FRACTIONAL-STEP SCHEMES FOR THE COUPLING OF DISTRIBUTED AND LUMPED MODELS IN HEMODYNAMICS

CRISTÓBAL BERTOGLIO†, ALFONSO CAIAZZO‡, AND MIGUEL A. FERNÁNDEZ†

Abstract. In three-dimensional (3D) blood flow simulations, lumped parameter models (0D) are often used to model the neglected parts of the downstream circulatory system. We consider two 3D-0D coupling approaches in which a fractional-step projection scheme is used in the fluid. Our analysis shows that explicit approaches might yield numerical instabilities, particularly in the case of realistic geometries with multiple outlets. We introduce and analyze an implicitly 3D-0D coupled formulation with enhanced stability properties and which requires a negligible additional computational cost. We also address the extension of these methods to fluid-structure interaction problems. The theoretical stability results are confirmed by meaningful numerical experiments in patient specific geometries coming from medical imaging.

Key words. blood flows, chorin-Temam projection method, fluid-structure interaction, lumped parameter model, 3D-0D coupling

AMS subject classifications. 35M20, 76D05, 74F10, 65L05, 76Z05

DOI. 10.1137/120874412

1. Introduction. In three-dimensional (3D) distributed models of blood flow (e.g., Navier–Stokes equations, fluid-structure interaction), downstream pressure boundary conditions are often used to represent the effect of the neglected portion of the vessels. In fact, since pressure measurements are invasive and not always available, downstream circulation is usually modeled through lumped parameter (or 0D) models. This results in a set of algebraic-differential equations, relating fluxes and pressures at each outlet boundary of the fluid domain (see, e.g., [12, 21, 22, 25] and the references therein). Widely used lumped parameter models are the so-called Windkessel systems (see, e.g., [12, 13, 23]).

The 3D-0D coupling between distributed and lumped models is operated by interface conditions that guarantee the continuity of the fluxes and pressures on the outlet boundaries (3D-0D interfaces). It is well known that when this coupling is treated explicitly in time, the whole system might suffer from numerical instability, regardless of the solution scheme in the 3D compartment. This enforces restrictions on the time step length that can considerably increase computational cost in realistic applications (see, e.g., the discussion in [22]). On the other hand, implicit 3D-0D coupling schemes overcome this instability issue at the expense of solving a fully coupled 3D-0D system at each time step. The so-called partitioned approaches solve this system by subiterating between the 3D and 0D models, which might be inefficient in practice (see, e.g., [19]). Alternatively, monolithic procedures solve both models simultaneously and yield system matrices with a modified sparsity pattern, which might lead to preconditioning issues (see, e.g., [20]).

†Submitted to the journal’s Computational Methods in Science and Engineering section April 20, 2012; accepted for publication (in revised form) February 6, 2013; published electronically May 2, 2013. This work was partially funded by the Seventh Framework Program (FP7/2007-2013) of the European Community under grant agreement 224495 (euHeart project).
http://www.siam.org/journals/sisc/35-3/87441.html
†Inria, REO team, Rocquencourt BP 105, 78153 Le Chesnay Cedex, France, and UPMC Univ Paris 06, REO team, UMR 7598 LJLL, 75005 Paris, France (cristobal.bertoglio@inria.fr, miguel.fernandez@inria.fr).
‡WIAS Berlin, Mohrenstrasse 39, 10117, Berlin, Germany (Alfonso.caiazzo@wias-berlin.de).
This work is devoted to the formulation and analysis of 3D-0D coupling schemes based on a fractional-step projection time-marching of the fluid (see, e.g., [6, 24, 16]). First, an advection-diffusion problem is solved to recover an intermediate approximation of the velocity field and then a suitable pressure field is recovered by solving a Poisson problem. We also consider the case of a 3D fluid-structure interaction (FSI) modeling, time discretized by the projection semi-implicit coupling scheme reported in [9, 10]. In this fractional-step framework, the explicit and implicit treatments of the 3D-0D coupling lead to different formulations of the pressure-Poisson projection step. This step can be discretized in an explicit fashion by time-advancing the 0D model from the previously computed intermediate velocity flux, which provides an explicit Dirichlet boundary data for the 3D pressure-Poisson problem. For a purely fluid 3D distributed modeling with two or more 3D-0D interfaces, our analysis shows that this approach might compromise stability. We also show that in the case of FSI, numerical instabilities might appear even with a sole downstream boundary. We propose to overcome these instability issues through the introduction of an implicit 3D-0D formulation with enhanced energy balance across the 3D-0D interface, for both a fluid and a FSI modeling of blood flow. A salient feature of the proposed schemes is that they preserve the two-step velocity/pressure splitting of the original fractional-step scheme, characterized by a 3D-0D coupled problem of reduced size. Moreover, we show that this coupling strategy can be efficiently implemented by considering a single unknown per 3D-0D interface. This yields a computational complexity comparable to an explicit scheme and, moreover, overcomes the mentioned stability and complexity issues.

The rest of the paper is organized as follows. In section 2 we introduce the 3D fluid equations, its corresponding time discretization (via a fractional-step projection scheme), and the considered Windkessel model. Section 3 is devoted to the formulation and analysis of explicit and implicit 3D-0D coupling schemes with a 3D distributed model based on the Navier–Stokes equations, while section 4 considers the FSI case. In section 5 we present and discuss the performed numerical experiments, that is, the Navier–Stokes flow in a realistic aortic geometry and with measured clinical data, and FSI in an idealized abdominal aortic aneurism (AAA). Section 6 draws some concluding remarks and lines of future work.

2. The 3D-0D model problem. We first summarize the main ingredients of a 3D-0D model of blood flow coupling the incompressible Navier–Stokes equation and a lumped parameter description of the downstream boundaries. We then present the
time-marching schemes considered in each subsystem and introduce some notation for the spatial discretization of the fluid equations.

2.1. 3D fluid equations. We consider a domain \( \Omega_f \subset \mathbb{R}^3 \) with the following partition of its boundary \( \partial \Omega_f = \Gamma^{\text{in}} \cup \Sigma \cup \Gamma^{\text{out}} \). In the context of blood flow simulations, \( \Omega_f \) will represent the lumen of the vessel (see Figure 2.1) with \( \Gamma^{\text{in}} \), \( \Sigma \), and \( \Gamma^{\text{out}} \) denoting, respectively, the inlet, vessel wall, and outlet boundaries. We now consider the incompressible Navier–Stokes equations for the velocity \( u \) : \( \Omega_f \times \mathbb{R}^+ \to \mathbb{R}^3 \) and the pressure \( p : \Omega_f \times \mathbb{R}^+ \to \mathbb{R} \):

\[
\begin{align*}
\rho_f \frac{\partial u}{\partial t} + \rho_f u \cdot \nabla u - \nabla \cdot \sigma(u, p) &= 0 \quad \text{in} \quad \Omega_f, \\
\nabla \cdot u &= 0 \quad \text{in} \quad \Omega_f, \\
u = u_{\text{in}} &\quad \text{on} \quad \Gamma^{\text{in}}, \\
u = 0 &\quad \text{on} \quad \Sigma, 
\end{align*}
\]

(2.1)

where \( \rho_f \) stands for the density of the fluid and the fluid Cauchy stress tensor is given by \( \sigma(u, p) \) def \( = -pI + 2\mu \varepsilon(u) \) and \( \varepsilon(u) \) def \( = \frac{1}{2}(\nabla u + \nabla u^\top) \), \( \mu \) being the dynamic viscosity of the fluid and \( u_{\text{in}} \) being a given inlet velocity field. The additional boundary conditions on \( \Gamma^{\text{out}} \) will be considered in the next subsection.

2.2. 0D Windkessel model. In what follows, the outlet boundary \( \Gamma^{\text{out}} \) is assumed to be made of \( n_{0D} \) components,

\[ \Gamma^{\text{out}} = \bigcup_{l=1}^{n_{0D}} \Gamma_l \]

such that \( \Gamma_i \cap \Gamma_j = \emptyset \) for \( i, j = 1, \ldots, n_{0D} \) with \( i \neq j \) (see, e.g., Figure 2.1). We will consider a three-element Windkessel model (see, e.g., [12, Chapter 10] and [13]), where the pressure \( P_l : \mathbb{R}^+ \to \mathbb{R} \) and the flux \( Q_l : \mathbb{R}^+ \to \mathbb{R} \) on the outlet \( \Gamma_l \) are related through the following algebraic-differential equations:

\[
\begin{align*}
C_{d,l} \frac{d\pi_l}{dt} + \frac{\pi_l}{R_{d,l}} &= Q_l, \\
P_l &= R_{p,l} Q_l + \pi_l
\end{align*}
\]

(2.2)

for \( l = 1, \ldots, n_{0D} \). Here, \( R_{p,l} \) and \( R_{d,l} \) model the resistance to the flow of the vascu-

lature proximal and distal to each outlet \( \Gamma_l \), respectively, and the capacity \( C_{d,l} \) takes into account the deformability of the downstream vessels. The values \( P_l \) and \( \pi_l \) are also called proximal and distal pressures, respectively.

Remark 2.1. We will mainly focus on a three-element Windkessel model that is one of the most popular choices in computational hemodynamics. However, this choice is purely illustrative and the methods and the analysis presented in the following sections can be extended to more complex lumped parameter models (see section 3.2.4 for details).

2.3. 3D-0D coupling conditions. The 3D-0D coupling between (2.1) and (2.2) is defined through the following relations on each \( \Gamma_l \):

\[
\begin{align*}
Q_l &= \int_{\Gamma_l} u \cdot n_l, \\
\sigma(u, p) n_l &= -P_l n_l \quad \text{on} \quad \Gamma_l
\end{align*}
\]

(2.3)

for \( l = 1, \ldots, n_{0D} \) and where \( n_l \) denotes the exterior unit-vector normal of \( \Omega_l \).
Energy balance. Let the quantity
\[ E(t) = \frac{\rho f}{2} \| u \|^2_{0,\Omega} + \sum_{l=1}^{n_{\text{in}}} \frac{C_{d,l}}{2} \pi^2 l^2 \]
denote the total (kinetic + potential) energy of the 3D-0D coupled system given by (2.1)–(2.3), while
\[ D(t) = 2\mu \int_0^t \| \epsilon(u(s)) \|^2_{0,\Omega} ds + \sum_{l=1}^{n_{\text{in}}} \int_0^t \left( \frac{\pi^2}{R_{d,l}} + R_{p,l}Q^2_l(s) \right) ds > 0 \]
represents the dissipative effects. Assuming that \( u_{\text{in}} = 0 \) (free system) and using a standard energy argument, we get the following identity:
\[ E(t) + D(t) + \int_0^t \left( \int_{\Gamma_{\text{out}}} \frac{\rho f}{2} |u(s)|^2 u(s) \cdot n_f \right) ds = E(0). \]

Remark 2.2. Since the last term of the left-hand side of (2.5) can be negative, this expression does not guarantee a correct energy balance across the 3D-0D interface \( \Gamma_{\text{out}} \). This issue is well known in computational hemodynamics. The interested reader is referred to [4] for a stabilization technique and to [11] for a different 3D-1D coupling. The methods introduced in this paper can be easily adapted to these alternative formulations.

