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KAPER, B

Published in:
Siam Journal on Applied Mathematics

DOI:
10.1137/0131046

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
1976

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

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PERTURBED NONLINEAR OSCILLATIONS*

B. KAPER†

Abstract. For a class of nonlinear oscillation problems described by a second order ordinary differential equation containing a small nonnegative perturbation parameter \( \varepsilon \) we determine asymptotic solutions which are uniformly valid as \( \varepsilon \downarrow 0 \) on intervals of the order of magnitude \( \varepsilon^{-1} \). These asymptotic solutions are obtained from formal asymptotic solutions subject to a condition in order that they approximate the exact solution asymptotically. These formal asymptotic solutions are constructed in the form of a finite generalized asymptotic power series involving functions of the periodic two variable type.

1. Introduction. We consider a class of perturbed nonlinear oscillations described by the second order ordinary differential equation containing a small nonnegative perturbation parameter \( \varepsilon \)

\[
\begin{align*}
\varepsilon^2 x'' + F(x, \varepsilon t) + \varepsilon f(x, x', \varepsilon t, \varepsilon) &= 0, \quad t \geq 0, \\
x(0, \varepsilon) &= \alpha_1(\varepsilon), \quad x'(0, \varepsilon) = \alpha_2(\varepsilon).
\end{align*}
\]

The force term \( F \) and its perturbation \( \varepsilon f \) include the independent time variable \( t \) in a slowly varying way. In general we may expect the solution of problem (1.1) to be an oscillating function with slowly varying amplitude and “frequency”. Asymptotically we may distinguish two different time scales: a local (or rapid) time scale in which the solution of problem (1.1) is periodic with a period of the asymptotic order of magnitude one and a slow (or stretched) time scale, characterized by the slow variable \( \tau = \varepsilon t \), which accounts for the slow modulations of the oscillation. In order to include these large scale variations in a study of the solutions we apply the change of variable \( \tau = \varepsilon t \) to problem (1.1). Setting \( x(\tau/\varepsilon, \varepsilon) = \tilde{x}(\tau, \varepsilon) \) and omitting the tilde (~) at the same time we get

\[
\begin{align*}
\varepsilon^2 \tilde{x}'' + F(x, \tau) + \varepsilon f(x, \varepsilon x', \tau, \varepsilon) &= 0, \quad \tau \geq 0, \\
\tilde{x}(0, \varepsilon) &= \alpha_1(\varepsilon), \quad \varepsilon \tilde{x}'(0, \varepsilon) = \alpha_2(\varepsilon),
\end{align*}
\]

where the primes indicate differentiation with respect to \( \tau \).

With respect to the perturbation parameter \( \varepsilon \) we determine asymptotic solutions of arbitrary order of the class of oscillation problems (1.2) uniformly valid on the finite \( \tau \)-interval \( I = [0, L] \). A function \( \tilde{\phi}_N \) is called an \( N \)th order asymptotic solution of (1.2) on \( I \) if

\[
|\tilde{x}(\tau, \varepsilon) - \tilde{\phi}_N(\tau, \varepsilon)| \leq K\varepsilon^{N+1}, \quad |\varepsilon \{x'(\tau, \varepsilon) - \tilde{\phi}'_N(\tau, \varepsilon)\}| \leq K\varepsilon^{N+1}, \quad \tau \in I,
\]

where \( \tilde{x} \) represents the exact solution of (1.2) and \( K \) is some positive number independent of \( \tau \) and \( \varepsilon \).

* Received by the editors May 13, 1974, and in revised form August 14, 1975.
† Department of Mathematics, University of Groningen, Groningen, the Netherlands.

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Asymptotic solutions will be obtained from so-called formal asymptotic solutions. A function $\phi_N$ is an $N$th order formal asymptotic solution of (1.2) if

$$\epsilon^2 \phi_N' + F(\phi_N, \tau) + \epsilon f(\phi_N, \epsilon \phi_N', \tau, \epsilon) = \epsilon^{N+1} g_N(\tau, \epsilon) = O(\epsilon^{N+1}), \quad \tau \in I,$$

$$\alpha_1(\epsilon) - \phi_N(0, \epsilon) = O(\epsilon^{N+1}), \quad \alpha_2(\epsilon) - \epsilon \phi_N'(0, \epsilon) = O(\epsilon^{N+1}).$$

($g_N$ is called the residual function of $\phi_N$). Moreover, we let the residual function $g_N$ satisfy some condition in order that $\phi_N$ approximate the exact solution uniformly on $I$. For the construction of such an $N$th order formal asymptotic solution $\phi_N$ we anticipate the form of $\phi_N$ as a finite generalized asymptotic power series for the construction of which we apply a two variable technique; cf. Cole [3], Nayfeh [9].

Apart from the variable $\tau$, which describes the slow modulations, a second variable $p$ is introduced which accounts for the local periodic behavior of the solution. This technique has been applied to a class of weakly nonlinear oscillation problems (for which the force term $F$ is linear in $x$) by Cole and Kevorkian [4], Mitropol’skii [8] and recently by Hoogstraten and Kaper [5]. In [7], Kuzmak extended the Cole–Kevorkian method to the class of nonlinear oscillation problems (1.2) with a quasi-linear perturbation function

$$\epsilon^2 x'' + F(x, \tau) + \epsilon^2 f(x, \tau) x' = 0, \quad \tau \in I.$$  

He constructed a first order formal asymptotic solution $\phi_1$ of the form

$$\phi_1(\tau, \epsilon) = y_0(\rho, \tau) + \epsilon y_1(\rho, \tau), \quad p = \epsilon^{-1} q(\tau),$$

where $q'(\tau) = O(1)$. In his article he did not include a proof of asymptotic correctness of $\phi_1$ with respect to the exact solution. For the class of oscillation problems (1.2) this form does not contain sufficient degrees of freedom to satisfy the boundedness condition to be imposed on the coefficients of a generalized asymptotic power series.

Mitropol’skii indicated in his book [8] two asymptotic methods for the class of problems (1.2), an indirect method of successive transformations and a direct method by substitution of the series

$$\phi(\tau, \epsilon) \sim u_0(\tau, \psi, a) + \epsilon u_1(\tau, \psi, a) + \cdots,$$

where $a$ and $\psi$ should be determined from the system of equations

$$\frac{da}{d\tau} \sim a_0(\tau, a) + \epsilon a_1(\tau, a) + \cdots,$$

$$\frac{d\psi}{d\tau} \sim \epsilon^{-1} \omega(\tau, a) + \psi_0(\tau, a) + \cdots.$$

The first term $u_0$ is a periodic solution of the $\tau$ parameter equation

$$\omega^2(\tau) u_{0,\psi\psi} + F(u_0, \tau) = 0,$$

where the subscripts $\psi$ denote partial differentiation with respect to $\psi$.

This method is an extension of a method developed for a class of weakly nonlinear oscillation problems. In the Introduction of [5] Hoogstraten and Kaper already mentioned the disadvantage of this method with respect to the different...
nonlinear first order equations for the amplitude $a$ to be solved in order to obtain asymptotic solutions of different orders. As the coefficients in the equations defining $u_1, u_2, \cdots$ are periodic functions Mitropol’skii does not concern himself with this method and restricts himself to the somewhat laborious indirect method of transformations to new variables. This method is by its contents equivalent to the method of averaging.

In this paper we arrive at an $N$th order asymptotic solution $\phi_N$ of (1.2) of the form

$$\begin{align*}
\phi_N(\tau, \varepsilon) &= \eta(\tau) + A_0(\tau)\Phi_0(p, \tau) + \sum_{\nu=1}^{N} \varepsilon^\nu [A_\nu(\tau)z_2^\nu(p, \tau) + \Phi_\nu(p, \tau)], \\
p &= e^{-1}S(\tau, \varepsilon),
\end{align*}$$

The first term $U_0 = \eta + A_0\Phi_0$ represents the solution of a $\tau$-parameter equation of the type (1.3) whereas the functions $U_\nu = A_\nu z_2^\nu + \Phi_\nu$ satisfy the inhomogeneous linear equations

$$\begin{align*}
\omega^2(\tau)U_{\nu pp} + F_x(U_0, \tau)U_\nu &= -\gamma_\nu, \\
\nu &= 1, \cdots, N
\end{align*}$$

($F_x$ denotes the partial derivative of $F$ respect to $x$). $U_0$ is $2\pi$-periodic and oscillates between $\eta \pm A_0$, $z_2^\nu$ is one of the two linearly independent homogeneous solutions $z_{1,2}^\nu$ of (1.5) and $\Phi_\nu$ represents that particular solution of (1.5) which satisfies

$$\int_{0}^{2\pi} \Phi_\nu(p, \tau)z_i^\nu(p, \tau) dp = 0, \quad i = 1, 2.$$ 

The expansion (1.4) of $\phi_N$ includes as a particular case the expansion developed by Hoogstraten and Kaper [5] for weakly nonlinear oscillation problems if we identify $\Phi_0(p, \tau) = z_2^\nu(p, \tau) = \cos p$. A detailed description of the construction method can be found in §4 of this paper. In §3 we deduce conditions on the residual function $g_N$ in order that $\phi_N$ approximate the exact solution uniformly on the interval $I$. In the final part of §4 we show that the construction of $\phi_N$ can be performed as an algorithm yielding successively a sequence of asymptotic solutions of increasing order. In §5 we shall apply the algorithm to a perturbed Duffing equation with slowly varying coefficients in order to obtain a zeroth order asymptotic solution. To a certain extent the results could be compared to the first order formal asymptotic solution $\phi_1$ obtained by Kuzmak of the Duffing equation with slowly varying coefficients and without a perturbation function (cf. [7]). We shall show some limitations of Kuzmak’s form in dealing with a more general class of problems as well as in dealing with higher order (formal) asymptotic solutions.

