So the system (13) is invariant under $t \to -t$, with the phase space involution $G : x \to x, y \to y, r \to 1-r$. Hence, this system is reversible with respect to the above G and the fixed manifold $Fix(G)$ is the plane $\{r = \frac{1}{2}\}$. Now we review the intrinsic property of the reversible system.

Lemma 4 (Periodic orbits for reversible systems [20]): Let $o(\mathbf{x})$ be an orbit of the flow of an autonomous vector filed with time-reversal symmetry G. Then an orbit $o(\mathbf{x})$ intersects $\text{Fix}(G)$ at precisely two points if and only if the orbit is periodic (and not a fixed point) and symmetric with respect to G.

We continue to analyze system (13) with the mentioned time-reversal symmetry G. First, we show that there is another invariant set for system (13).

Lemma 5 (Invariant set): For system (13), the open region, $\Omega = (0,1)^3$, is an (positively and negatively) invariant set.

Proof: First it can be easily checked in the plane $\{r = 1\}$ the point $(0,0,1)$ is asymptotically stable for (13). Then, denote one trajectory starting from an arbitrary point p in the domain $\Omega = (0,1)^3$ by $\varphi(t,0,p)$. If it approaches the face $\{r = 1\}$ at some time t_1, then it will always converges to the point $(0,0,1)$ as $t \to \infty$ because of the continuity of $\dot{x}$ and $\dot{y}$. Consider the deleted neighborhood M of $(0,0,1)$ in Ω , i.e., $M \subset \Omega$. One can check that $\dot{r}$ is always negative in M. Thus, the trajectory cannot approach the point $(0,0,1)$. One can also analogously prove that the trajectory $\varphi(t,0,p)$ cannot reach other faces either. Hence, the positive invariance of Ω is proved.

If the trajectory starts from the region $\mathbb{R}^3 \backslash \Omega$, it may reach the boundary of $[0,1]^3$. But it cannot get into the interior, because the 6 faces of the cube are positively invariant. This corresponds to the negative invariance of Ω. In conclusion, we have proved that the open set $\Omega = (0,1)^3$ is positively and negatively invariant. ■

Now we are ready to prove the main result.

Theorem 6: The interior of the phase space, $\Omega = (0,1)^3$, is filled with infinitely many independent periodic orbits, each centered at an interior equilibrium point.

Proof: First we divide the phase space $\Omega = (0,1)^3$ into 4 regions by the two planes $\{r = \frac{1}{2}\}$ and $\{(1+\theta_1)x + (1+\theta_2)y = 2\}$. Denote the four regions by

$$\begin{aligned}
\Omega_1 &: \{\tfrac{1}{2} < r < 1, (1+\theta_1)x + (1+\theta_2)y > 2\}; \\
\Omega_2 &: \{\tfrac{1}{2} < r < 1, (1+\theta_1)x + (1+\theta_2)y < 2\}; \\
\Omega_3 &: \{0 < r < \tfrac{1}{2}, (1+\theta_1)x + (1+\theta_2)y < 2\}; \\
\Omega_4 &: \{0 < r < \tfrac{1}{2}, (1+\theta_1)x + (1+\theta_2)y > 2\}.
\end{aligned}$$

From (13) it is easy to determine that the sign of $\dot{r}$ is positive in Ω_1, Ω_4 and negative in Ω_2, Ω_3, respectively. And the signs of $\dot{x}, \dot{y}$ are available as well.

Now, we consider an arbitrary trajectory $\varphi(t,0,q)$ starting from a point $q = (x_0, y_0, r_0)$ in Ω_2. By using Lemma 5 one can exclude the trajectory going towards the faces, as a result it will always stay inside the cube.

Then, in Ω_2 we choose a closed subset $S = \{\frac{1}{2} \le r \le r_0\} \cap \Omega_2$.

It is clear that $\dot{r}$ in subset S remains negative. As the function on the right-hand side of equation $\dot{r}$ is continuous, there must exist a small number $a > 0$ such that $\dot{r} \leq -a$. If the trajectory does not go across the plane $\{r = \frac{1}{2}\}$, then the projection of $\varphi(t,0,q)$ to the r axis at time τ will be

$$r(\tau) = r(0) + \int_0^\tau \dot{z}dt \leq r(0) - a(\tau - 0)$$

Then when the time τ goes to infinity, we get

$$\lim_{\tau\to\infty} r(\tau) \leq r(0) - \lim_{\tau\to\infty} a(\tau - 0) = -\infty$$

This result contradicts the boundedness of the subset. Therefore, the trajectory will traverse the plane $\{r = \frac{1}{2}\}$ and always goes into the region Ω_3. Following similar steps, one can verify the trajectory will cross the plane $\{(1+\theta_1)x + (1+\theta_2)y = 2\}$ by checking the signs of $\dot{x}$ or $\dot{y}$, and enters into the region Ω_4, and then it continues to cross the plane $\{r = \frac{1}{2}\}$ again. The geometric view of the whole process is depicted in Fig. (3).

In view of Lemma 4, the trajectory will finally return to the starting point and form a closed orbit. Hence, the periodic trajectory is proved. Immediately one can claim that every closed orbit is neutrally stable since no other trajectories converge to it. ∎

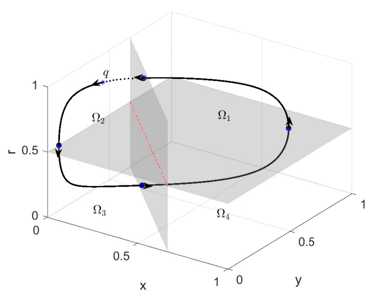


Fig. 3: Formation of one periodic orbit with initial point q.

Then we can deduce the property of the internal equilibrium points.

Theorem 7: The interior equilibrium points are neutrally stable for the nonlinear system (13).

By showing all the trajectories around the equilibria are independently periodic orbits, it is intuitive to get the neutrally stability of the corresponding equilibrium points.

The periodic trajectory of system (13) presents an unusual phenomenon, namely the states of two populations and environment oscillate dynamically. Under the payoff matrices of (10) and (11), the population profiles with the environment can evolve periodically without the extinction of any specify type of strategic players. The occurrence of closed orbits is more complicated than convergence to limit cycles or equilibria, because how the co-evolutionary dynamics evolve depends highly on the system's initial states.

IV. CONCLUSIONS

We have proposed a new framework to study multi-population evolutionary game dynamics with environmental feedback and investigated whether and how convergence and coexistence take place in the new closed-loop system. The influence of dynamic and asymmetric payoff matrices have been studied in depth for two interacting populations. Two situations, convergence and dynamically coexistence, have been clarified through different approaches. In particular, one scenario where oscillation offers the best predictions of long-term behavior has been identified with PD game. The obtained results can be very useful to describe the evolution of multi-community societies in which individuals' payoffs and societal feedback interact. In the future, we will generalize the framework to multi-population or networked populations situations. It is also of great interest to study different payoff matrices.

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