2.4. Time semidiscretization. We consider a fractional-step time-marching of the fluid equations (2.1) and a backward Euler scheme for the lumped parameter model (2.2). In what follows, the parameter \( \tau \) denotes the time step size, we set \( t_n = n\tau \) for \( n \in \mathbb{N} \), and \( \partial_x^n \equiv (x^n - x^{n-1})/\tau \) stands for the first-order backward difference.

2.4.1. Fractional-step fluid time-marching. Several variants of the original Chorin–Temam projection scheme [6, 24] have been proposed in the literature (see, e.g., [16] for a recent review). The methods presented and analyzed in section 3 below do not a priori depend on the specific formulation considered for the projection scheme. To fix the ideas and without generality loss, we consider the nonincremental pressure-correction version (see, e.g., [16, section 4]). Hence, the time semidiscrete approximation of 2.1 is performed as follows. We set \( \tilde{u}^0 = u^0 = u_0 \) and, for \( n \geq 1 \), we compute \((u^n, p^n, \tilde{u}^n)\) by solving the following:
1. Viscous step:
\[
\begin{align*}
\frac{\rho_f}{\tau} (\tilde{u}^n - u^{n-1}) + \rho_f \tilde{u}^{n-1} \cdot \nabla \tilde{u}^n - 2\mu \nabla \cdot \epsilon(\tilde{u}^n) &= 0 \quad \text{in} \quad \Omega_f, \\
\tilde{u}^n &= u_{\text{in}}(t_n) \quad \text{on} \quad \Gamma_{\text{in}}, \\
\tilde{u}^n &= 0 \quad \text{on} \quad \Sigma.
\end{align*}
\]
2. Projection step:
\[
\begin{align*}
\frac{\rho_f}{\tau} (u^n - \tilde{u}^n) + \nabla p^n &= 0 \quad \text{in} \quad \Omega_f, \\
\nabla \cdot u^n &= 0 \quad \text{in} \quad \Omega_f, \\
u^n \cdot n_f &= u_{\text{in}}(t_n) \cdot n_f \quad \text{on} \quad \Gamma_{\text{in}}, \\
u^n &= 0 \quad \text{on} \quad \Sigma.
\end{align*}
\]
From the implementation point of view, the projection step (2.7) is usually reformulated as the pressure-Poisson problem

\[\begin{align*}
\frac{-\tau}{\rho_l} \Delta p^n &= -\nabla \cdot \bar{u}^n \quad \text{in} \quad \Omega_l, \\
\frac{\tau}{\rho_l} \partial p^n / \partial n_l &= 0 \quad \text{on} \quad \Gamma_{\text{in}} \cup \Sigma,
\end{align*}\]  

(2.8)

which requires further regularity on the pressure \((H^1(\Omega_l))\) instead of \(L^2(\Omega_l)\) in practice. Then, the divergence-free (or end-of-step) velocity \(u^n\) can be eliminated in (2.6) using the following relation (from (2.7)):

\[u^n = \bar{u}^n - \frac{\tau}{\rho_l} \nabla p^n.\]  

(2.9)

It should be noted that the boundary conditions on \(\Gamma_{\text{out}}\) have been omitted deliberately in (2.6)–(2.8), since they depend on the type of 3D-0D coupling scheme considered. In particular, they lead to different formulations of the pressure-Poisson equation (2.8) (see section 3.1).

2.4.2. Backward Euler Windkessel time-marching. Without loss of generality, we consider a backward Euler time discretization of (2.2), which yields

\[\begin{align*}
C_{d,l} \partial \tau_i^n + \frac{\pi_l^n}{R_{d,l}} &= Q_{d,l}^n, \\
P_{d,l}^n &= \gamma Q_{d,l}^n + \alpha_i \pi_i^{n-1},
\end{align*}\]  

(2.10)
or, equivalently,

\[\begin{align*}
\pi_i^n &= \alpha_i \pi_i^{n-1} + \beta_i Q_{d,l}^n, \\
P_{d,l}^n &= \gamma Q_{d,l}^n + \alpha_i \pi_i^{n-1},
\end{align*}\]  

(2.11)

with the notation \(\alpha_i \equiv R_{d,l} C_{d,l} / R_{d,l} C_{d,l} + \tau, \beta_i \equiv R_{d,l} (1 - \alpha_i)\) and \(\gamma \equiv R_{p,l} + \beta_l\).

2.5. Spatial discretization. In what follows, we will consider the usual Sobolev space \(H^1(\Omega)\) for a given domain \(\Omega \subset \mathbb{R}^3\). Then, for \(X \subset \partial \Omega\) (with \(\text{meas}(X) > 0\)), we define \(H^1_\Sigma(\Omega)\) the subspace of \(H^1(\Omega)\) with vanishing trace on \(X\). The scalar product in \(L^2(\Omega)\) is denoted by \((\cdot, \cdot)_\Omega\) and its associated norm by \(|\cdot|_{0,\Omega}\).

We consider a family of triangulations \(\{T_{h,k}\}_{0 < h \leq 1}\) of the domain \(\Omega\) satisfying the usual requirements of finite element approximations (see, e.g., [7]). The subscript \(h \in (0,1]\) refers to the level of refinement of the triangulations. In order to ease the presentation, we assume that the family of triangulations is quasi-uniform. For the discretization in space of (2.1), we consider continuous Lagrange finite element approximations \(V_h\) and \(R_h\) of \([H^1(\Omega_l)]^3\) and \(H^1(\Omega_l)\), respectively. Other choices of approximation spaces are possible for the projection method (see [17] for a discussion). For a given \(X \subset \partial \Omega\) (with \(\text{meas}(X) > 0\)), we set

\[V_{X,h} \equiv V_h \cap [H^1_\Sigma(\Omega_l)]^3, \quad R_{X,h} \equiv R_h \cap H^1_\Sigma(\Omega_l).\]

3. Fractional-step time-marching and 3D-0D coupling schemes. In this section, we describe two coupling schemes (explicit and implicit) resulting from appropriate time discretizations of the coupling conditions (2.3).

3.1. Explicit 3D-0D coupling scheme. In this case the 3D-0D coupling conditions (2.3) are time discretized as follows:
for all $l = 1, \ldots, n_{0D}$. Note that the continuity of fluxes (3.3) is treated explicitly by using the flux of the latest computed viscous velocity. For the relation (3.4) we consider a Dirichlet boundary condition for the pressure, while the viscous part of the fluid stresses is set to zero. This is a standard procedure to decouple the projection and viscous steps in the framework of projection schemes with natural boundary conditions (see, e.g., [15] and [16, section 10]).

Remark 3.1. It is well known that the artificial Dirichlet boundary condition (3.1) could lead to suboptimal approximations since, in general, the relation (3.1) is not consistent with the solution of continuous problem. For further details on this issue, we refer to the analysis reported [15], which suggests the use of the rotational version of the pressure-correction scheme.

The resulting fully discrete time-marching procedure is reported in Algorithm 1. In the viscous step (3.2) we have considered the standard Temam’s consistent version of the pressure-correction scheme. This 3D-0D explicit coupling given by (3.1) allows a fully uncoupled computation of the Windkessel state, fluid pressure, and velocity. This is particularly appealing from the implementation and computational efficiency point of view. Nevertheless, as suggested in section 3.1.1 (and then confirmed by numerical experiments in section 5), Algorithm 1 may suffer from stability issues.

**Algorithm 1** (explicit 3D-0D coupling scheme).

Let $u^0 \doteq u_0$, $\mathbf{u}^0 \in V_h$, and $\pi^0_1, \ldots, \pi^0_{n_{0D}} \in \mathbb{R}$ be given initial data. For $n \geq 1$ perform the following:

1. **Viscous step:** Find $\mathbf{u}^n \in V_{\Sigma,h}$ such that

\[
\begin{aligned}
\left\{ \begin{array}{l}
\mathbf{u}^n|_{\Gamma_i} = u_{in}(t^n), \\
\frac{\rho^0}{\tau} (\mathbf{u}^n, \mathbf{v})_{\Omega_t} + \rho^0 (\mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n, \mathbf{v})_{\Omega_t} + \frac{\rho^0}{2} (\nabla \cdot \mathbf{u}^{n-1}) \mathbf{u}^n, \mathbf{v})_{\Omega_t} \\
+ 2\mu (\mathbf{e}(\mathbf{u}^n), \mathbf{e}(\mathbf{v}))_{\Omega_t} = \frac{\rho^0}{\tau} (\mathbf{u}^{n-1}, \mathbf{v})_{\Omega_t}
\end{array} \right.
\end{aligned}
\]

for all $\mathbf{v} \in V_{\Sigma \cup \Gamma^w \cup}$. Thereafter set $Q^n_1 \doteq \int_{\Gamma_1} \mathbf{u}^n \cdot \mathbf{n}_f$.

2. **Windkessel step:** For $l = 1, \ldots, n_{0D}$, compute $(Q^n_1, \pi^n_1, P^n_1) \in \mathbb{R}^3$ from

\[
(3.3) \quad Q^n_1 = Q^n_1, \quad C_{d,l} \partial t \pi^n_1 + \frac{\pi^n_1}{R_{d,l}} = Q^n_1, \quad P^n_1 = R_{p,l} Q^n_1 + \pi^n_1.
\]

3. **Projection step:** Find $p^n \in \mathcal{P}_h$ such that

\[
\begin{aligned}
\left\{ \begin{array}{l}
p^n|_{\Gamma_i} = P^n_1, \quad l = 1, \ldots, n_{0D}, \\
\frac{\tau}{\rho^0} (\nabla p^n, \nabla q)_{\Omega_t} = -(\nabla \cdot \mathbf{u}^n, q)_{\Omega_t}
\end{array} \right.
\end{aligned}
\]

for all $q \in \mathcal{P}_{0D}$, Thereafter set $u^n \doteq u^n - \frac{1}{\rho^0} \nabla p^n \in [L^2(\Omega_t)]^d$. 

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
3.1.1. Stability analysis. Let the quantities

\[ E^n \overset{\text{def}}{=} \frac{\rho_i}{2} \left\| \tilde{u}^n \right\|_{0, \Omega_t}^2 + \sum_{l=1}^{n_{oD}} \frac{C_{d,l}}{2} |\pi_{l}^n|^2, \]

\[ D^n \overset{\text{def}}{=} 2\mu \sum_{m=1}^{n} \tau \| \varepsilon(\tilde{u}^m) \|_{0, \Omega_t}^2 + \sum_{m=1}^{n} \sum_{l=1}^{n_{oD}} \tau \left( \frac{|\pi_{l}^m|^2}{R_{d,l}} + R_{u,l} |Q_{l}^m|^2 \right), \]

for \( n \geq 1 \), denote the energy and physical dissipation of the discrete system. Let us also set

\[ E^0 \overset{\text{def}}{=} \frac{\rho_i}{2} \left\| u^0 \right\|_{0, \Omega_t}^2 + \sum_{l=1}^{n_{oD}} \frac{C_{d,l}}{2} |\pi_{l}^0|^2. \]

We then have the following energy based result (whose proof can be found in [5, Chapter 2]).