### §2. Notations and definitions.

Throughout this paper all quantities are assumed to be real, $\varepsilon$ is a small nonnegative perturbation parameter. For vectors and matrices the subscript $\varepsilon$ is used to indicate dependence on $\varepsilon$. All considerations apply for sufficiently small positive $\varepsilon$, unless stated otherwise. The positive
numbers $\delta_0$, $\varepsilon_0$ and $M_0$ are generic constants, that is, they are not necessarily the same number each time they appear. By $I$ we denote the closed interval $[0, L]$, where $L$ is an arbitrary fixed positive number. The norm of a vector or a matrix is defined as the sum of the absolute values of the elements. The linear space of real $n \times n$ matrices with unit element $E$ is denoted by $\mathbb{R}^{n \times n}$.

The asymptotic order symbol $O$ has its usual meaning and is always understood to be related to the limit process $\varepsilon \downarrow 0$. All asymptotic order relations involving a function of $\tau$ hold uniformly for $\tau \in I$, unless stated otherwise. When a vector or a matrix function satisfies an order relation it is to be understood that the norm satisfies the order relation.

We define two classes of functions, $P^\infty$ and $P_S^\infty$ whose elements will be referred to as “functions of the periodic two variable type”. By $P^\infty$ we denote the class of scalar functions $\chi^*$ of $p$, $\tau$ and $\varepsilon$ for which hold

1. $\chi^* \in C^\infty(\mathbb{R} \times I \times [0, \varepsilon_0], \mathbb{R})$,
2. $\chi^*(p + 2\pi, \tau, \varepsilon) = \chi^*(p, \tau, \varepsilon)$ for $(p, \tau, \varepsilon) \in \mathbb{R} \times I \times [0, \varepsilon_0]$.

$P^\infty$ also contains the class of scalar functions which depend only on $p$ and $\tau$ and which satisfy conditions (i) and (ii) omitting the $\varepsilon$ dependence.

Let $S$ be a scalar function which belongs to the class $C^\infty(I \times [0, \varepsilon_0], \mathbb{R})$ and which has a strictly positive derivative with respect to $\tau$ on $I$; thus

$$\frac{\partial}{\partial \tau} S(\tau, \varepsilon) \geq \delta_0 > 0 \quad \text{for} \quad (\tau, \varepsilon) \in I \times [0, \varepsilon_0].$$

Then $P_S^\infty$ consists of all functions $\chi$ of $\tau$ and $\varepsilon$ for which hold

1. $\chi \in C^\infty(I \times [0, \varepsilon_0], \mathbb{R})$,
2. $\chi(\tau, \varepsilon) = \chi^*(p, \tau, \varepsilon)$, $\quad p = \varepsilon^{-1} S(\tau, \varepsilon)$,

where $\chi^* \in P^\infty$.

A vector or a matrix function is said to belong to $P^\infty$ or $P_S^\infty$ if all its elements belong to $P^\infty$, respectively $P_S^\infty$. If a function belongs to $P_S^\infty$, its “two variable counterpart” in $P^\infty$ will be indicated by the same symbol with an asterisk. Let $H_\varepsilon$ be a vector function where $H_\varepsilon \in C^i(\mathbb{R}^n \times I, \mathbb{R}^n)$ and consider the first order ordinary differential equation in $\mathbb{R}^n$

$$\varepsilon y'_\varepsilon = H_\varepsilon(y_\varepsilon, \tau), \quad \tau \in I,$$

with initial condition

$$y_\varepsilon(0) = \alpha_\varepsilon$$

**DEFINITION 1.** Assume that problem (2.1) has a unique solution $y_\varepsilon$ on $I$. A function $\tilde{u}_\varepsilon$ is an $N$-th order asymptotic solution of (2.1) on $I$ if

1. $\tilde{u}_\varepsilon(\tau) \in \mathbb{R}^n$ for $\tau \in I$,
2. $y_\varepsilon - \tilde{u}_\varepsilon = O(\varepsilon^{N+1})$. 

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DEFINITION 2. A function $u_\varepsilon$ is an $N$-th order formal asymptotic solution of problem (2.1) on $I$ if

(i) $u_\varepsilon \in C^1(I, \mathbb{R}^n)$,
(ii) $u_\varepsilon = O(1)$,
(iii) $\varepsilon u'_\varepsilon(\tau) - H_\varepsilon\{u_\varepsilon(\tau), \tau\} \overset{\text{def}}{=} \varepsilon^{N+1}g_\varepsilon(\tau) = O(\varepsilon^{N+1})$,
(iv) $\alpha_\varepsilon - u_\varepsilon(0) \overset{\text{def}}{=} \varepsilon^{N+1}\tilde{\alpha}_\varepsilon = O(\varepsilon^{N+1})$.

The function $g_\varepsilon$ will be called the residual function of $u_\varepsilon$. Accordingly we may define asymptotic solutions of the second order problem (1.2).

DEFINITION 3. A scalar function $\phi_N$ is an $N$th order formal asymptotic solution of (1.2) on $I$ if

(i) $\phi_N \in C^2(I, \mathbb{R})$,
(ii) $\phi_N$ and $\varepsilon\phi_N' = O(1)$,
(iii) $\varepsilon^2 \phi_N'' + F(\phi_N, \tau) + \varepsilon f(\phi_N, \varepsilon\phi_N', \tau, \varepsilon) \overset{\text{def}}{=} \varepsilon^{N+1}g_N(\tau, \varepsilon) = O(\varepsilon^{N+1})$, $\tau \in I$,
(iv) $\alpha_1(\varepsilon) - \phi_N(0, \varepsilon) \overset{\text{def}}{=} \varepsilon^{N+1}\tilde{\alpha}_1(\varepsilon) = O(\varepsilon^{N+1})$,
$\alpha_2(\varepsilon) - \varepsilon\phi_N'(0, \varepsilon) \overset{\text{def}}{=} \varepsilon^{N+1}\tilde{\alpha}_2(\varepsilon) = O(\varepsilon^{N+1})$.

It is seen that the vector function $u_\varepsilon$, $u_\varepsilon(\tau) = \text{col}(\phi_N(\tau, \varepsilon), \varepsilon\phi_N'(\tau, \varepsilon))$, is an $N$th order formal asymptotic solution (in the sense of Definition 2) of the system of first order equations in $\mathbb{R}^2$ associated with (1.2) if we take $y_\varepsilon(\tau) = \text{col}(x(\tau, \varepsilon), \varepsilon x'(\tau, \varepsilon))$.

3. Formal asymptotic solutions of the periodic two variable type. In this section we deduce some theorems that eventually lead to conditions on the residual function $g_N$ of an $N$th order formal asymptotic solution $\phi_N$ of the periodic two variable type of problem (1.2). We shall repeat a fundamental theorem on a first order system of differential equations in $\mathbb{R}^n$ that has been proved by Hoogstraten and Kaper in [5].

THEOREM 1. Consider the first order initial value problem in $\mathbb{R}^n$

$$
\begin{align*}
\varepsilon y'_\varepsilon &= H_\varepsilon(y_\varepsilon, \tau), \quad \tau \in I, \\
y_\varepsilon(0) &= \alpha_\varepsilon,
\end{align*}
$$

where the following assumptions are made about $H_\varepsilon(z, \tau) = \text{col}(H_1(z, \tau, \varepsilon), \ldots, H_n(z, \tau, \varepsilon))$:

(i) $H_i$ and the elements $\{\nabla H_\varepsilon(z, \tau)\}_{i,j} = (\partial H_i/\partial z_j)(z, \tau, \varepsilon)$, $i, j = 1, \ldots$, of the Jacobian $\nabla H_\varepsilon$ belong to $C^0(\mathbb{R}^n \times I, \mathbb{R})$. 

for each \( R > 0 \) there exists a number \( M \), depending on \( R \) and \( L \) only, such that, for all \((u, \tau)\) and \((v, \tau) \in \mathbb{R}^n \times \mathbb{I}\) with \(|u|, |v| \leq R\),

\[
\frac{\partial H_i}{\partial z_j}(u, \tau, \varepsilon) - \frac{\partial H_i}{\partial z_j}(v, \tau, \varepsilon) \leq M|u - v|, \quad i, j = 1, \ldots, n.
\]

Let \( u_\varepsilon \) be an \( N \)th order formal asymptotic solution of problem (3.1) on \( \mathbb{I} \) with residual function \( g_\varepsilon \). Let \( \Psi_\varepsilon \) be the fundamental matrix solution of the first variational equation of (3.1) with respect to \( u_\varepsilon \),

\[(3.2) \quad \varepsilon Z_\varepsilon' \Psi_\varepsilon = \nabla H_\varepsilon \{ u_\varepsilon(\tau), \tau \} z_\varepsilon, \quad \tau \in \mathbb{I},\]

satisfying \( \Psi_\varepsilon(0) = \mathbb{I} \).

If the asymptotic order relations

\[(3.3) \quad \Psi_\varepsilon = O(1), \quad \Psi_\varepsilon^{-1} = O(1)\]

and

\[(3.4) \quad \int_0^\tau \Psi_\varepsilon^{-1}(s) g_\varepsilon(s) \, ds = O(\varepsilon)\]

hold on \( \mathbb{I} \), and if \( N \geq 1 \), then problem (3.1) has a unique solution \( y_\varepsilon \) on \( \mathbb{I} \) and, moreover, \( u_\varepsilon \) is an \( N \)-th order asymptotic solution of problem (3.1) on \( \mathbb{I} \).

It is clear that in order to verify conditions (3.3) and (3.4) it suffices to have a zeroth order asymptotic approximation of \( \Psi_\varepsilon \) that satisfies (3.3) and (3.4). We notice that \( \Psi_\varepsilon \) is the solution of a linear matrix initial value problem in \( \varepsilon Z_\varepsilon' \mathbb{I} \) of the following type:

\[(3.5a) \quad \varepsilon Z_\varepsilon' = A_\varepsilon(\tau) Z_\varepsilon, \quad \tau \in \mathbb{I},\]

\[(3.5b) \quad Z_\varepsilon(0) = \mathbb{I},\]

where \( A_\varepsilon \in C^0(\mathbb{I}, \mathbb{R}^{n \times n}) \). Moreover, we make the additional assumption

\[(3.6) \quad \int_0^\tau \text{tr} A_\varepsilon(s) \, ds = O(\varepsilon).\]

From the theory of linear ordinary differential equations (cf., e.g., Coddington and Levinson [2]) we know that problem (3.5) has a unique solution \( \Psi_\varepsilon \) which belongs to \( C^1(\mathbb{I}, \mathbb{R}^{n \times n}) \). Assumption (3.6) implies that \( \det \Psi_\varepsilon(\tau) \),

\[
\det \Psi_\varepsilon(\tau) = \exp \left[ \frac{1}{\varepsilon} \int_0^\tau \text{tr} A_\varepsilon(s) \, ds \right]
\]

(cf. [2]), is uniformly bounded away from zero; thus

\[
\det \Psi_\varepsilon(\tau) \geq \delta_0 > 0, \quad \tau \in \mathbb{I}.
\]

This means that if \( \psi_\varepsilon = O(1) \), then also \( \psi_\varepsilon^{-1} = O(1) \).

The following lemma on the boundedness and asymptotic approximation of \( \Psi_\varepsilon \) will be needed in the sequel.
LEMMA 1. Let the matrix $W_e$ be a zeroth order formal asymptotic solution of (3.5a), that is,

(i) $W_e \in C^1(I, \mathbb{R}^{n\times n}),$

(ii) $W_e = O(1),$

(iii) $\varepsilon W_e'(\tau) - A_e(\tau)W_e(\tau) \overset{\text{def}}{=} \varepsilon G_e(\tau) = O(\varepsilon),$

(iv) $|\det W_e(\tau)| \geq \delta > 0$ for $\tau \in I.$

If there exists a matrix $C_e$ belonging to $C^1(I, \mathbb{R}^{n\times n})$ which satisfies the asymptotic order relations

(3.7a) $\int_0^\tau [W_e^{-1}(s)G_e(s)C_e(s) + C_e'(s)]ds = O(\varepsilon),$

(3.7b) $C_e(0) = W_e^{-1}(0) + O(\varepsilon),$

then $\Psi_e$ (and consequently $\Psi_e^{-1}$) is $O(1)$ and, moreover, the matrix function $\tilde{W}_e,$

$\tilde{W}_e(\tau) = W_e(\tau)C_e(\tau),$

is a zeroth order asymptotic solution of problem (3.5), that is,

$\Psi_e(\tau) = \tilde{W}_e(\tau) + O(\varepsilon).$

Proof. We shall show that $\Psi_e$ can be written in the form

$\Psi_e = W_e(C_e + \varepsilon R_e),$

where $R_e$ and $C_e$ are both $O(1).$ It is evident that the lemma has been proved then. The matrix $R_e$ should satisfy the differential equation

(3.8a) $R_e' = -W_e^{-1}G_eR_e - \frac{1}{\varepsilon} [W_e^{-1}G_eC_e + C_e'], \quad \tau \in I,$

with initial condition

(3.8b) $R_e(0) = \frac{1}{\varepsilon} [W_e^{-1}(0) - C_e(0)].$

Note that $R_e(0) = O(1).$ The initial value problem (3.8) can be transformed into the linear Volterra integral equation

(3.9) $R_e(\tau) = R_e(0) - \frac{1}{\varepsilon} \int_0^\tau [W_e^{-1}(s)G_e(s)C_e(s) + C_e'(s)] ds$

$- \int_0^\tau W_e^{-1}(s)G_e(s)R_e(s) ds.$

From the assumed properties of $W_e$ and condition (3.7a) we obtain the inequality

$|R_e(\tau)| \leq |R_e(0)| + M_0 + M_0^2 \int_0^\tau |R_e(s)| ds \quad \text{for} \quad \tau \in I.$

Upon application of Gronwall's lemma [2, Probl. 1, p. 37] we get

$|R_e(\tau)| \leq \{ |R_e(0)| + M_0 \} \exp [M_0^2 L], \quad \tau \in I,$
from which it follows that $R_\epsilon = O(1)$. From relation (3.7) we may deduce that a positive number $M_0$ exists such that

$$|C_\epsilon(\tau)| \leq M_0(1 + \epsilon) + M_0^2 \int_0^\tau |C_\epsilon(s)| \, ds, \quad \tau \in I.$$ 

Applying again Gronwall's lemma we get the estimate

$$|C_\epsilon(\tau)| \leq M_0(1 + \epsilon) \exp [M_0^2 L],$$

valid for $\tau \in I$. Hence $C_\epsilon = O(1)$. This completes the proof of Lemma 1.

We now assume that the $N$th order formal asymptotic solution $u_\epsilon$ of the initial value problem (3.1) belongs to $P_S^\infty$ and that the linear matrix equation associated with the first variational equation (3.2) has a zeroth order formal asymptotic solution $W_\epsilon$ which belongs to $P_S^\infty$. With the help of Lemma 1 we transform the conditions (3.3) and (3.4) of Theorem 1 for asymptotic correctness of $u_\epsilon$ into a condition which involves $W_\epsilon$ instead of $\Psi_\epsilon$. This is shown by the following.

**THEOREM 2.** Consider problem (3.1). Let $u_\epsilon$ be an $N$-th order formal asymptotic solution of (3.1) on $I$ which belongs to $P_S^\infty$ and for which

(3.10) $$\int_0^\tau \text{tr} H_\epsilon\{u_\epsilon(s), s\} \, ds = O(\epsilon).$$

Let $W_\epsilon$ be a zeroth order formal asymptotic solution in the sense of Lemma 1 of the first variational matrix equation in $\mathbb{R}^{n \times n}$,

(3.11) $$\epsilon Z_\epsilon' = \nabla H_\epsilon\{u_\epsilon(\tau), \tau\} Z_\epsilon, \quad \tau \in I,$$

and let $W_\epsilon$ belong to $P_S^\infty$ also.

If the asymptotic order relation

(3.12) $$\int_0^{2\pi} W_\epsilon^{*\tau}(p, \tau) g_\epsilon^*(p, \tau) \, dp = O(\epsilon)$$

holds on $I$, where $W_\epsilon^{*\tau}(p, \tau) = W_\epsilon(\tau)$, $g_\epsilon^{*\tau}(p, \tau) = g_\epsilon(\tau)$ for $p = \epsilon^{-1} S(\tau, \epsilon)$, and if $N \geq 1$, then problem (3.1) has a unique solution $y_\epsilon$ on the interval $I$. Moreover, $u_\epsilon$ is an $N$-th order asymptotic solution of the periodic two variable type of problem (3.1) on $I$.

**Proof.** Because of condition (3.10) and the assumed properties of $H_\epsilon$ and $u_\epsilon$, the solution $\Psi_\epsilon$ of equation (3.11) with $\Psi_\epsilon(0) = E$ satisfies a matrix initial value problem of the type (3.5). We shall show first that a matrix function $C_\epsilon \in C^\infty(I \times [0, \epsilon_0], \mathbb{R}^{n \times n})$ exists such that

(3.13) $$\Psi_\epsilon = W_\epsilon C_\epsilon + O(\epsilon).$$

Since $W_\epsilon$ and $\nabla H_\epsilon\{u_\epsilon(\tau), \tau\}$ both belong to $P_S^\infty$, we also have that the residual function $G_\epsilon$, which corresponds to $W_\epsilon$, belongs to $P_S^\infty$. Hence we may write

$$G_\epsilon(\tau) = G_\epsilon^*(p, \tau), \quad p = \epsilon^{-1} S(\tau, \epsilon).$$
Let $C_{e}$ be the solution of the matrix initial value problem in $\mathbb{R}^{n \times n}$

\begin{align}
(3.14a) & \quad C_{e}'(\tau) = B_{e}(\tau)C_{e}(\tau), \quad \tau \in I, \\
(3.14b) & \quad C_{e}(0) = W_{e}^{-1}(0),
\end{align}

where

\[ B_{e}(\tau) = \frac{-1}{2\pi} \int_{0}^{2\pi} W_{e}^{* -1}(p, \tau)G_{e}^{*}(p, \tau) \, dp. \]

Since $B_{e} \in C^{\infty}(I \times [0, \varepsilon_0], \mathbb{R}^{n \times n})$ the existence of a unique solution $C_{e} \in C^{\infty}(I \times [0, \varepsilon_0], \mathbb{R}^{n \times n})$ of problem (3.14) is guaranteed by the theory of linear ordinary differential equations. Because of the properties of $W_{e}$ and $B_{e}$ we have

\[ |\det C_{e}(\tau)| = |\det W_{e}^{-1}(0)| \exp \left[ \int_{0}^{\tau} \text{tr} B_{e}(s) \, ds \right] \geq \delta_0 > 0, \quad \tau \in I, \]

and hence $C_{e}^{-1}$ also belongs to $C^{\infty}(I \times [0, \varepsilon_0], \mathbb{R}^{n \times n})$.