**Theorem 3.2.** Let \( \{(\tilde{u}^n, p^n)\}_{n \geq 1} \) and \( \{(Q_{l}^n, \pi_{l}^n, P_{l}^n)_{1 \leq l \leq n_{oD}}\}_{n \geq 1} \) be the solution given by Algorithm 1 and assume that \( u_m = 0 \) (free system). The following inequality holds for \( n \geq 1 \):

\[ E^n + D^n + \sum_{m=1}^{n} \frac{\rho_i}{2} \tau (\tilde{u}^{m-1} \cdot n_l, |\tilde{u}^m|^2)_{\Omega_t} \leq E^0 - \sum_{m=1}^{n-1} \frac{\tau^2}{\rho_i} \left\| \nabla p^m \right\|_{0, \Omega_t}^2 \]
\[ + \sum_{m=1}^{n-1} \tau (\nabla \cdot \tilde{u}^m, \phi^m)_{\Omega_t} + \sum_{m=1}^{n} \frac{\tau^2}{\rho_i} (\nabla p^m, \nabla \phi^m)_{\Omega_t}, \]

where \( \phi^n \in R_h \) is an arbitrary discrete lifting of the (unknown) proximal pressures, namely,

\[ \phi^n = P^n_{l} \quad \text{on} \quad \Gamma_l, \quad l = 1, \ldots, n_{oD}. \]

The left-hand side of estimate (3.5) corresponds to the discrete counterpart of (2.5). Nevertheless, the artificial power introduced by the last two terms of (3.5),

\[ \sum_{m=1}^{n-1} \tau (\nabla \cdot \tilde{u}^m, \phi^m)_{\Omega_t} + \sum_{m=1}^{n} \frac{\tau^2}{\rho_i} (\nabla p^m, \nabla \phi^m)_{\Omega_t}, \]

cannot be controlled, so that this estimate does not guarantee the energy stability of the approximations provided by Algorithm 1. Two remarks are now in order.

**Remark 3.3.** It is worth mentioning that the term (3.7) corresponds to the residual of the projection step (3.4) (note that \( \phi^n \notin R_{\Gamma_{out,3}} \)). In fact, for the space continuous counterpart of (3.4) we have

\[(\nabla \cdot \tilde{u}^n, \phi^n)_{\Omega_t} + \frac{\tau}{\rho_i} (\nabla p^n, \nabla \phi^n)_{\Omega_t} = \frac{\tau}{\rho_i} \int_{\Gamma_{out}} \frac{\partial p^n}{\partial n_l} \phi^n_{l} + \frac{\tau}{\rho_i} \int_{\Gamma_{out}} \frac{\partial p^n}{\partial n_l} \phi^n_{l} \]
\[ = - \sum_{l=1}^{n_{oD}} \left( \int_{\Gamma_l} \tilde{u}^n \cdot n_l - \int_{\Gamma_l} \tilde{u}^n \cdot n_l \right) P^n_{l}. \]

Hence, the uncontrolled artificial power involved in the energy estimate (3.5) is due to the time lag in the flux (\( \int_{\Gamma_l} \tilde{u}^n \cdot n_l \)) instead of \( \int_{\Gamma_l} u^\kappa \cdot n_l \) introduced by the explicit
treatment of the continuity of fluxes (3.1) on the 3D-0D interfaces \( \Gamma_l \). It should be noted that a 3D-0D explicit coupling with a monolithic time-marching scheme in the fluid yields a similar time lag, but now with \( \int_{\Gamma_l} \mathbf{u}^{n-1} \cdot \mathbf{n} \) instead of \( \int_{\Gamma_l} \mathbf{u}^n \cdot \mathbf{n} \).

Remark 3.4. In the case of a single outlet (i.e., \( n_{0D} = 1 \)), we can take \( \phi^n = P^n \) in \( \Omega_l \). From the proof of Proposition 3.2 (see [5, Chapter 2]) one then recovers the following energy estimate for the fluid:

\[
\frac{\rho_l}{2} \| \tilde{\mathbf{u}}^n \|_{0,\Omega_l}^2 + 2\mu \sum_{m=1}^n \tau \| \mathbf{e}(\tilde{\mathbf{u}}^m) \|_{0,\Omega_l}^2 + \sum_{m=1}^n \frac{\rho_l}{2} \tau (\tilde{\mathbf{u}}^{m-1} \cdot \mathbf{n}_l, |\tilde{\mathbf{u}}^m|^2)_{\Gamma_{out}} \\
\leq \frac{\rho_l}{2} \| \mathbf{u}^0 \|_{0,\Omega_l}^2 - \sum_{m=1}^{n-1} \frac{\tau^2}{2\rho_l} \| \nabla p^m \|_{0,\Omega_l}^2.
\]

As a result, the energy stability of the fluid does not depend on the imposed outlet pressure.

3.2. Implicit 3D-0D coupling scheme. In this case the 3D-0D coupling conditions (2.3) are time discretized as follows:

\[
\begin{aligned}
Q^n_l &= \int_{\Gamma_l} \mathbf{u}^n \cdot \mathbf{n}_l, \\
p^n &= P^n_l \quad \text{on} \quad \Gamma_l, \\
2\mu\mathbf{e}((\tilde{\mathbf{u}}^n))\mathbf{n}_l &= 0 \quad \text{on} \quad \Gamma_l
\end{aligned}
\]

for \( l = 1, \ldots, n_{0D} \). Note that in comparison with (3.1), the above coupling scheme treats implicitly the continuity of fluxes on the outlet boundaries. This feature enhances stability, as we will show in section 3.2.1. However, from the computational point of view, the relations (3.8) and (2.10) apparently couple the evaluation of (2.8) and (2.9). Fortunately, this difficulty can be circumvented via an appropriate reformulation of the pressure boundary condition (3.8) for the projection step (2.8). Indeed, by inserting (2.9) into (3.8), we get

\[
Q^n_l = \tilde{Q}^n_l - \frac{\tau}{\rho_l} \int_{\Gamma_l} \frac{\partial p^n}{\partial n_l},
\]

which with (2.11) and (3.8) yields the following (implicit) boundary condition for the outlet pressures:

\[
p^n|_{\Gamma_l} = \gamma_l \tilde{Q}^n_l - \frac{\gamma_l \tau}{\rho_l} \int_{\Gamma_l} \frac{\partial p^n}{\partial n_l} + \alpha_l \tilde{\sigma}^{n-1}_l
\]

for \( l = 1, \ldots, n_{0D} \). Note that this expression still enforces \( p^n \) as a constant on each \( \Gamma_l \).

Multiplying (2.8) by \( q \in R_h \), integrating by parts, using (2.8), and the fact that \( q|_{\Gamma_l} \) is constant, we get

\[
\frac{\tau}{\rho_l} (\nabla p^n, \nabla q)_{\Omega_l} - \frac{\tau}{\rho_l} \sum_{i=1}^{n_{0D}} \left( \int_{\Gamma_l} \frac{\partial p^n}{\partial n_l} \right) q|_{\Gamma_l} = - (\nabla \cdot \tilde{\mathbf{u}}^n, q)_{\Omega_l}
\]

for all \( q \in R_h \). We can eliminate the normal derivative of the pressure using (3.10), which yields the following modified variational formulation for the projection step:
Find $p^n \in R_h$ such that

\[
\frac{\tau}{\rho_l} (\nabla p^n, \nabla q)_{\Omega_l} + \sum_{l=1}^{\text{nD}} \left( \frac{p^n|_{\Gamma_l}}{\gamma_l} \right) (q|_{\Gamma_l}) = \sum_{l=1}^{\text{nD}} \left( \tilde{Q}^n_l + \frac{\alpha_l \pi^n_{l-1}}{\gamma_l} \right) q|_{\Gamma_l} - (\nabla \cdot \tilde{u}^n, q)_{\Omega_l}
\]

for all $q \in R_h$. We can then set $P^n_l = p^n|_{\Gamma_l}$ and retrieve $(Q^n_l, \pi^n_l)$ from (3.11) for $l = 1, \ldots, n_{\text{nD}}$.

**Remark 3.5.** The well-posedness of the pressure-Poisson problem (3.11) follows from a generalized Poincaré inequality, which guarantees the coercivity of the left-hand side of (3.11) in $R_h$.

**Remark 3.6.** Testing (3.11) with $q = 1$, and since $P^n_l = p^n|_{\Gamma_l}$, we have

\[
\sum_{l=1}^{\text{nD}} \frac{P^n_l - \alpha_l \pi^n_{l-1}}{\gamma_l} = \sum_{l=1}^{\text{nD}} \tilde{Q}^n_l + \int_{\Omega_l} \nabla \cdot \tilde{u}^n.
\]

Hence, integrating by parts in the last term, using (2.11), and owing to (2.6)$_{2,3}$ we get the following mass conservation for the Windkessel fluxes:

\[
\sum_{l=1}^{\text{nD}} Q^n_l = - \int_{t^n} u_{\text{in}}(t_n).
\]

The complete time-marching procedure is reported in Algorithm 2.

**Algorithm 2 (3D-0D implicit coupling scheme).**

Let $u^0 = u_0$, $\tilde{u}^0 \in V_h$, and $\pi^0_1, \ldots, \pi^0_{n_{\text{nD}}} \in \mathbb{R}$ be given initial data. For $n \geq 1$, perform the following:

1. **Viscous step:** Find $\tilde{u}^n \in V_{\Sigma,h}$ such that

   \[
   \left\{
   \begin{aligned}
   &\tilde{u}^n|_{\Gamma_{\infty}} = u_{\text{in}}(t_n), \\
   &\frac{\rho_l}{\tau} (\tilde{u}^n, \nu)_{\Omega_l} + \rho_l (\tilde{u}^{n-1} \cdot \nabla \tilde{u}^n, \nu)_{\Omega_l} + 2\mu (\tilde{u}^n, \tilde{\nu}(\tilde{u}^n, \nu))_{\Omega_l} \\
   &+ \frac{\rho_l}{\tau} (\tilde{u}^{n-1}, \nu)_{\Omega_l}
   \end{aligned}
   \right.
   \]

   for all $\nu \in V_{\Sigma,\Gamma_{\infty}}$. Thereafter set $\tilde{Q}^n_l \overset{\text{def}}{=} \int_{\Gamma_l} \tilde{u}^n \cdot \nu_l$.

2. **Projection-Windkessel step:** Find $p^n \in R_h$ and such that

   \[
   \frac{\tau}{\rho_l} (\nabla p^n, \nabla q)_{\Omega_l} + \sum_{l=1}^{\text{nD}} \left( \frac{p^n|_{\Gamma_l}}{\gamma_l} \right) (q|_{\Gamma_l}) = \sum_{l=1}^{\text{nD}} \left( \tilde{Q}^n_l + \frac{\alpha_l \pi^{n-1}_l}{\gamma_l} \right) q|_{\Gamma_l} - (\nabla \cdot \tilde{u}^n, q)_{\Omega_l}
   \]

   for all $q \in R_h$. Thereafter, set $P^n_l = p^n|_{\Gamma_l}$ and compute $(Q^n_l, \pi^n_l) \in \mathbb{R}^2$ from the relations

   \[
   Q^n_l = \frac{P^n_l - \alpha_l \pi^{n-1}_l}{\gamma_l}, \quad \pi^n_l = \alpha_l \pi^{n-1}_l + \beta_l Q^n_l, \quad l = 1, \ldots, n_{\text{nD}},
   \]

   and set $u^n \overset{\text{def}}{=} \tilde{u}^n - \frac{\tau}{\rho_l} \nabla p^n \in [L^2(\Omega_t)]^d$.