According to Lemma 2 (cf. Appendix A), the fact that $C_{e}$ satisfies equation (3.14a) is just a sufficient condition that the integral

\[ \int_{0}^{\tau} \left[ W_{e}^{* -1}(p, s)G_{e}^{*}(p, s)C_{e}(s) + C_{e}'(s) \right] ds, \quad p = \varepsilon^{-1}S(s, \varepsilon), \]

be $O(\varepsilon)$. Hence it is seen that both conditions (3.7a,b) of Lemma 1 are satisfied. Thus, by virtue of Lemma 1, we have shown relation (3.13). Moreover, we know by Lemma 1 that $\psi_{e}$ and $\psi_{e}^{-1}$ are both uniformly $O(1)$ for $\tau \in I$. Next we shall show that condition (3.4) is satisfied. Then, by virtue of Theorem 1, we have proved Theorem 2. Using (3.13) it is evident that (3.4) is satisfied if

\begin{equation}
(3.15) \quad \int_{0}^{\tau} C_{e}^{-1}(s)W_{e}^{-1}(s)g_{e}(s) \, ds = O(\varepsilon).
\end{equation}

Note that the integrand in relation (3.15) belongs to $P_{e}^{\infty}$. Then, using Lemma 2 (cf. Appendix A) we see that a sufficient condition for (3.15) to hold is

\[ C_{e}^{-1}(\tau) \int_{0}^{2\pi} W_{e}^{* -1}(p, \tau)G_{e}^{*}(p, \tau) \, dp = O(\varepsilon). \]

This latter condition is satisfied because of condition (3.12). This completes the proof of Theorem 2.

Next we shall apply Theorem 2 to the second order initial value problem (1.2) in order to deduce conditions on the residual function $g_{N}$ of an $N$th order formal asymptotic solution $\phi_{N}$. Problem (1.2) may be brought into the vector form (3.1) by setting

\[ y_{e}(\tau) = \begin{pmatrix} x(\tau, \varepsilon) \\ \varepsilon x'(\tau, \varepsilon) \end{pmatrix}, \quad \alpha_{e} = \begin{pmatrix} \alpha_{1}(\varepsilon) \\ \alpha_{2}(\varepsilon) \end{pmatrix}, \]

\[ H_{e}(y_{e}, \tau) = \begin{pmatrix} \varepsilon x' \\ -F(x, \tau) - \varepsilon f(x, \varepsilon x', \tau, \varepsilon) \end{pmatrix}. \]
We assume that \( F \in C^\infty(R \times I, R) \), \( f \in C^\infty(R^2 \times I \times [0, \varepsilon_0], R) \) and \( \alpha_i \in C^\infty([0, \varepsilon_0]) \), \( i = 1, 2 \). It is clear that \( H \) has the properties required in Theorem 1 (and consequently in Theorem 2).

In § 4 we shall develop under certain assumptions a construction technique for an \( N \)th order formal asymptotic solution \( \phi_N \). It will be of the form

\[
(3.16a) \quad \phi_N(p, \tau, \varepsilon) = \phi_N^*(p, \tau, \varepsilon) + \epsilon \tilde{U}(p, \tau, \varepsilon), \quad p = \epsilon^{-1} S(\tau, \varepsilon),
\]

\[
(3.16b) \quad S(\tau, \varepsilon) = S_{-1}(\tau) + \epsilon \tilde{S}(\tau, \varepsilon),
\]

where \( \phi_N^*, U_0 \) and \( \tilde{U} \) belong to \( P^\infty \), \( S_{-1} \) belongs to \( C^\infty(I, R) \) and \( \tilde{S} \) to \( C^\infty(I \times [0, \varepsilon_0], R) \). \( S_{-1, \tau} \) is a strictly positive function on \( I \), thus

\[
S_{-1, \tau} \equiv \delta_0 > 0, \quad \tau \in I.
\]

Then the residual function \( g_N \) of \( \phi_N \) will belong to \( P_S \). The function \( U_0 \) is a solution of the ordinary differential equation

\[
(3.17) \quad S_{-1, \tau}^2(\tau) U_{0, pp} + F(U_0, \tau) = 0,
\]

in which \( \tau \) should be considered as a fixed parameter. Consider the first variational equation of (3.17) with respect to \( U_0 \),

\[
(3.18) \quad S_{-1, \tau}^2(\tau) z_{pp}^* + F_z(U_0(p, \tau), \tau) z^* = 0
\]

in which \( \tau \) should be considered again as a fixed parameter. Let \( z_i^*, i = 1, 2 \), be two linearly independent solutions of (3.18) which both belong to \( P^\infty \) and have a Wronskian \( D \),

\[
D(z_1^*, z_2^*) = z_1^* z_{2, p}^* - z_1^* z_2^* = 1.
\]

A complete description of \( z_1^* \) and \( z_2^* \) in terms of \( U_0 \) is given in Appendix B. From the functions \( z_1 \) and \( z_2 \) defined by

\[
z_i(\tau, \varepsilon) = z_i^*(p, \tau), \quad p = \epsilon^{-1} S(\tau, \varepsilon), \quad i = 1, 2,
\]

we form the matrix function \( W_\varepsilon \),

\[
W_\varepsilon(\tau) = \begin{pmatrix}
    z_1(\tau, \varepsilon) & z_2(\tau, \varepsilon) \\
    \epsilon^{-1} z_1'(\tau, \varepsilon) & \epsilon^{-1} z_2'(\tau, \varepsilon)
\end{pmatrix},
\]

which obviously belongs to \( P_S^\infty \).

We shall show that \( W_\varepsilon \) is a zeroth order formal asymptotic solution with residual function \( G_\varepsilon \) of a linear matrix equation of the type (3.11) where

\[
\nabla H_\varepsilon(u_\varepsilon, \tau) = \begin{pmatrix}
    0 & 1 \\
    -F_\varepsilon(\phi_N, \tau) & 0
\end{pmatrix} + \epsilon \begin{pmatrix}
    0 & 0 \\
    -f_1(\phi_N, \epsilon \phi_N', \tau, \varepsilon) & -f_2(\phi_N, \epsilon \phi_N', \tau, \varepsilon)
\end{pmatrix}
\]

The subscripts 1 and 2 indicate the partial derivatives with respect to the first, respectively second, independent variable of \( f \). Before calculating \( G_\varepsilon \) we note that

\[
(3.19) \quad W_\varepsilon(\tau) = W_\varepsilon^*(p, \tau) = \begin{pmatrix}
    z_1^* & z_2^* \\
    S_{-1, \tau} z_{1, p}^* & S_{-1, \tau} z_{2, p}^*
\end{pmatrix} + O(\varepsilon), \quad p = \epsilon^{-1} S(\tau, \varepsilon),
\]
and

$$\nabla H_e \{u_e(\tau), \tau \} = \nabla H_e \{u_e^+(p, \tau), \tau \} = \begin{pmatrix} 0 \\ -F_x \{ U_0(p, \tau), \tau \} \end{pmatrix} + O(\epsilon), \quad p = \epsilon^{-1} S(\tau, \epsilon).$$

Then we obtain for $G_e$,

$$\epsilon G_e(\tau) = \epsilon W_e(\tau) - \nabla H_e \{u_e(\tau), \tau \} W_e(\tau)$$

$$= S_{-1, r} W_{e, p}^* + \epsilon W_{e, r}^* - \nabla H_e \{u_e^+(p, \tau), \tau \} W_e^*$$

$$= S_{-1, r} \left( \begin{array}{c}
    z_{1, p}^* \\
    S_{-1, r} z_{1, pp}^* - S_{-1, r} z_{1, p}^* \end{array} \right) + O(\epsilon)$$

Thus we have

$$\det W_e(\tau) = S_{-1, r} D(z^*, z^*) + \epsilon (z^* z^* - z^* z^*) + O(\epsilon),$$

Hence, $\det W_e(\tau) \equiv \delta_0 > 0$ for $\tau \in I$. Thus we have shown that $W_e$ is a zeroth order formal asymptotic solution of a linear matrix equation of the type (3.11), which belongs to $P_s^\infty$. Furthermore, we note that condition (3.10) is satisfied in the present case.