**3.2.1. Stability analysis.** The focus of this section is to present the stability result of the formulation (3.12)–(3.14) summarized in the following proposition.
THEOREM 3.7. Let \( \{ (\tilde{u}^n, p^n) \}_{n \geq 1} \) and \( \{ (Q^l_t, \pi^l_t, P^l_t)_{1 \leq l \leq n_{\text{abs}}} \}_{n \geq 1} \) be the approximated solution given by Algorithm 2 and assume that \( u_{\text{in}} = 0 \) (free system). The following energy inequality holds:

\[
E^n + D^n + \sum_{m=1}^{n} \frac{\rho_t}{2} (\tilde{u}^{m-1} \cdot \mathbf{n}_t, |\tilde{u}^m|^2)_{\Gamma_{\text{out}}} \leq E^0 - \sum_{m=1}^{n-1} \frac{\tau}{2\rho_t} \| \nabla p^m \|^2_{0, \Omega_t}.
\]

Proof. We first test (3.12) with \( v = \tilde{u}^n \) and integrate by parts the convective term. This yields the identity

\[
\frac{\rho_t}{2} \partial_t \| \tilde{u}^n \|^2_{0, \Omega_t} + \frac{\rho_t}{2\tau} \| \tilde{u}^n - \tilde{u}^{n-1} \|^2_{0, \Omega_t} + 2\mu \| \epsilon(\tilde{u}^n) \|^2_{0, \Omega_t} + (\nabla p^{n-1}, \tilde{u}^n)_{\Omega_t} + \frac{\rho_t}{2} (\tilde{u}^{n-1} \cdot \mathbf{n}_t, |\tilde{u}^n|^2)_{\Gamma_{\text{out}}} = 0
\]

for \( n \geq 2 \) and, for \( n = 1 \), we get

\[
\frac{\rho_t}{2\tau} \left( \| \tilde{u}^1 \|^2_{0, \Omega_t} - \| \tilde{u}^0 \|^2_{0, \Omega_t} \right) + \frac{\rho_t}{2\tau} \| \tilde{u}^1 - \tilde{u}^0 \|^2_{0, \Omega_t} + 2\mu \| \epsilon(\tilde{u}^1) \|^2_{0, \Omega_t} + \frac{\rho_t}{2} (\tilde{u}^0 \cdot \mathbf{n}_t, |\tilde{u}^1|^2)_{\Gamma_{\text{out}}} = 0.
\]

Thereafter, taking (3.13) at time step \( n - 1 \), testing with \( q = p^{n-1} \), and integrating by parts in its right-hand side yields

\[
\frac{\tau}{\rho_t} \| \nabla p^{n-1} \|^2_{0, \Omega_t} + \sum_{l=1}^{n_{\text{abs}}} \frac{P^{n-1}_l - \alpha_l \pi^{n-2}_l}{\gamma_l} P^{n-1}_l = (\tilde{u}^{n-1}, \nabla p^{n-1})_{\Omega_t}
\]

for \( n \geq 2 \). Hence, the addition and subtraction of suitable terms and the application of the Cauchy–Schwarz and arithmetic-geometric inequalities yields

\[
\frac{\tau}{\rho_t} \| \nabla p^{n-1} \|^2_{0, \Omega_t} + \sum_{l=1}^{n_{\text{abs}}} \frac{P^{n-1}_l - \alpha_l \pi^{n-2}_l}{\gamma_l} P^{n-1}_l \leq 0.
\]

As a result, the summation of (3.16) and (3.18) gives

\[
\frac{\rho_t}{2} \partial_t \| \tilde{u}^n \|^2_{0, \Omega_t} + 2\mu \| \epsilon(\tilde{u}^n) \|^2_{0, \Omega_t} + \frac{\rho_t}{2} (\tilde{u}^{n-1} \cdot \mathbf{n}_t, |\tilde{u}^n|^2)_{\Gamma_{\text{out}}} + \sum_{l=1}^{n_{\text{abs}}} \frac{P^{n-1}_l - \alpha_l \pi^{n-2}_l}{\gamma_l} P^{n-1}_l \leq -\frac{\tau}{2\rho_t} \| \nabla p^{n-1} \|^2_{0, \Omega_t}
\]

for \( n \geq 2 \). At last, from (3.14) and its equivalence to (2.11), we have

\[
\frac{P^n_l - \alpha_l \pi^{n-1}_l}{\gamma_l} P^n_l = Q^n_l P^n_l = R_{p,l}|Q^n_l|^2 + Q^n_l \pi^n_l \geq \frac{C_{d,l}}{2} \partial_t |\pi^n_l|^2 + \frac{1}{R_{d,l}} |\pi^n_l|^2 + R_{p,l}|Q^n_l|^2
\]
for \( n \geq 1 \). Hence, by inserting the last inequality of (3.20) into (3.19), multiplying by \( \tau \), and summing over \( m = 2, \ldots, n \) we get the estimate

\[
E^n + D^n + \sum_{m=2}^{n} \frac{\theta_t}{2} (\tilde{u}^{m-1} \cdot \tilde{n}_t, |\tilde{u}^m|^2)_{\Omega \setminus \Gamma_{\text{out}}} \leq E^1 - \sum_{m=1}^{n-1} \frac{\tau^2}{2\rho_t} |\nabla p^m|^2_{0, \Omega_t}
\]

for \( n \geq 2 \). We recover the estimate (3.15) by simply adding to this inequality the expression (3.17) multiplied by \( \tau \), which completes the proof.

The estimate (3.15) corresponds to the discrete counterpart of (2.5). Note that the right-hand side of (3.15) is a pure numerical dissipation term (the natural pressure stabilization). Therefore, the 3D-0D coupling reported in Algorithm 2 does not introduce any uncontrolled artificial power and, hence, a guarantee of numerical stability. This feature will be illustrated in section 5 via numerical experiments.

### 3.2.2. Implementation details

In this section we discuss the implementation of the pressure problem (3.13) in a finite element framework. For simplicity, and without loss of generality, we limit the discussion to the case of a single outlet. We define the arrays \( P, V \in \mathbb{R}^N \) corresponding to the degrees of freedom (d.o.f.) of the pressure \( p^n \in Q_h \) and of a general test function \( q \in Q_h \), respectively. The bilinear form \( (\nabla p^n, \nabla q)_{\Omega_t} \), without imposing Dirichlet boundary conditions to \( p^n \), can be written in matrix form as

\[
(\nabla p^n, \nabla q)_{\Omega_t} = V^T A P = [V^T_I \quad V^T_O] \begin{bmatrix} A_{II} & A_{IO} \\ A_{OI} & A_{OO} \end{bmatrix} \begin{bmatrix} P_I \\ P_O \end{bmatrix},
\]

where the subindexes \( O \) and \( I \) indicate the elements of the array corresponding to the d.o.f. on \( \Gamma_{\text{out}} \) and \( \Omega\setminus\Gamma_{\text{out}} \), respectively. Hence, the pressure-Poisson projection step with explicit Dirichlet data can be formulated as

\[
A_{II} P_I = \tilde{F}_I - A_{IO} P^n_{\Omega \setminus \Gamma_{\text{out}}},
\]

where the notation in right-hand side is such that

\[
[V^T_I \quad V^T_O] \begin{bmatrix} \tilde{F}_I \\ \tilde{F}_O \end{bmatrix} = \frac{\rho_t}{\tau} (\nabla \cdot \bar{u}^n, q)_{\Omega_t}
\]

with \( I_O \in \mathbb{R}^{N_O} \) denoting a vector of ones and with \( N_O \) being the number of pressure d.o.f. on \( \Gamma_{\text{out}} \). The linear system (3.21) is usually solved by means of a preconditioned conjugate gradient method (PCG) with the preconditioning operator \( A^{-1}_{II} \) given, for example, by an incomplete Cholesky factorization of \( A_{II} \). (Alternative preconditioners could be used.) With the notations introduced above, the stiffness matrix of the implicit formulation (3.13) can be derived straightforwardly. Indeed, since \( V_O = \mathbb{1}_O q|_{\Gamma_{\text{out}}} \), and \( P_O = \mathbb{1}_O P^n_{\Omega \setminus \Gamma_{\text{out}}} \), we obtain

\[
\begin{bmatrix} A_{II} & \mathbf{a} \\ \mathbf{a}^T & b \end{bmatrix} \begin{bmatrix} P_I \\ P^n_{\Omega \setminus \Gamma_{\text{out}}} \end{bmatrix} = \begin{bmatrix} \mathbb{1}_O^T \tilde{F}_I + \frac{\rho_t}{\tau} (\bar{Q}^n + \frac{\alpha^n n^{-1}}{\gamma}) \\ \mathbb{1}_O^T \tilde{F}_O \end{bmatrix}
\]

with \( \mathbf{a} \stackrel{\text{def}}{=} A_{IO} \mathbb{1}_O \) and \( b \stackrel{\text{def}}{=} \mathbb{1}_O^T A_{OO} \mathbb{1}_O + \rho_t/\gamma \). It should be noted that the matrix in (3.22) has a nonstandard sparsity pattern. However, since we use a Krylov linear
solver this matrix is never assembled in practice (only matrix-vector products are evaluated). In the numerical experiments, we considered the block-preconditioner given by

\[
\begin{bmatrix}
\tilde{A}^{-1}_{II} & 0 \\
0 & \frac{1}{\tau}
\end{bmatrix},
\]

which yielded practically the same number of PCG iterations as in the solution of (3.21). In the general case of a domain \( \Omega_t \) with multiple outlets, the aforementioned considerations can be extended by considering one additional equation for each 3D-0D interface. Concerning the computational cost, in our numerical simulations we did not observe any relevant difference between the implicit and the explicit coupling.

### 3.2.3. Extension to higher-order time-splitting schemes

Although widely used in practice, the original Chorin–Temam projection scheme might exhibit limited accuracy in time and through spurious boundary layers, due to the unphysical homogenous Neumann boundary condition (2.8)\(_\delta\). Among the several variants available (see, e.g., [16] for an overview), in this section we describe a possible extension of the implicit 3D-0D coupling (Algorithm 2) in the context of an incremental pressure projection scheme with a second-order time discretization.

Following [16], we decompose the time iteration in a BDF2 time discretization for the viscous step

\[
\begin{aligned}
\left\{ \begin{array}{l}
\rho_i \frac{3\tilde{u}^n - 4u^{n-1} + u^{n-2}}{2\tau} + \rho_i \tilde{u}^{n-1} \cdot \nabla \tilde{u}^n - 2\mu \nabla \cdot \epsilon(\tilde{u}^n) + \nabla p^{n-1} = 0 \quad \text{in} \quad \Omega_t, \\
\tilde{u}^n = u_{in}(t_n) \quad \text{on} \quad \Gamma_{in}, \\
\tilde{u}^n = 0 \quad \text{on} \quad \Sigma,
\end{array} \right.
\end{aligned}
\]

a projection step for the increment of pressure

\[
\begin{aligned}
\left\{ \begin{array}{l}
-\frac{\tau}{\rho_i} \Delta p^n = -\frac{3}{2} \nabla \cdot \tilde{u}^n \quad \text{in} \quad \Omega_t, \\
\frac{\tau}{\rho_i} \frac{\partial p^n}{\partial n_l} = 0 \quad \text{on} \quad \Gamma_{in} \cup \Sigma, \\
p^n = P^n_{l} - P_{l}^{n-1} \quad \text{on} \quad \Gamma_l, \quad l = 1, \ldots, n_{0D},
\end{array} \right.
\end{aligned}
\]

(3.23)

completed by the end-of-step updates: \( p^n = p^{n-1} + \delta p^n \) and \( u^n = \tilde{u}^n - \frac{2\tau}{3\rho_i} \nabla \delta p^n \).

To have a second-order time discretization for the whole 3D-0D problem, we also discretize the 0D model with a BDF2 scheme, namely,

\[
\pi_l^n = \hat{\alpha}_l \pi_l^{n-1} - \frac{\hat{\alpha}_l}{4} \pi_l^{n-2} + \hat{\beta}_l Q_l^n, \quad P_l^n = \hat{\gamma}_l Q_l^n + \hat{\alpha}_l \pi_l^{n-1} - \frac{\hat{\alpha}_l}{4} \pi_l^{n-2}
\]

(3.24)

with \( \hat{\alpha}_l = \frac{2R_{l,t}C_{l,t}}{(3/2)R_{l,t}C_{l,t} + R_{l,t}}, \hat{\beta}_l = \frac{R_{l,t}}{(3/2)R_{l,t}C_{l,t} + R_{l,t}}, \hat{\gamma}_l = R_{p,t} + \hat{\beta}_l \).

Hence, using (3.24)\(_2\) to define the implicit coupling with the projection step, we obtain the following time-stepping method:
1. Viscous step: Find \( \tilde{u}^n \in V_{\Sigma,h} \) such that
\[
\begin{aligned}
\tilde{u}^n|_{\Gamma^n} &= u_{\text{in}}(t_n), \\
\frac{\rho_{\ell}}{2\tau} (3\tilde{u}^n, v)_{\Omega_{\ell}} + \rho_{\ell}(\tilde{\nabla}n \cdot \nabla \tilde{u}^n, v)_{\Omega_{\ell}} &+ 2\mu (e(\tilde{u}^n), e(v))_{\Omega_{\ell}} \\
&\quad + \frac{\rho_{\ell}}{2}((\nabla \cdot \tilde{u}^n)\tilde{u}^n, v)_{\Omega_{\ell}} - (\nabla p^n, v) + \frac{\rho_{\ell}}{2\tau} (4u^{n-1} - u^n, v)_{\Omega_{\ell}}
\end{aligned}
\]
for all \( v \in V_{\Sigma,\mathcal{T}_{h-\mathcal{A}}^\infty,h} \).
Thereafter set \( \tilde{Q}_l^n = \int_{\Gamma} \tilde{u}^n \cdot n_l \).

2. Increment-Windkessel step: Find \( \delta p^n \in R_h \) such that
\[
\frac{\tau}{\rho_l} (\nabla \delta p^n, \nabla q)_{\Omega_{\ell}} + \sum_{l=1}^{n_{\text{OD}}} \frac{(\delta p^n)_{\Gamma_l}}{\gamma_l} (q|_{\Gamma_l}) = 0
\]
for all \( q \in R_h \).

3. End of step: Set \( p^n = p^{n-1} + \delta p^n, u^n = \tilde{u}^n - \frac{2\tau}{3\rho_l} \nabla \delta p^n \in [L^2(\Omega_{\ell})]^d, \)
\( P^n_{\ell} = p^n|_{\Gamma_{\ell}}, \) and compute \( (Q^n, \pi^n) \in \mathbb{R}^{2} \) from the relations (3.14).

Remark 3.8. It is well known that the stability of the pressure-incremental projection scheme requires that the finite element spaces for velocity and pressure satisfy an inf-sup condition. (See [16] for an extended discussion.)

### 3.2.4. Extension to more complex lumped parameter models

The algorithm and analysis presented above can be straightforwardly extended to more complex networks of lumped parameter models, made of resistances, capacitances, and inductances. These networks can be obtained, for instance, by connecting several Windkessel elements (see, e.g., [18, 20, 23] and the references therein).

Let us consider the general lumped parameter model

\[
\begin{aligned}
C \frac{dP}{dt} + RP &= Q + H\Psi, \\
L \frac{d\Psi}{dt} &= -H^T P
\end{aligned}
\]
with
\[
P^T \overset{\text{def}}{=} \left[ P_1 \ldots P_{n_{\text{OD}}} \pi_1 \ldots \pi_{n_{\pi}} \right],
\]
the collection of outlet \( P_l, l = 1, \ldots, n_{\text{OD}}, \) and distal pressures \( \pi_j, j = 1, \ldots, n_{\pi}, \) and where \( \Psi \) are the fluxes through the inductances, both representing the dynamical state of the lumped parameter model. Moreover, we assume that \( Q(t) \in \mathbb{R}^{n_{\pi}} \) has the form
\[
Q^T \overset{\text{def}}{=} \left[ Q_1 \ldots Q_{n_{\text{OD}}} 0 \ldots 0 \right]
\]
with \( Q_l \) the input flux at the outlet \( \Gamma_l \).

In (3.25), \( C, R, L \) denote symmetric capacitances, resistances, and inductances matrices. This ensures the correct energy balance of the system in the 3D-0D time-space continuous formulation, namely,
\[
\frac{d}{dt} \left( \frac{1}{2} P^T CP + \frac{1}{2} \Psi^T L \Psi \right) = -P^T RP + P^T Q,
\]
since that the last term of the right-hand side cancels out when coupling (3.25) with (2.1). As an example, the three-element Windkessel presented above, with an additional inductance \( L_p \) parallel to \( R_p \), can be rewritten in this format, obtaining the following expressions for the system matrices:

\[
C = \begin{bmatrix} 0 & 0 \\ 0 & C_d \end{bmatrix}, \quad R = \begin{bmatrix} 1/R_p & -1/R_p \\ -1/R_p & 1/R_p + 1/R_d \end{bmatrix}, \quad H = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad L = L_p.
\]

More complex models can be represented in this format analogously.

Discretizing (3.25) in time (e.g., using a backward Euler scheme), we obtain

\[
P^n = A P^{n-1} + B Q^n + B H \Psi^{n-1}
\]

with

\[
B \overset{\text{def}}{=} (C/\tau + R)^{-1}, \quad A \overset{\text{def}}{=} B(C/\tau + \tau H L^{-1} H^\top).
\]

In order to derive the 3D-0D implicit coupling scheme (Algorithm 2), we need explicit expressions for \( Q^n_l, l = 1, \ldots, n_{0D} \). These can be computed algebraically from (3.26) by first isolating the d.o.f. on the outlets via

\[
\tilde{B} \begin{bmatrix} Q^n_1 \\ \vdots \\ Q^n_{n_{0D}} \end{bmatrix} = \begin{bmatrix} P^n_1 \\ \vdots \\ P^n_{n_{0D}} \end{bmatrix} - \tilde{A} P^{n-1} - \tilde{B} H \Psi^{n-1}
\]

with \( \tilde{()} \) denoting the first \( n_{0D} \) rows and \( \tilde{()} \) the first \( n_{0D} \) rows and columns, respectively. Hence, we obtain the expression

\[
\begin{bmatrix} Q^n_1 \\ \vdots \\ Q^n_{n_{0D}} \end{bmatrix} = G \begin{bmatrix} P^n_1 \\ \vdots \\ P^n_{n_{0D}} \end{bmatrix} - G \tilde{A} P^{n-1} - G \tilde{B} H \Psi^{n-1}
\]

with \( G = (\tilde{B})^{-1} \). Finally, by combining the latter with (3.9), we obtain the following formulation of the pressure projection-Windkessel step (compare with (3.13)): Find \( p^n \in R_h \) such that

\[
\frac{\tau}{\rho_l} (\nabla p^n, \nabla q)_{\Omega_t} + \sum_{k,l=1}^{n_{0D}} g_{k,l}(p^n | r_k)(q | r_l)
\]

\[
= \sum_{l=1}^{n_{0D}}(\tilde{Q}_l^n + b_l^{n-1} q | r_l)(q | r_l) - (\nabla \cdot \tilde{u}_l^n, q)_{\Omega_t}
\]

for all \( q \in R_h \) and with \( g_{k,l} \overset{\text{def}}{=} [G]_{k,l}, b_l^n \overset{\text{def}}{=} [G \tilde{A} P^n + G \tilde{B} H \Psi^n]_l \). Note that in this general formulation, the pressure at the outlets can be coupled through the lumped parameter model since generally \( g_{k,l} \neq 0 \) for \( k \neq l \).

Remark 3.9. Using the same arguments as in section 3.2.1, the unconditional stability results of Theorem 3.7 can be extended to a fractional-step scheme with the generalized pressure projection formulation (3.27).

4. Incompressible FSI. Fractional-step time-marching schemes have been a valuable tool for the design of efficient solution methods for incompressible FSI.
problems, yielding the projection semi-implicit coupling schemes \[1, 10\]. This coupling approach is based on the following three basic ideas:

- Treat explicitly the geometrical nonlinearities and the viscous-structure coupling, which reduces computational complexity.
- Treat implicitly the pressure-structure coupling, which avoids numerical instability.
- Perform this explicit-implicit splitting through a projection scheme in the fluid.

So far, the stability of this method has been analyzed within a simplified framework which enforces null pressure on the outlet boundaries (see \[1, 10\]). In this section the analysis is extended to the case of a lumped parameter modeling of the outlet boundaries, with a pressure-Poisson formulation of the projection step, based on the methods of section 3.

### 4.1. Model problem

For the analysis (see \[10\]), we consider as a model problem a coupled linear system in which the fluid is described by the Stokes equations, in the fixed domain \(\Omega_f\), and the structure either by the classical linear elastodynamics equations or by equations based on linear thin-solid models (e.g., plate, shell). The reference domain of the solid is denoted by \(\Omega_s\). It will be either a domain or a 2-manifold of \(\mathbb{R}^3\). (In this later case in which the elastic domain is identified to its midsurface.) We denote by \(\Sigma \equiv \partial \Omega_s \cap \partial \Omega_f\) the FSI. In the case in which the structure is described by a thin-solid model we have \(\Sigma_s = \Sigma\) (see Figure 4.1). The resulting coupled system, describing the fluid velocity \(\mathbf{u} : \Omega_f \times \mathbb{R}^+ \to \mathbb{R}^3\), fluid pressure \(p : \Omega_f \times \mathbb{R}^+ \to \mathbb{R}\), and solid displacement \(\mathbf{y} : \Omega_s \times \mathbb{R}^+ \to \mathbb{R}^3\), is given by

\[
\begin{align*}
\rho_f & \frac{\partial t}{\partial t} \mathbf{u} - \nabla \cdot \sigma(\mathbf{u}, p) = 0 & \text{in } \Omega_f, \\
\nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega_f, \\
\mathbf{u} &= \mathbf{u}_{in} & \text{on } \Gamma_{in}, \\
\mathbf{u} &= \mathbf{y} & \text{in } \Sigma, \\
\end{align*}
\]

\[
\begin{align*}
\rho_s \left( \partial_t \dot{\mathbf{y}}, \mathbf{v}_s \right)_{\Omega_s} + a_s(\mathbf{y}, \mathbf{v}_s) &= - \left( \sigma(\mathbf{u}, p) n_f, \mathbf{v}_s \right)_\Sigma & \text{for all } \mathbf{v}_s \in \mathbf{W}, \\
\dot{\mathbf{y}} &= \partial_t \mathbf{y} & \text{in } \Omega_s,
\end{align*}
\]

completed with the lumped-parameter modeling (2.2)–(2.3) on the outlet boundary \(\Gamma_{out}\) (see Figure 4.1). Here, \(\rho_s\) denotes the solid density, the abstract bilinear form \(a_s : \mathbf{W} \times \mathbf{W} \to \mathbb{R}\) describes the elastic behavior of the structure, and \(\mathbf{W}\) stands

![Figure 4.1. Examples of geometric configuration \(\Pi_s \neq \Sigma\) (\(\Pi_s = \Sigma\); see Figure 2.1).](image-url)
for its space of admissible displacements. It should be noted that the solid problem (4.2) has been written in weak form, which allows us to treat in the same formulation the case of thin and thick solid models. The relations (4.1) and (4.2) enforce the kinematic and kinetic interface coupling conditions, respectively. Note that the latter also represents the variational formulation of the structure subproblem. Though simplified, problem (4.1)–(4.2) features some of the main numerical issues that appear in complex nonlinear FSI problems involving an incompressible fluid (see, e.g., [8]).