We are now in a position to apply Theorem 2 to the system of first order equations in $\mathbb{R}^2$ associated with the second order problem (1.2). Making use of (3.19) we may rewrite condition (3.12) of Theorem 2 in the following way:

$$\int_0^{2\pi} \begin{pmatrix} S_{-1, r} z_{2, pp}^* - z_{2, p}^* \\
    -S_{-1, r} z_{1, pp}^* + z_{1, p}^* \\
    -S_{-1, r} z_{1, p}^* + z_{1, p}^* \\
    -S_{-1, r} z_{2, p}^* + z_{2, p}^* \\
    -S_{-1, r} z_{2, p}^* + z_{2, p}^* \\
    -S_{-1, r} z_{2, p}^* + z_{2, p}^* \\
    0 \\
    z_{1, p}^* \\
    z_{2, p}^* \\
    z_{1, p}^* \\
    z_{2, p}^* \\
    z_{1, p}^* \\
    z_{2, p}^* \\
    -z_{1, p}^* \\
    -z_{2, p}^* \\
    0 \\
    z_{1, p}^* \\
    z_{2, p}^* \\
    z_{1, p}^* \\
    z_{2, p}^* \\
    z_{1, p}^* \\
    z_{2, p}^* \\
    -z_{1, p}^* \\
    -z_{2, p}^* \\
    0 \\
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    z_{2, p}^* \\
    -z_{1, p}^* \\
    -z_{2, p}^* \\
    0 \\
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    -z_{2, p}^* \\
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    z_{2, p}^* \\
    -z_{1, p}^* \\
    -z_{2, p}^* \\
    0 \\
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    z_{2, p}^* \\
    z_{1, p}^* \\
    z_{2, p}^* \\
    z_{1, p}^* \\
    z_{2, p}^* \\
    -z_{1, p}^* \\
    -z_{2, p}^* \\
    0 \\
    z_{1, p}^* \\
Hence, in general, one should obtain first a first order asymptotic solution, \( \tilde{\phi}_1 \). The asymptotic validity of \( \phi_0 \) may be established by showing that

\[
\phi_0 - \tilde{\phi}_1 \quad \text{and} \quad \varepsilon (\phi'_0 - \tilde{\phi}'_1)
\]

are \( O(\varepsilon) \) on \( I \).

**4. A two variable construction technique.** We consider the class of nonlinear oscillation problems (1.2),

\[
\begin{align*}
(1.2a) & \quad e^2x'' + F(x, \tau) + \varepsilon f(x, \varepsilon x', \tau, \varepsilon) = 0, \quad \tau \geq 0, \\
(1.2b) & \quad x(0, \varepsilon) = \alpha_1(\varepsilon), \quad \varepsilon x'(0, \varepsilon) = \alpha_2(\varepsilon),
\end{align*}
\]

where

(i) \( F \in C^\infty(\mathbb{R} \times I, \mathbb{R}) \), \( f \in C^\infty(\mathbb{R}^2 \times I \times [0, \varepsilon_0], \mathbb{R}) \), \( \alpha_i \in C^\infty([0, \varepsilon_0]), i = 1, 2 \),

(ii) \( xF(x, \tau) > 0 \) for all \( \tau \in I \) and \( x \neq 0 \).

In the first part of this section we develop a two variable construction technique for an \( N \)th order formal asymptotic solution \( \phi_N \) of problem (1.2) on \( I \). It is this function for which we deduced in § 3 conditions on the residual function \( g_N \),

\[
\int_0^{2\pi} g_N^*(p, \tau, 0)z_1^*(p, \tau) \, dp = 0, \quad i = 1, 2,
\]

in order that \( \phi_N \) be an \( N \)th order asymptotic solution. In the second part we shall describe an algorithmic procedure for successively generating asymptotic solutions of increasing order.

The form of \( \phi_N \) is anticipated as a finite generalized asymptotic power series involving uniformly bounded functions of two variables \( p \) and \( \tau \):

\[
\begin{align*}
(4.1a) & \quad \phi_N(\tau, \varepsilon) = \sum_{\nu=0}^{N} \varepsilon^\nu U_\nu(p, \tau), \quad p = \varepsilon^{-1}S(\tau, \varepsilon; N), \\
(4.1b) & \quad S(\tau, \varepsilon; N) = \sum_{j=-1}^{N} \varepsilon^{j+1}S_j(\tau),
\end{align*}
\]

where we require that

(i) \( U_\nu \in C^\infty(\mathbb{R} \times I, \mathbb{R}), \nu = 0, 1, \ldots, N \),

(ii) \( U_\nu \) and all its partial derivatives with respect to \( p \) and \( \tau \) are uniformly bounded on \( \mathbb{R} \times I \),

(iii) \( S_j \in C^\infty(i, \mathbb{R}), j = -1, 0, \ldots, N \),

(iv) \( S_{-1, \tau} \equiv \delta_0 > 0 \) for \( \tau \in I \),

(v) \( S_{-1}(0) = 0 \).
We insert the series (4.1) into the left-hand side of equation (1.2a) and expand (1.2a) in powers of $\varepsilon$ in the following way:

$$
\varepsilon^2 \phi_N + F(\phi_N, \tau) + \varepsilon f(\phi_N, \varepsilon \phi_N, \tau, \varepsilon) = (S_{-1,\tau} + \varepsilon S_0 + \cdots)^2(U_{0,pp} + \varepsilon U_{1,pp} + \cdots) + \varepsilon(S_{-1,\tau} + \varepsilon S_0 + \cdots) \\
\cdot (U_{0,p} + \varepsilon U_{1,p} + \cdots) + 2\varepsilon(S_{-1,\tau} + \varepsilon S_0 + \cdots) \\
\cdot (U_{0,pr} + \varepsilon U_{1,pr} + \cdots) + \varepsilon^2(U_{0,pp} + \varepsilon U_{1,pp} + \cdots) \\
+ F(U_0, \tau) + \varepsilon F_x(U_0, \tau) U_1 + \cdots + \varepsilon f(U_0, S_{-1,\tau} U_{0,p}, \tau, 0) \\
+ \varepsilon^2 f_1(U_0, S_{-1,\tau} U_{0,p}, \tau, 0) U_1 + \varepsilon^2 f_2(U_0, S_{-1,\tau} U_{0,p}, \tau, 0) \\
\cdot (U_{0,\tau} + \varepsilon S_{-1,\tau} U_{1,p} + S_0 \tau U_{0,p}) \\
+ \varepsilon^2 f_3(U_0, S_{-1,\tau} U_{0,p}, \tau, 0) + \cdots,
$$

(4.2)

where $f_i$ denotes the partial derivative of $f$ with respect to its $i$th argument ($i = 1, 2, 4$). According to the definition of the $N$th order formal asymptotic solution of problem (1.2), the right-hand side of (4.2) should be $O(\varepsilon^{N+1})$. As a first step, therefore, we equate to zero the $O(1)$-contribution in (4.2). This yields an ordinary differential equation for $U_0$,

$$
S_{-1,\tau} U_{0,pp} + F(U_0, \tau) = 0, \quad (p, \tau) \in \mathbb{R} \times I,
$$

(4.3)

in which $\tau$ is to be considered as a fixed parameter.

Let us examine the existence of periodic solutions of equation (4.3). With respect to the independent variable $p$, (4.3) describes a conservative system with one degree of freedom. Because of assumption (ii) on $F$, the potential energy $V_0(U_0, \tau)$ has an absolute minimum at $U_0 = 0$ for all $\tau \in I$. The total mechanical energy $E_0$ of the system,

$$
E_0 = \frac{1}{2} U_{0,p}^2 + S_{-1,\tau} V_0(U_0, \tau),
$$

is said to be noncritical if $E_0$ is different from any of the values of the potential energy at its critical points uniformly for $\tau$ on $I$, that is,

$$
|S_{-1,\tau} V_0(U_0, \tau)| \equiv \delta_0 > 0 \quad \text{for all } \tau \text{ on } I
$$

at any point $U_0$ where $E_0 = S_{-1,\tau} V_0(U_0, \tau)$. Arnold [1] proved a theorem on the existence of periodic solutions of equations of the above type (4.3) with respect to $p$ for a noncritical value of the energy $E_0$. On a compact interval $[U^{(1)}, U^{(2)}]$ of the

---

1 A critical point of a function is a point where the derivative of that function vanishes.
$U_0$-axis for which

$$E_0 = S_{-1, \tau}^{-2} V_0(U_0^{(1)}, \tau) = S_{-1, \tau}^{-2} V_0(U_0^{(2)}, \tau)$$

and

$$S_{-1, \tau}^{-2} V_0(U_0, \tau) < E_0 \text{ for } U_0^{(1)} < U_0 < U_0^{(2)},$$

the solution of (4.3) is a periodic function oscillating between $U_0^{(1)}$ and $U_0^{(2)}$. The values of $E_0$ follow from the initial values of $U_0$ and $U_0, \tau$ with respect to $\tau$ which among other things depend on the initial conditions $\alpha_{1,2}(\epsilon)$ of the perturbed system. In view of assumption (ii) on $F$ there are noncritical values of the energy $E_0$ which imply a compact interval of the above type. At this moment we assume that the formal solution $\phi_N$ leads to such a value of $E_0$ for $\tau$ on $I$, e.g., by a suitable choice of the initial conditions $\alpha_{1,2}(\epsilon)$. Then (4.3) has periodic solutions which moreover oscillate between a negative and a positive value.

Equation (4.3) can be solved by quadratures. The “constants of integration” may be functions of $\tau$ and $\epsilon$. However, because of the required uniform boundedness of $U_0$ and its partial derivatives, the period of $U_0$ as a function of $\tau$ should be independent of $\tau$ and $\epsilon$. For instance, if the period $P$ depends on $\tau$, then we derive from

$$U_0(p + nP(\tau), \tau) = U_0(p, \tau), \quad n \text{ an integer},$$

that

$$U_0,_{\tau}(p, \tau) = nP, U_0,_{\tau}(p + nP(\tau), \tau) + U_0,_{p}(p + nP(\tau), \tau)$$

which is not uniformly bounded for $(p, \tau) \in \mathbb{R} \times I$ when $n$ tends to infinity. In solving equation (4.3) we shall show that this condition is satisfied for a special choice of $S_{-1}$.