Let the quantity $E(t) \overset{\text{def}}{=} \frac{\alpha}{2} \| u \|_{0, \Omega}^2 + \frac{\beta}{2} \| y \|_{0, \Omega}^2 + \frac{\gamma}{2} \| y \|^2_{l, \Omega} + \sum_{t=1}^{\tau_{\text{top}}} C_{\text{ani}} \pi_t^2$ denote the total (kinetic + potential) energy of the FSI-0D coupled system given by (4.1)–(4.2) and (2.2)–(2.3). Here, $\| \cdot \|_s$ stands for the elastic energy norm of the solid, defined as $\| y \|^2_s \overset{\text{def}}{=} a_s(y, y)$. Assuming that $u_{ih} = 0$ (free system) and using a standard energy argument, we get the identity

$$E(t) + D(t) = E(0)$$

with the dissipative term $D(t) > 0$ given as in (2.4).

Note that in this model problem we do not consider the convective term in the fluid, hence we have to solve the fluid in a fixed domain in order to get the right energy balance across the FSI $\Sigma$.

### 4.1.2. Semi-implicit FSI scheme with implicit 3D-0D coupling.

In this case we consider numerical approximations of the coupled problem FSI-0D system (4.1)–(4.2) and (2.2)–(2.3) by combining the projection semi-implicit coupling scheme reported in [10] with the 3D-0D implicit coupling of section 3.2. The resulting fully discrete time-marching procedure is reported in Algorithm 3. In the solid subproblem (4.9), the fluid residual terms are given by

$$\langle R_\mu (\bar{u}^n), v \rangle \overset{\text{def}}{=} \frac{\rho_t}{\tau} \langle \bar{u}^n, v \rangle_{\Omega_t} + 2\mu (\epsilon(\bar{u}^n), \epsilon(v))_{\Omega_t} - \frac{\rho_t}{\tau} \langle u^{n-1}, v \rangle_{\Omega_t},$$

$$\langle R_p (p^n, u^n), v \rangle \overset{\text{def}}{=} \frac{\rho_t}{\tau} \langle u^n, v \rangle_{\Omega_t} - \frac{\rho_t}{\tau} \langle \bar{u}^n, v \rangle_{\Omega_t} - \langle p^n, \nabla \cdot v \rangle_{\Omega_t}.$$
Algorithm 3 (semi-implicit FSI scheme with implicit 3D-0D coupling).

Let \( u^0 = u_0, n_{1,0}, \ldots, n_{n_{0D}} \in \mathbb{R} \) and \( y^0, \dot{y}^0 \in W_h \) be given initial data. For \( n \geq 1 \) perform the following:

1. Viscous step: Find \( \tilde{u}^n \in V_h \) such that
   \[
   \begin{aligned}
   \tilde{u}^n|_\Sigma &= \dot{y}^{n-1}|_\Sigma, \\
   \tilde{u}^n|_{\Gamma^n} &= u_{in}(t_n), \\
   \frac{\rho_f}{\tau} (\tilde{u}^n, v)_\Omega + 2\mu (\tilde{e}(\tilde{u}^n), \tilde{e}(v))_\Omega &= \frac{\rho_f}{\tau} (u^{n-1}, v)_\Omega,
   \end{aligned}
   \]
   for all \( v \in V_{\Sigma,\Omega_{\Gamma^n}} \). Thereafter set \( \tilde{Q}^n_t \equiv \int_{\Omega_l} \tilde{u}^n \cdot n_l \).

2. Implicit step (projection-Windkessel-solid step): Find \( (p^n, u^n, y^n) \in R_h \times V_h \times W_h \) with \( \dot{y}^n = \partial_y y^n \) and such that
   \[
   \begin{aligned}
   \frac{\tau}{\rho_f} (\nabla p^n, \nabla q)_\Omega + \sum_{l=1}^{n_{0D}} \left( \frac{\rho_f}{\gamma_l} | \dot{y}^n |_{\Gamma_l} \right) \left( q |_{\Gamma_l} \right) &= \sum_{l=1}^{n_{0D}} \left( \tilde{Q}^n_l + \frac{\alpha_l}{\gamma_l} n_{\Gamma_l} \right) q |_{\Gamma_l}, \\
   - (\nabla \cdot \tilde{u}^n, q)_\Omega &= -((\dot{y}^n - \tilde{u}^n) \cdot n_l, q)_\Sigma, \\
   u^n |_{\Gamma_{\text{in}}} &= u_{\text{in}}(t_n), \\
   u^n |_\Sigma &= \dot{y}^n |_\Sigma, \\
   \frac{\rho_f}{\tau} (u^n, v)_\Omega &= \frac{\rho_f}{\tau} (\tilde{u}^n, v)_\Omega - (\nabla p^n, v)_\Omega,
   \end{aligned}
   \]
   for all \( (q, v, \dot{v}_s) \in R_h \times V_{\Sigma,\Omega_{\Gamma^n}} \times W_h \).

Thereafter, set \( P^n_l = p^n |_{\Gamma_l} \) and compute \( (Q^n_l, \pi^n_l) \in \mathbb{R}^2 \) from the relations
   \[
   Q^n_l = \frac{P^n_l - \alpha_l \pi^n_{l-1}}{\gamma_l}, \quad \pi^n_l = \alpha_l \pi^n_{l-1} + \beta_l Q^n_l, \quad l = 1, \ldots, n_{0D}.
   \]

Let the quantities
   \[
   E^n \equiv \frac{\rho_f}{2} \| u^n |_{0, \Omega}^2 + \rho_f \frac{\gamma}{2} \| \dot{y}^n |_{0, \Omega}^2 + \frac{1}{2} \| y^n |_{\Omega}^2 + \sum_{l=1}^{n_{0D}} C_{d,l} \| \pi^n_l \|_{\Gamma_l}^2,
   \]
   \[
   D^n \equiv 2\mu \sum_{m=1}^{n} \tau \| \tilde{e}(u^m) \|^2_{0, \Omega} + \sum_{m=1}^{n-1} \sum_{l=1}^{n_{0D}} \tau \left( \| \tilde{e}(\dot{u}^m) \|^2_{\Omega_l} + R_{d,l} \| \dot{Q}^n_l \|_{\Gamma_l}^2 \right),
   \]
   for \( n \geq 1 \), denote the energy and physical dissipation of the discrete FSI-0D system. We then have the following energy based result.

**Theorem 4.1.** Let \( \{ (\tilde{u}^n, p^n, y^n, \dot{y}^n) \}_{n \geq 1} \) and \( \{ (Q^n_l, \pi^n_l, P^n_l)_{1 \leq l \leq n_{0D}} \}_{n \geq 1} \) be the approximated solution given by Algorithm 3, and assume that \( u_{\text{in}} = 0 \) (free system) and that the following condition holds:
   \[
   \rho_s \geq 3C_L \left( \rho_t h^{1-\alpha} + \frac{\mu T}{h^{1+\alpha}} \right)
   \]
with a given by (4.5). Then, the following energy inequality holds:

\begin{equation}
E^n + D^n \lesssim E^0 - \sum_{m=1}^{n-1} \frac{\tau^2}{\rho_l} \| \Pi^\perp_h (\nabla p^n) \|_{0, \Omega_t}^2,
\end{equation}

where \( \Pi_h : L^2(\Omega^t) \to V_{\Sigma,\Gamma^{in},h} \) stands for the \( L^2 \)-projection operator into \( V_{\Sigma,\Gamma^{in},h} \) and \( \Pi^\perp_h \overset{\text{def}}{=} I - \Pi_h \) for the corresponding orthogonal projection.

Proof. First, testing the viscous step (4.6) with \( v = \tilde{u}^n - \mathcal{L}_h \hat{y}^{n-1} \) yields

\begin{equation}
\frac{\rho_l}{2\tau} \left( \| \tilde{u}^n \|_{0, \Omega_t}^2 - \| u^{n-1} \|_{0, \Omega_t}^2 \right) + \frac{\mu}{2\tau} \| \tilde{u}^n - u^{n-1} \|_{0, \Omega_t}^2 \\
+ 2\mu \| \varepsilon(\tilde{u}^n) \|_{0, \Omega_t}^2 - \langle R_\mu(\tilde{u}^n), \mathcal{L}_h \hat{y}^{n-1} \rangle = 0.
\end{equation}

On the other hand, testing (4.8) with \( v = u^n - \mathcal{L}_h \hat{y}^n \) yields

\begin{equation}
\frac{\rho_l}{2\tau} \left( \| u^n \|_{0, \Omega_t}^2 - \| \tilde{u}^n \|_{0, \Omega_t}^2 \right) + \frac{\rho_s}{2\tau} \| u^n - \tilde{u}^n \|_{0, \Omega_t}^2 \\
+ \langle \nabla p^n, u^n \rangle_{\Omega_t} - \langle p^n, \hat{y}^n \cdot n_t \rangle_{\Sigma} - \langle R_p(p^n, u^n), \mathcal{L}_h \hat{y}^n \rangle = 0.
\end{equation}

and taking \( v = \hat{y}^n \) in (4.9) yields

\begin{equation}
\frac{\rho_s}{2\tau} \partial_t \| \hat{y}^n \|_{0, \Omega_t}^2 + \frac{\rho_s}{2\tau} \| \hat{y}^n - \hat{y}^{n-1} \|_{0, \Omega_t}^2 \\
+ \langle \nabla p^n, u^n \rangle_{\Omega_t} - \langle p^n, \hat{y}^n \cdot n_t \rangle_{\Sigma} - \langle R_p(p^n, u^n), \mathcal{L}_h \hat{y}^n \rangle \\+ \langle R_\mu(\tilde{u}^n), \mathcal{L}_h \hat{y}^n \rangle = - \langle R_\mu(\hat{u}^n), \mathcal{L}_h \hat{y}^n \rangle.
\end{equation}

As a result, by adding the equalities (4.13)–(4.15) we get

\begin{equation}
\frac{\rho_l}{2\tau} \partial_t \| u^n \|_{0, \Omega_t}^2 + 2\mu \| \varepsilon(\tilde{u}^n) \|_{0, \Omega_t}^2 + \frac{\rho_s}{2\tau} \| u^n - \tilde{u}^n \|_{0, \Omega_t}^2 + \frac{1}{2} \partial_t \| y^n \|_s^2 + \frac{1}{2\tau} \| y^n - y^{n-1} \|_s^2 \\
+ \langle \nabla p^n, u^n \rangle_{\Omega_t} - \langle p^n, \hat{y}^n \cdot n_t \rangle_{\Sigma} - \langle R_\mu(\tilde{u}^n), \mathcal{L}_h \hat{y}^n \rangle - \langle R_p(p^n, u^n), \mathcal{L}_h \hat{y}^n \rangle \\= - \langle R_\mu(\hat{u}^n), \mathcal{L}_h \hat{y}^n \rangle - \langle R_p(p^n, u^n), \mathcal{L}_h \hat{y}^n \rangle.
\end{equation}

Following the argument used in [1, Appendix A], from (4.7) we infer that

\begin{equation}
\tilde{u}^n = u^n + \Pi^\perp_h (\tilde{u}^n - \mathcal{L}_h \hat{y}^n) + \frac{\tau}{\rho_l} \Pi_h (\nabla p^n).
\end{equation}

Thereafter, taking \( q = p^n \) in (4.7), integrating by parts in its right-hand side, and since \( P^n_l = p^n |_{\Gamma^l} \), we have

\[
\frac{\tau}{\rho_l} \| \nabla p^n \|_{0, \Omega_t}^2 + \sum_{l=1}^{n_{\text{ad}}} \frac{P^n_l - \alpha_l \pi^{n-1}_l}{\gamma_l} P^n_l - \langle \tilde{u}^n, \nabla p^n \rangle_{\Omega_t} + \langle p^n, \hat{y}^n \cdot n_t \rangle_{\Sigma} = 0.
\]

Now, by inserting (4.17) into this expression and using (3.20), we get

\[
\sum_{l=1}^{n_{\text{ad}}} \frac{P^n_l - \alpha_l \pi^{n-1}_l}{\gamma_l} P^n_l - \langle \Pi^\perp_h (\tilde{u}^n - \mathcal{L}_h \hat{y}^n), \nabla p^n \rangle_{\Omega_t} \\
\geq \frac{\tau}{\rho_l} \| \Pi^\perp_h (\nabla p^n) \|_{0, \Omega_t}^2 + \frac{C_{d,l}}{2} \partial_t |\pi^n_l|^2 + \frac{1}{R_{d,l}} |\pi^n_l|^2 \\
- \langle \Pi^\perp_h (\tilde{u}^n - \mathcal{L}_h \hat{y}^n), \nabla p^n \rangle_{\Omega_t}.
\]
Therefore, by applying this lower bound to (4.16) we get

\begin{align*}
(4.18) \quad & \frac{\rho_t}{2} \partial_t \|u^n\|^2_{0,\Omega_t} + \frac{\rho_t}{2\tau} \|u^n - u^{n-1}\|^2_{0,\Omega_t} + 2\mu \|\epsilon(u^n)\|^2_{0,\Omega_t} + \frac{\rho_s}{2} \partial_x \|y^n\|^2_{0,\Omega_s} \\
& + \frac{1}{2} \partial_x \|y^n\|^2_x + \frac{\rho_s}{2\tau} \|y^n - y^{n-1}\|^2_{0,\Omega_s} + \tau \|\Pi_h^n(\nabla p^n)\|^2_{0,\Omega_t} + \frac{C_{d,t}}{2} \partial_x |\pi^t|^2 \\
& + \frac{1}{R_{d,t}} |\pi^t|^2 + R_{p,t}|Q^n_t|^2 \leq T_2 + T_3.
\end{align*}

Term $T_2$ can be bounded as in [10], using (4.4), which yields

\begin{align*}
(4.19) \quad & T_2 \leq \frac{\rho_t}{\tau} \|u^n - u^{n-1}\|_{0,\Omega_t} \|\mathcal{L}_h(y^n - y^{n-1})\|_{0,\Omega_t} \\
& + 2\mu \|\epsilon(u^n)\|_{0,\Omega_t} \|\epsilon(\mathcal{L}_h(y^n - y^{n-1}))\|_{0,\Omega_t} \\
& \leq \varepsilon_1 \frac{\rho_t}{2\tau} \|u^n - u^{n-1}\|^2_{0,\Omega_t} + \varepsilon_2 \mu \|\epsilon(u^n)\|^2_{0,\Omega_t} \\
& + C_L \left( \frac{\rho_t}{2\tau \varepsilon_1} h^{1-\alpha} + \frac{\mu}{\varepsilon_2} h^{1-\alpha} \right) \|y^n - y^{n-1}\|^2_{0,\Omega_t}.
\end{align*}

Term $T_3$ can be bounded following the argument used in [1], which yields

\begin{align*}
(4.20) \quad & T_3 = (\Pi_h^n(\mathcal{L}_h(y^n - y^n)), \Pi_h^n(\nabla p^n))_{\Omega_t} \\
& \leq \varepsilon_3 \frac{\tau}{2\varepsilon_3} \|\Pi_h^n(\nabla p^n)\|^2_{0,\Omega_t} + \frac{\rho_t}{2\varepsilon_3} C_L h^{1-\alpha} \|y^n - y^{n-1}\|^2_{0,\Omega_t}.
\end{align*}

Hence, by inserting (4.19)–(4.20) into (4.18) we get the energy estimate

\begin{align*}
& \frac{\rho_t}{2} \partial_t \|u^n\|^2_{0,\Omega_t} + \frac{\rho_t}{2\tau} (1 - \varepsilon_1) \|u^n - u^{n-1}\|^2_{0,\Omega_t} + \mu (2 - \varepsilon_2) \|\epsilon(u^n)\|^2_{0,\Omega_t} + \frac{\rho_s}{2} \partial_x \|y^n\|^2_{0,\Omega_s} \\
& + \frac{1}{2} \partial_x \|y^n\|^2_x + \left[ \frac{\rho_s}{2\tau} - C_L \frac{\rho_t}{2\tau} h^{1-\alpha} \left( \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3} \right) - C_L \frac{\mu}{\varepsilon_2} h^{1-\alpha} \right] \|y^n - y^{n-1}\|^2_{0,\Omega_s} \\
& + \frac{\tau}{\rho_t} \left( 1 - \varepsilon_3 \right) \|\Pi_h^n(\nabla p^n)\|^2_{0,\Omega_t} + \sum_{l=1}^{n_{ad}} \left( \frac{C_{d,t}}{2} \partial_x |\pi^t|^2 + \frac{1}{R_{d,t}} |\pi^t|^2 + R_{p,t}|Q^n_t|^2 \right) \leq 0.
\end{align*}

At last, the energy estimate (4.12) follows by taking in the latter $\varepsilon_1 = \frac{1}{2}$, $\varepsilon_2 = \varepsilon_3 = 1$, summing over $n$ and using (4.11), which completes the proof.

Proposition 4.1 guarantees the conditional stability of Algorithm 3. Note that the stability condition is similar to the one obtained in [10] with a Darcy-like formulation of the projection step. The estimate (4.12) corresponds to the discrete counterpart of (4.3), and the right-hand side of (4.12) is a dissipative numerical term related to the natural pressure stabilization of the projection scheme. Moreover, extensive numerical simulations with the semi-implicit FSI coupling scheme in physiological regimes (see [10, 8, 21]) suggest that condition (4.11) is not necessary for stability.

### 4.1.3. Semi-implicit FSI scheme with explicit 3D-0D coupling.

We now consider numerical approximations of the coupled problem FSI-0D system (4.1)–(4.2) and (2.2)–(2.3) by combining the projection semi-implicit coupling scheme reported in [10] with the 3D-0D explicit coupling of section 3.1. The resulting fully discrete time-marching procedure is reported in Algorithm 4.
Algorithm 4 (semi-implicit FSI scheme with explicit 3D-0D coupling).

Let \( \mathbf{u}^0 = \mathbf{u}_0, \pi_1^0, \ldots, \pi_{n_{\text{FD}}}^0 \in \mathbb{R} \) and \( \mathbf{y}^0, \mathbf{y}^0 \in W_h \) be given initial data. For \( n \geq 1 \) perform the following:

1. **Viscous step:** Find \( \tilde{\mathbf{u}}^n \in V_h \) such that

\[
\begin{align*}
\tilde{\mathbf{u}}^n|_{\Sigma} &= \mathbf{y}^{n-1}|_{\Sigma}, \\
\tilde{\mathbf{u}}^n|_{\Gamma_{\text{in}}} &= \mathbf{u}_{\text{in}}(t_n), \\
\frac{\rho_k}{\tau} (\tilde{\mathbf{u}}^n, \mathbf{v})_{\Omega_t} + 2\mu (\mathbf{e}(\tilde{\mathbf{u}}^n), \mathbf{e}(\mathbf{v}))_{\Omega_t} &= \frac{\rho_k}{\tau} (\mathbf{u}^{n-1}, \mathbf{v})_{\Omega_t}
\end{align*}
\]

for all \( \mathbf{v} \in V_{\Sigma, t=\tau} \). Thereafter set \( \bar{Q}_t^n \overset{\text{def}}{=} \int_{\Omega_t} \tilde{\mathbf{u}}^n \cdot \mathbf{n}_t \).

2. **Windkessel step:** For \( l = 1, \ldots, n_{\text{FD}} \), compute \((Q_l^n, \pi_l^n, P_l^n) \in \mathbb{R}^3 \) from

\[
\begin{align*}
Q_l^n &= \bar{Q}_t^n, \\
C_{\text{d},l} \partial_t \pi_l^n + \frac{\pi_l^n}{K_{\text{d},l}} &= Q_l^n, \\
P_l^n &= R_{p,l} Q_l^n + \pi_l^n.
\end{align*}
\]

3. **Implicit projection-solid step:** Find \((p^n, \mathbf{u}^n, \mathbf{y}^n) \in R_h \times V_h \times W_h \) with \( \mathbf{y}^n = \partial_t \mathbf{y}^n \) and such that

\[
\begin{align*}
p^n|_{\Gamma_1} &= P_l^n, \quad l = 1, \ldots, n_{\text{FD}}, \\
\frac{\tau}{\rho_l} (\nabla p^n, \nabla q)_{\Omega_t} &= - (\nabla \cdot \tilde{\mathbf{u}}^n, q)_{\Omega_t} - (\langle \mathbf{y}^n - \mathbf{u}^n \rangle \cdot \mathbf{n}_t, q)_{\Sigma}, \\
\mathbf{u}^n|_{\Gamma_{\text{in}}} &= \mathbf{u}_{\text{in}}(t_n), \\
\mathbf{u}^n|_{\Sigma} &= \mathbf{y}^n|_{\Sigma}, \\
\frac{\rho_l}{\tau} (\mathbf{u}^n, \mathbf{v})_{\Omega_t} &= \frac{\rho_l}{\tau} (\tilde{\mathbf{u}}^n, \mathbf{v})_{\Omega_t} - (\nabla p^n, \mathbf{v})_{\Omega_t}, \\
\rho_s (\partial_t \mathbf{y}^n, \mathbf{v}_s)_{\Omega_t} + a_s (\mathbf{y}^n, \mathbf{v}_s) &= - \langle \mathcal{R}_p (\tilde{\mathbf{u}}^n), \mathcal{L}_h \mathbf{v}_s \rangle - \langle \mathcal{R}_p (\mathbf{u}^n, p^n), \mathcal{L}_h \mathbf{v}_s \rangle
\end{align*}
\]

for all \((q, \mathbf{v}, \mathbf{v}_s) \in R_{t=\tau} \times V_{\Sigma, t=\tau} \times W_h \).