We now set

$$U_0(p, \tau) = \eta(\tau) + A_0(\tau) \Phi_0(p, \tau),$$

where $A_0$ figures as a constant of integration and where $\Phi_0$ is a periodic function of its first argument which oscillates between the values $-1$ and $+1$. The function $A_0$ has the meaning of an amplitude function and $\eta$ is the algebraic average of the extreme values of $U_0$. Note that $\eta$ is not equal to the averaged values of $U_0$ over one period in $p$ in general. Note furthermore that the second constant of integration which corresponds to a shift in the phase $p$ has been absorbed in $p = \epsilon^{-1} S$. This may be done without loss of generality since $S$ itself has not yet been determined at this stage of the construction of $\phi_N$.

Substituting (4.4) into (4.3), multiplying by $A_0 \Phi_{0,p}$ and integrating once with respect to $p$, we get

$$\frac{1}{2} S_{-1, \tau}^{-2} S^2 A_0 \Phi_{0,p}^2 + V_0(\eta + A_0 \Phi_0, \tau) = V_0(\eta + A_0, \tau),$$

The functional dependence of $\eta$ on $A_0$ and $\tau$ follows from the equality

$$V_0(\eta + A_0, \tau) = V_0(\eta - A_0, \tau)$$

and will be indicated by

$$\eta(\tau) = \tilde{\eta}(A_0(\tau), \tau).$$
Normalizing the period of $\Phi_0$ to $2\pi$ we get the dispersion relation between $S_{-1,\tau}$ and $A_0$ which is obtained by integration of (4.5)

$$ S_{-1,\tau}(\tau) = \tilde{\omega}(A_0(\tau), \tau) = \omega(\tau), $$

where

$$ \tilde{\omega}(A_0, \tau) = \pi \sqrt{2} \left( \int_{-1}^{1} A_0(\xi) \frac{d\xi}{\sqrt{V_0(\tilde{\eta}(A_0, \tau) + A_0, \tau)}} \right)^{-1}. $$

(4.7a)

(4.7b)

The function $\Phi_0$ is of the form

$$ \Phi_0(p, \tau) = \Phi_0[p, A_0(\tau), \tau], $$

where $\Phi_0(p, A_0, \tau)$ is an even $2\pi$-periodic function of $p$ for which the following implicit representation may be obtained by integration of (4.5):

$$ p = 2^{-1/2} \tilde{\omega}(A_0, \tau) \int_{\Phi_0}^{1} A_0 d\xi/[V_0(\tilde{\eta}(A_0, \tau) + A_0, \tau)]^{1/2}, \quad \tau \in [0, \pi]. $$

(4.8a)

(4.8b)

We require that $A_0$ belongs to the class $C^\infty(I, \mathbb{R})$. Then it is seen that $U_0$ belongs to $P^\infty$ because of the smoothness of all functions involved.

Summarizing, we have up to now the following expression for $\phi_N$:

$$ \phi_N(\tau, \varepsilon) = \eta(\tau) + A_0(\tau)F_0(p, \tau) + \sum_{\nu=1}^{N} \varepsilon^\nu U_\nu(p, \tau), \quad p = \varepsilon^{-1} S(\tau, \varepsilon; N), $$

(4.9a)

$$ S(\tau, \varepsilon; N) = \int_{0}^{\tau} \omega(\sigma) d\sigma + \varepsilon \sum_{j=0}^{N} \varepsilon^j S_j(\tau) $$

(4.9b)

Using (4.9), we may replace (4.2) by

$$ \varepsilon^2 \phi'' + F(\phi_N, \tau) + \varepsilon f(\phi_N, \omega \phi_N', \tau, \varepsilon) $$

(4.10)

$$ = \sum_{\nu=1}^{N} \varepsilon^\nu [\omega^2 U_{\nu,pp} + F_x(\eta + A_0 \Phi_0, \tau) U_{\nu} + \gamma_{\nu}] + \varepsilon^{N+1} \delta^{N+1}, $$

where

$$ \gamma_1(p, \tau) = 2\omega S_{0,0} A_0 \Phi_{0,pp} + \omega_0 A_0 \Phi_{0,p} + 2\omega (A_0 \Phi_{0,p})_\tau + f^{(0)}(p, \tau), $$

$$ \gamma_2(p, \tau) = (2\omega S_{1,0} + S_{0,0}^2) A_0 \Phi_{0,pp} + 2\omega S_{0,0} U_{1,pp} + S_{0,0} (A_0 \Phi_{0,p})_\tau + 2S_{0,0} (A_0 \Phi_{0,p}) + \omega U_{1,p} + 2\omega U_{1,\tau} + \eta_{\tau} + (A_0 \Phi_0)_{\tau\tau} + f^{(1)}(p, \tau) + F^{(2)}(p, \tau), $$

---

In the theory of wave propagation (governed by partial differential equations), a dispersion relation is a relation between typical quantities of a uniform wave, like frequency, wavenumber(s), amplitude, etc. In the present case this relation only involves the frequency and the amplitude.
\[ \gamma_{\nu}(p, \tau) = (2\omega S_{\nu-1,\tau} + \sum_{i+j=\nu-2} S_{i,\nu} S_{j,\tau}) A_0 \Phi_{0,pp} + \sum_{i+j=\nu-1} (2\omega S_{\nu-1,\tau} + \sum_{k=1}^{\nu-1} S_{k,\tau} S_{1,\tau}) U_{i,pp} + S_{\nu-2,\tau} A_0 \Phi_{0,p} + 2S_{\nu-2,\tau} (A_0 \Phi_{0,p})_\tau + \omega \Upp{\nu}{p} + 2\omega \Upp{\nu-1}{p} + \sum_{i+j=\nu-2} (S_{i,\tau} U_{i,p} + 2S_{i,\tau} U_{i,\tau}) + \Upp{\nu-1}{\tau} + F^{(\nu)}(p, \tau), \quad \nu = 3, \ldots, N + 1, \]
\[ g_{N+1}(p, \tau, \epsilon) = \gamma_{N+1}(p, \tau) + O(\epsilon), \text{ uniformly for } (p, \tau) \in \mathbb{R} \times I. \]

The functions \( f^{(\nu)}, \nu = 0, 1, \ldots, N \), denote the coefficients in the expansion of \( f \)

\[ f(\phi_N, \epsilon \phi'_N, \tau, \epsilon) = \sum_{\nu=0}^{N} \epsilon^{\nu + 1} r_{N+1}^*(p, \tau, \epsilon) \]

with remainder \( \epsilon^{N+1} r_{N+1}^*(\epsilon^{-1}S(\tau, \epsilon; N), \tau, \epsilon) = r_N(\tau, \epsilon) = O(1) \) holds. The expansion of \( F \) is as follows:

\[ F(\phi_N, \tau) = F(\eta + A_0 \Phi_0, \tau) + \epsilon F_x(\eta + A_0 \Phi_0, \tau) U_1 + \sum_{\nu=1}^{N} \epsilon^{\nu} [F_x(\eta + A_0 \Phi_0, \tau) U_\nu + F^{(\nu)}(p, \tau)] + \epsilon^{N+1} F^{(N+1)}(p, \tau) + \epsilon^{N+2} R_{N+1}^*(p, \tau, \epsilon), \]

with remainder \( \epsilon^{N+2} R_{N+1}^* \) for which \( R_{N+1}^*(\epsilon^{-1}S(\tau, \epsilon; N), \tau, \epsilon) = R_{N+1}(\tau, \epsilon) = O(1) \) holds. It should be noted that, although \( p \) represents a function depending also on \( \epsilon \), the functions \( \Phi_0, U_\nu(\nu = 1, \ldots, N) \) and their partial derivatives with respect to \( p \) and \( \tau \) have not been expanded in powers of \( \epsilon \). This is allowed because of the generalized asymptotic power series expansions used in this construction procedure. In view of the conditions on \( f \) and \( F \) and the requirements made for all quantities in (4.9) it follows that \( f^{(\nu)}, \nu = 0, 1, 2, \ldots, N, \) and \( F^{(\nu)}, \nu = 2, 3, 4, \ldots, N + 1, \) satisfy assumptions (i) and (ii) for \( U_\nu \).

Since

\[ f^{(0)}(p, \tau) = f(\eta + A_0 \Phi_0, \omega A_0 \Phi_{0,p}, \tau, 0), \]

it follows that \( f^{(0)} \) belongs to \( P^\infty \). In view of (4.6), (4.7) and (4.8) we may write

\[ f^{(0)}(p, \tau) = f^{(0)}(p, A_0(\tau), \tau). \]

Proceeding with the construction, we equate to zero successively the contributions of \( O(\epsilon), \ldots, O(\epsilon^N) \) to the right-hand side of (4.10). This yields a set of recurrent equations for \( U_1, \ldots, U_N \). If these equations have solutions \( U_1, \ldots, U_N \) satisfying requirements (i) and (ii), then the function \( g_N, \)

\[ g_N(\tau, \epsilon) = g_N^*(p, \tau, \epsilon) = \epsilon^{-1} \int_{0}^{\tau} \omega(\sigma) d\sigma + \sum_{j=0}^{N} \epsilon^j S_j(\tau), \]

...
is uniformly bounded on $I$. This is a necessary condition for $\phi_N$ to be an $N$th order formal asymptotic solution of problem (1.2) on $I$.