The following proposition provides an energy estimate for the approximations provided by Algorithm 4.

**Theorem 4.2.** Let \( \{ (\bar{\mathbf{u}}^n, p^n, \mathbf{y}^n, \mathbf{y}^n) \}_{n \geq 1} \) and \( \{ (Q_l^n, \pi_l^n, P_l^n)_{1 \leq l \leq n_{\text{FD}}} \}_{n \geq 1} \) be the approximated solution given by Algorithm 4 and assume that \( \mathbf{u}_{\text{in}} = \mathbf{0} \) (free system). Then, under condition (4.11), the following energy inequality holds:

\[
E^n + D^n \leq E^0 - \sum_{m=1}^{n-1} \frac{\tau^2}{2 \rho_l} \left\| \Pi_h (\nabla \mathbf{p}_m) \right\|_{0, t_m}^2 + \sum_{m=1}^{n-1} \tau (\nabla \cdot \tilde{\mathbf{u}}^m, \phi^m)_{\Omega_t} + \sum_{m=1}^{n-1} \tau \left( \mathbf{y}^m \cdot \mathbf{n}_t, \phi^m \right)_{\Sigma}.
\]
Proof. The result follows by combining the arguments involved in the proofs of Propositions 3.2 and 4.1. Hence, only partial details are given. The main difference lies on the estimation of term $T_1$ in (4.16). Since $q \in R_{\Gamma_{\text{out}, h}}$, integration by parts in (4.23) gives

$$
(\nabla p^n, \nabla q)_{\Omega_t} = \frac{\rho_f}{\tau} (\tilde{u}^n, \nabla q)_{\Omega_t} - (\tilde{y}^n \cdot n_t, q)_{\Sigma},
$$

so that by testing with $q = (p^n - \phi^n) \in R_{\Gamma_{\text{out}, h}}$ we get

$$
(4.27) \quad \frac{\tau}{\rho_f} |\nabla p^n|_{0, \Omega_t}^2 - (\tilde{u}^n, \nabla p^n)_{\Omega_t} + (\tilde{y}^n \cdot n_t, p^n)_{\Sigma} + \left( \tilde{u}^n - \frac{\tau}{\rho_f} \nabla p^n, \nabla \phi^n \right)_{\Omega_t} - (\tilde{y}^n \cdot n_t, \phi^n)_{\Sigma} = 0.
$$

As in the proof of Proposition 4.1, from (4.24) we get (4.17). Hence, inserting this expression into (4.27) we get

$$
(4.28) \quad - (\nabla p^n, u^n)_{\Omega_t} + (p^n, \tilde{y}^n \cdot n_t)_{\Sigma} + \frac{\tau}{\rho_f} |\Pi_h^+ (\nabla p^n)|_{0, \Omega_t}^2 - (\Pi_h^+ (\tilde{u}^n - L_h \tilde{y}^n), \nabla p^n)_{\Omega_t}
$$

$$
= - \left( \tilde{u}^n - \frac{\tau}{\rho_f} \nabla p^n, \nabla \phi^n \right)_{\Omega_t} + (\tilde{y}^n \cdot n_t, \phi^n)_{\Sigma}
$$

$$
- \sum_{l=1}^{N_0} Q^P_n e^P_l + (\nabla \cdot \tilde{u}^n, \phi^n)_{\Omega_t} + \frac{\tau}{\rho_f} (\nabla p^n, \nabla \phi^n)_{\Omega_t} + (\tilde{y}^n \cdot n_t, \phi^n)_{\Sigma},
$$

and estimate (4.26) follows using the same arguments as in the proof of Proposition 4.1.

Remark 4.3. A comparison of the energy estimates (3.5) and (4.26) suggests that FSI introduces an additional destabilizing effect in the explicit splitting of the 3D-0D coupling (3.1) due to the presence of the artificial interface term

$$
\sum_{m=1}^{n-1} \tau \left( \tilde{y}^m \cdot n_t, \phi^m \right)_{\Sigma}.
$$

In particular, it is worth noting that the observation made in Remark 3.4 for the case of a single outlet is not valid in the FSI framework, since the above term does not vanish for $\phi^m = P^m$ in $\Omega_t$. This point will be illustrated through numerical experiments in section 5.2.

5. Numerical experiments. In this section we present two numerical experiments that confirm the analysis of the previous sections.

5.1. Blood flow in a patient-specific aorta. Our first example is a pure Navier–Stokes flow within a patient-specific aorta with repaired coarctation (see Figure 5.1, left and center). The geometry comes from the euHeart database.\(^1\) A segment growing registration algorithm (see [2, 3]) was used for the segmentation of the geometry from the medical image. The resulting surface was preprocessed with 3-matic from Materialise and the final mesh was generated with Gmsh [14]. The inflow

\(^1\)www.euheart.eu.
Fig. 5.1. Patient-specific aorta. Left: geometry. Center: surface mesh. Right: simulation results for the aorta (vector velocity field and pressure distribution).

Fig. 5.2. Mean pressures at inlet (black) and outlets 1 to 4 (blue, green, red, cyan). Explicit (left) and implicit (right) schemes.

Table 5.1
Parameters for the Windkessel model. The outlets are ordered in direction of the flow.

<table>
<thead>
<tr>
<th></th>
<th>Outlet 1</th>
<th>Outlet 2</th>
<th>Outlet 3</th>
<th>Outlet 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_p$ (dyn · s · cm$^{-5}$)</td>
<td>250</td>
<td>683</td>
<td>615</td>
<td>94</td>
</tr>
<tr>
<td>$R_d$ (dyn · s · cm$^{-5}$)</td>
<td>$4 \times 10^{-4}$</td>
<td>$1.296 \times 10^4$</td>
<td>$1.1664 \times 10^4$</td>
<td>$0.1794 \times 10^4$</td>
</tr>
<tr>
<td>$C$ (cm$^5$, dyn$^{-1}$)</td>
<td>$2 \times 10^{-4}$</td>
<td>$2 \times 10^{-4}$</td>
<td>$2 \times 10^{-4}$</td>
<td>$14 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

curve used as boundary condition (Figure 5.2, right, black line) was obtained from the same patient with phase contrast MRI. The initial constant pressure was set to 47 mmHg, and the Windkessel parameters (reported in Table 5.1) were calibrated in order to approximate the measured pressure at the coarctation and the measured flow on each outlet. The physical parameters of the fluid are $\mu = 0.035$ Po and $\rho_f = 1$ g/cm$^3$. For the numerical simulation, we use $P_1$ finite elements for both pressure and velocity fields (SUPG stabilization in the viscous-convective step, where we assume that we first perform the Chorin–Temam splitting and then we stabilize) and a time step $\tau = 10^{-3}$ s. A snapshot of the simulation results is shown in Figure 5.1, right. Figures 5.2 and 5.3 show the pressure and flow results for both schemes (Algorithms 1 and 2, respectively). The spurious oscillations in the approximation provided by the explicit 3D-0D coupling scheme are clearly visible for all the outlet pressures and (less pronounced) for some outlet fluxes, while the implicit formulation guarantees stability within the whole cardiac cycle. Hence, these results are in
agreement with the stability estimates provided by Theorems 3.2 and 3.7. As further validation of the numerical scheme, Figure 5.4 shows the pressures and flows curves computed using a monolithic solver, implicitely coupled with the 0D-model, showing a satisfactory agreement of the results.

5.2. FSI in an idealized AAA. We consider an idealized AAA of length 22.95 cm, minimal diameter 1.7 cm (tubular part), maximal diameter 4.98 cm (aneurysm), and wall thickness 0.2 cm (see Figure 5.5, left). The structure is described by the linear elastodynamics equations and the fluid by the incompressible Navier–Stokes equations in a moving domain (ALE formalism). We considered the nonlinear counterpart of Algorithms 3 and 4 using the projection semi-implicit coupling scheme reported in [10]. The solid has Young’s modulus 1 MPa, Poisson ratio is 0.46 and density 1.2 g/cm$^3$. The fluid viscosity is $\mu = 0.035$ Po and its density is 1 g/cm$^3$. The inlet and the outlet parts of the solid are clamped. In the fluid, a parabolic velocity profile is enforced at the inlet with a realistic inflow (peak velocity $\approx 96$ cm/s). The Windkessel parameters are $R_p = 700$ dyn $\cdot$ s $\cdot$ cm$^{-5}$, $R_d = 5 \cdot 10^3$ dyn $\cdot$ s $\cdot$ cm$^{-5}$, and $C = 2 \cdot 10^{-4}$ cm$^5$ $\cdot$ dyn$^{-1}$. At $t = 0$, the pressure is constant and equal to 80 mmHg, whereas all the other state variables are zero. During the whole simulation, the stress received by the structure is corrected by the initial one, and the solid responds only to the difference with the diastolic phase. In this way, the load applied to the structure is kept reasonably small so that the linearity assumptions holds.
Fig. 5.5. Left: snapshot of the velocity field (left) inside the idealized AAA; the curves show outlet pressure and displacements of the aneurysm wall. Right: Windkessel pressure for the AAA, comparison between implicit (solid line) and explicit (dashed line) fluid-Windkessel coupling.

The results are summarized in Figure 5.5, right, showing the Windkessel pressures in time for a time step $P$ for $\tau = 0.001\ s$. Note that the semi-implicit algorithm with explicit 0D-3D coupling (Algorithm 4) is unstable, whereas with the implicit 3D-0D treatment (Algorithm 3) the numerical solution remains stable. In fact, from the results one can infer that the interface term outlined in Remark 4.3, namely, $(\dot{y}^n, P^n n_f)_{\Sigma}$, injects a positive artificial power into the system (an increased pressure $P^n > 0$ leads to $\dot{y}^n \cdot n_f > 0$).

6. Conclusions. In this paper we formulated and analyzed the treatment of a 3D-0D coupling between the 3D distributed (fluid, fluid-structure interaction) models and a set of Windkessel 0D models. The key ingredient in the proposed schemes is the use of a fractional-step time-marching in the 3D compartment. For purely fluid problems with multiple outlets, our energy-based stability analysis showed that numerical instabilities might appear when using an explicit 3D-0D coupling. Interestingly, this result also holds with a single outlet in the case of FSI. We proposed to overcome these issues via an implicit treatment of the 3D-0D coupling, which involves a negligible additional computational cost with respect to the explicit strategy. These theoretical expectations were confirmed by numerical experiments in realistic geometries and physiological data.

Acknowledgments. The authors are thankful to Jean-Frédéric Gerbeau and Romain Guibert (INRIA) for valuable discussions, Philipp Beerbaum and Israel Valverde (King’s College London) for the acquisition of the patient data, and David Barber, Rod Hose, and Cristina Staicu (University of Sheffield) for the segmented geometry.

REFERENCES

3D-0D COUPLING SCHEMES IN HEMODYNAMICS