The required uniform boundedness of $U_\nu$, $\nu = 1, \ldots, N$, leads to equations for $A_0, S_0, S_1, \ldots, S_{N-1}$. However, this set of functions does not suffice to achieve the uniform boundedness of $U_\nu$, $\nu = 1, \ldots, N$. In the construction procedure other unknown functions $A_1, \ldots, A_N$ appear which should belong to $C^\infty(I, \mathbb{R})$. They provide a sufficient additional degree of freedom. We will illustrate this explicitly by the determination of $U_1$.

The function $U_1$ satisfies the linear ordinary differential equation

$$\omega^2 U_{1,pp} + F_2(\eta + A_0 \Phi_0, \tau) U_1 = -\gamma_1(p, \tau).$$

In consequence of the results obtained above $\gamma_1$ belongs to the class $P^\infty$. The homogeneous equation corresponding to (4.11) is the first variational equation of equation (4.3) with respect to $U_0$. In the Appendix B two linearly independent solutions $z_1^*$ and $z_2^*$ of the homogeneous equation are determined in terms of $U_0$. The functions $z_1^*$ and $z_2^*$ are odd, respectively even, $2\pi$-periodic functions of $p$ with a Wronskian equal to 1. Both $z_1^*$ and $z_2^*$ belong to $P^\infty$ and have the alternative representation

$$z_i^*(p, \tau) = z_i^i(p, A_0(\tau), \tau), \quad i = 1, 2.$$

The requirement that $U_1$ should satisfy the uniform boundedness condition (ii) leads to the condition on $\gamma_1$,

$$\int_0^{2\pi} \gamma_1(p, \tau) z_i^*(p, \tau) \, dp = 0, \quad i = 1, 2.$$

This condition, known as the suppression of "secular terms" in the determination of $U_1$, gives first order differential equations from which $A_0$ and $S_0$ may be determined. We define

$$\int_0^{2\pi} f^{(0)}(p, A_0, \tau) z_i^*(p, A_0, \tau) \, dp = \hat{\lambda}_i^{(1)}(A_0, \tau), \quad i = 1, 2,$$

$$\frac{1}{A_0} \int_0^{2\pi} \{z_i^*(p, A_0, \tau)\}^2 \, dp = \bar{L}(A_0, \tau),$$

and we note that

$$A_0 \int_0^{2\pi} \Phi_{0,pp}(p, A_0, \tau) z_2^*(p, A_0, \tau) \, dp = \int_0^{2\pi} z_1^i p z_2^i \, dp$$

$$= -\int_0^{2\pi} z_1^i z_2^i \, dp = -\int_0^{2\pi} (1 + z_1^i p z_2^i) \, dp,$$

so that

$$A_0 \int_0^{2\pi} \Phi_{0,pp}(p, A_0, \tau) z_2^*(p, A_0, \tau) \, dp = -\pi.$$
Then, taking into account that the function $\gamma_1$ consists of even terms in $p$ as well as odd terms, the suppression of secular terms reduces to the equations

\begin{align}
(4.12a) & \quad \frac{d}{d\tau}\{A_0 \omega(A_0, \tau) \tilde{L}(A_0, \tau)\} = -\lambda_1^{(1)}(A_0, \tau), \\
(4.12b) & \quad 2\pi \omega(A_0, \tau) S_{0,\tau} = \tilde{S}_2^{(1)}(A_0, \tau).
\end{align}

Equation (4.12a) is a nonlinear equation for $A_0$ whereas (4.12b) is a linear one for $S_0$. It is essential for the construction procedure that equation (4.12a) possess a solution on $I$ which belongs to $C^\infty(I, \mathbb{R})$.

In order to determine a particular solution $\Phi_1$ of (4.11) we define functions $a_i^{(1)}(p, \tau)$ by

\begin{equation}
(4.13) \quad a_i^{(1)} = \frac{1}{\omega_2} \gamma_1 z_i^*, \quad i = 1, 2.
\end{equation}

then the method of variation of constants yields

\[ \Phi_1(p, \tau) = a_2^{(1)}(p, \tau) z_1^*(p, \tau) - a_1^{(1)}(p, \tau) z_2^*(p, \tau). \]

Because of the suppression of secular terms in $\gamma_1$, $\Phi_1$ is a $2\pi$-periodic function of $p$. The "constants of integration" in $a_i^{(1)}$ can be chosen in such a way that

\[ \int_0^{2\pi} \Phi_1(p, \tau) z_i^*(p, \tau) \, dp = 0, \quad i = 1, 2, \]

in other words, $\Phi_1$ does not contain secular terms. This is convenient in later stages of the construction procedure.

The solution $U_1$ of (4.11) is taken to be

\[ U_1(p, \tau) = A_1(\tau) z_2^*(p, \tau) + \Phi_1(p, \tau). \]

If we require that $A_1$ belong to $C^\infty(I, \mathbb{R})$, then $U_1$ belongs to $P^\infty$ because of the smoothness of all functions involved and the suppression of secular terms in $\gamma_1$.

It should be noted that a homogeneous solution proportional to $z_1^* = A_0 \Phi_0 p$ need not be included in $U_1$ since this has already been taken into account in $U_0$ because of the assumed expansion (4.1b) of $S$.

The determination of the functions $U_\nu, \nu = 2, 3, \cdots, N$, is completely analogous to that of $U_1$. Each time we first suppress the secular terms in the right-hand side $-\gamma_\nu$ which depends on $U_0, U_1, \cdots, U_{\nu-1}$. This yields equations for $S_{\nu-1}$ and $A_{\nu-1}$. In addition to the particular solution $\Phi_\nu$, which should satisfy the conditions

\[ \int_0^{2\pi} \Phi_\nu(p, \tau) z_i^*(p, \tau) \, dp = 0, \quad i = 1, 2, \]

the function $U_\nu$ should contain a term proportional to $z_2^*$

\[ U_\nu(p, \tau) = A_\nu(\tau) z_2^*(p, \tau) + \Phi_\nu(p, \tau). \]

If we require that $A_\nu \in C^\infty(I, \mathbb{R})$, then $U_\nu$ belongs to $P^\infty$. Together with $S_\nu$, the function $A_\nu$ may serve as a means of suppressing the secular terms in the right-hand side of the equation for $U_{\nu+1}$.
We shall restrict ourselves to an explicit determination of the equations for $S_1$ and $A_1$. These equations follow from the suppression of secular terms in $\gamma_2$. In view of the results obtained above $\gamma_2$ is of the form

$$\gamma_2(p, \tau) = (2\omega S_{0,1} + S_{0,2}^2)A_0\Phi_{0,pp} + 2\omega S_{0,1}A_1z_{2,pp}^* + S_{0,\tau}A_0\Phi_{0,p}$$

$$+ 2S_{0,\tau}(A_0\Phi_{0,p}) + \omega_1A_1z_{2,p}^* + 2\omega(A_1z_{2,p}^*)\tau + \eta_{\tau} + (A_0\Phi_0)_{\tau\tau}$$

$$+ f^{(1)}(p, \tau) + F^{(2)}(p, \tau)$$

+ functions proportional to $\Phi_1$ and its partial derivatives,

where

$$f^{(1)}(p, \tau) = f_1(\eta + A_0\Phi_0, \omega A_0\Phi_{0,p}, \tau, 0)(A_1z_{2}^* + \Phi_1)$$

$$+ f_2(\eta + A_0\Phi_0, \omega A_0\Phi_{0,p}, \tau, 0) \cdot \{ \eta_{\tau} + (A_0\Phi_0)_\tau + S_{0,\tau}A_0\Phi_{0,p} + \omega_1A_1z_{2,p}^* + \omega\Phi_{1,p} \}$$

$$+ f_4(\eta + A_0\Phi_0, \omega A_0\Phi_{0,p}, \tau, 0),$$

$$F^{(2)}(p, \tau) = \frac{1}{2}F_{xx}(\eta + A_0\Phi_0, \tau)(A_1z_{2}^* + \Phi_1)^2.$$

The functions proportional to $\Phi_1$ and its partial derivatives which have been left unspecified in the expression for $\gamma_2$ do not play a role in the suppression of secular terms in $\gamma_2$. We define

$$\int_0^{2\pi} f^{(1)}(p, \tau)z_{1}^*(p, \tau) \, dp = \lambda_i^{(2)}(\tau), \quad i = 1, 2,$$

and

$$\int_0^{2\pi} F^{(2)}(p, \tau)z_{1}^*(p, \tau) \, dp = \Lambda_i^{(2)}(\tau), \quad i = 1, 2.$$

Then the suppression of secular terms in $\gamma_2$ yields the following equations for $S_1$ and $A_1$:

$$2\pi\omega A_{1,1} + \pi\omega A_1 + 2\omega A_1 \int_0^{2\pi} z_{2,pp}^*z_{1}^* \, dp = \frac{d}{d\tau}(A_0S_{0,\tau}\tilde{\tau}) - \Lambda_1^{(2)} - \lambda_1^{(2)},$$

$$-2\pi\omega S_{1,1} + 2\omega A_1S_{0,\tau} \int_0^{2\pi} z_{2,pp}^*z_{2}^* \, dp = -\int_0^{2\pi} (A_0\Phi_0)_{\tau\tau}z_{2}^* \, dp - \Lambda_2^{(2)} - \lambda_2^{(2)}.$$

Since $F^{(2)}$ and hence, $\Lambda_i^{(2)}$, depend on $A_1$ in a nonlinear way, (4.14a) for $A_1$ is a nonlinear equation. On the other hand, (4.14b) for $S_1$ is linear. It is straightforward to check that the equations for $A_\nu$ and $S_\nu$, $\nu = 2, 3, \cdots, N - 1$, are linear.

When the calculation of $U_1, \cdots, U_N$ and the determination of the equations for $A_0, S_0, A_1, S_1, \cdots, A_{N-1}, S_{N-1}$ have been performed, we obtain an $N$th order formal asymptotic solution $\phi_N$ of the form

$$\phi_N(\tau, \varepsilon) = \eta(\tau) + A_0\Phi_0(\tau, \varepsilon) + \sum_{\nu=1}^N \varepsilon^\nu [A_\nu(\tau)z_{\nu}^*(p, \tau) + \Phi_\nu(p, \tau)],$$

$$p = \varepsilon^{-1}S(\tau, \varepsilon; N) = \varepsilon^{-1} \int_0^\tau \omega(\sigma) \, d\sigma + \sum_{j=0}^N \varepsilon^j S_j(\tau),$$

where

$$S_j(\tau) = \int_0^\tau \omega(\sigma) \, d\sigma.$$
in which no equations for $A_N$ and $S_N$ have been determined yet. In view of the requirements that $A_\nu$ and $S_\nu$, $\nu = 0, \ldots, N$, belong to $C^\infty(I, \mathbb{R})$, $\phi_N$ belongs to $P_2^\infty$. Consequently, the residual function $g_N$ of $\phi_N$ belongs to $P_2^\infty$. From the definition of $g_N$ in the expansion (4.10) the condition (3.20) in order that $\phi_N$ be an asymptotic solution uniformly valid on $I$ corresponds to the suppression of secular terms in $\gamma_{N+1}$:

$$
\int_0^{2\pi} \gamma_{N+1}(p, \tau) z_i^*(p, \tau) \, dp = 0, \quad i = 1, 2.
$$

This yields two linear ordinary differential equations for $A_N$ and $S_N$. In order to obtain the initial values for $A_\nu$ and $S_\nu$, $\nu = 0, \ldots, N$, we insert the expansion (4.15) into the expressions

$$
\alpha_1(\varepsilon) - \phi_N(0, \varepsilon) \quad \text{and} \quad \alpha_2(\varepsilon) - \varepsilon \phi'_N(0, \varepsilon)
$$

and expand them completely in powers of $\varepsilon$. Then the required initial conditions follow from equating to zero the contributions of $O(\varepsilon^r)$, $\nu = 0, \ldots, N$, to these expansions.

If the nonlinear initial value problems for $A_0$ and $A_1$ have solutions belonging to $C^\infty(I, \mathbb{R})$, then, by virtue of the theory of linear ordinary differential equations, we know that $A_\nu$, $\nu = 2, \ldots, N$, and $S_\nu$, $\nu = 0, \ldots, N$, are functions belonging to $C^\infty(I, \mathbb{R})$. Moreover, if, e.g., for a suitable choice of $\alpha_{1,2}(\varepsilon)$, the nonlinear problem (4.12a) for $A_0$ could be solved in such a way that the total mechanical energy $E_0$ is noncritical and that $E_0$ implies the bounded interval $(\eta - A_0, \eta + A_0)$ of the $U_0$-axis on which the potential energy is less than $E_0$ for all $\tau$ on $I$, then the solution $U_0$ of (4.3) is a periodic function. This completes the construction of an $N$th order asymptotic solution $\phi_N$ of (1.2) for $N \geq 1$. According to the remark following Theorem 3 the correctness of $\phi_0 = \eta + A_0 \Phi_0$ as a zeroth order asymptotic solution is established if we are able to show that the equation for $A_1$ has a solution belonging to $C^\infty(I, \mathbb{R})$.

From the $N$th order asymptotic solution $\phi_N$ constructed above we may obtain any lower order asymptotic solution $\phi_k$ ($k < N$) by deleting from $\phi_N$ the terms with $\Phi_{k+1}, \ldots, \Phi_N, A_{k+1}, \ldots, A_n$ and $S_{k+1}, \ldots, S_N$. If we compare the expansion for $\phi_k$ obtained in this way with the expansion of $\phi_k$ obtained by a calculation starting from the beginning as described for $\phi_N$ in this section, then we see that both results are exactly the same. This is a result of the fact that all of the functions $\Phi_\nu$, $A_\nu$, $\nu = 0, \ldots, N$, and $S_\nu$, $\nu = -1, \ldots, N$, do not depend on $N$. There is only a difference in the argument by which the equations for $A_k$ and $S_k$ are determined. In the result obtained by truncating the expansions in $\phi_N$ the equations follow from the suppression of secular terms in the right-hand side of the equation for $U_{k+1}$, whereas in the independent calculation of $\phi_k$ starting from the beginning the equations for $A_k$ and $S_k$ follow from condition (3.20) of Theorem 3 on the residual function. In view of this equality in the determination of asymptotic solutions we may indicate an algorithm which successively generates a sequence of asymptotic solutions $\phi_0, \phi_1, \ldots$, etc. This can be done by
putting $N = \infty$ in (4.9):

\begin{align}
(4.16a) \quad \phi(\tau, \varepsilon) & \sim \eta(\tau) + A_0(\tau) \Phi_0(p, \tau) + \sum_{\nu=1}^{\infty} \varepsilon^\nu U_\nu(p, \tau), \\
(4.16b) \quad S(\tau, \varepsilon) & \sim \int_0^{\tau} \omega(\sigma) \, d\sigma + \varepsilon \sum_{j=0}^{\infty} \varepsilon^j S_j(\tau),
\end{align}

where $\eta$, $\omega$ and $\Phi_0$ are given by (4.6), (4.7) and (4.8), respectively. The calculation proceeds in the same way as described above for $\phi_N$. Successively, we determine the functions $A_0$, $S_0$, $A_1$, $S_1$, $\Phi_2$, \ldots.

In order to obtain a $k$th order asymptotic solution, the algorithm is terminated after the determination of $A_k$ and $S_k$, and $\phi_k$ is then given by (4.16) truncated after $\nu = k$ and $j = k$.

5. A perturbed Duffing equation. As an application of the algorithmic procedure of § 4 we determine a zeroth order asymptotic solution of a perturbed Duffing equation with slowly varying coefficients

\begin{align}
(5.1a) \quad \varepsilon^2 x'' + a(\tau)x + b(\tau)x^3 + \varepsilon c(\tau)x^2 + \varepsilon^2 d(\tau)x' = 0, \quad \tau \in I,
\end{align}

with initial conditions

\begin{align}
(5.1b) \quad x(0, \varepsilon) = \alpha_1(\varepsilon), \quad \varepsilon x'(0, \varepsilon) = 0.
\end{align}

We assume that $a$, $b$, $c$ and $d$ belong to $C^\infty(I, \mathbb{R})$ and that $a$ and $b$ are strictly positive on $I$. Then, equation (5.1) is an example of the class of oscillation problems (1.2) as considered in § 4. The cases in which $a$ and $b$ have signs on $I$ which differ from the above could be treated in a similar way, cf. Kaper [6]. In [7] Kuzmak constructed a first order formal asymptotic solution of the Duffing equation with slowly varying coefficients ($c = d = 0$ in (5.1)). He succeeded in the suppression of secular terms in the equation for $U_1$ with the introduction of only one degree of freedom. This is due to the special form of his example and without a reference to the special choice of the initial conditions. This may be verified below in the equation for $S_0$ when $c = d = 0$ and $S_0(0) = 0$ which implies $S_0 = 0$.

Because of the symmetry of $V_0$,

\begin{align}
V_0(x, \tau) = \frac{1}{2} a(\tau)x^2 + \frac{1}{4} b(\tau)x^4,
\end{align}

with respect to $x = 0$, the function $\eta$ in the infinite series (4.16) is equal to zero. The even, $2\pi$-periodic solution $\Phi_0$ of the $\tau$-parameter equation

\begin{align}
(5.2) \quad \omega^2(\tau)A_0(\tau)\Phi_{0,pp} + a(\tau)A_0(\tau)\Phi_0 + b(\tau)A_0^3(\tau)\Phi_0^3 = 0
\end{align}

can be expressed in terms of the Jacobian elliptic function $cn$ with modulus $\lambda$, $0 < \lambda^2 < 1$. The differential equation satisfied by $cn$ is

\begin{align}
(cn'' + (1 - 2\lambda^2)cn + 2\lambda^2 cn^3 = 0.
\end{align}
If we put
\[ \Phi_0(p, \tau) = \text{cn} \left[ \frac{2K(\lambda)}{\pi} p; \lambda \right], \]
then \( \Phi_0 \) is the even, \( 2\pi \)-periodic solution of (5.2) if
\begin{align*}
(5.3a) \quad a - (1 - 2\lambda^2) \omega^2 \frac{4K^2}{\pi^2} &= 0, \\
(5.3b) \quad bA_0^2 - 2\lambda^2 \omega^2 \frac{4K^2}{\pi^2} &= 0,
\end{align*}
where \( K \) denotes the complete elliptic integral of the first kind with modulus \( \lambda \),
\[ K(\lambda) = \int_0^1 \frac{(1 - \xi^2)(1 - \lambda^2 \xi^2)^{-1/2}}{d\xi}. \]
Eliminating \( \lambda \) from (5.3) we get the dispersion relation between \( \omega \) and \( A_0 \)
\begin{align*}
(5.5a) \quad \omega(\tau &= \tilde{\omega}\{A_0(\tau), \tau\}, \\
(5.5b) \quad \lambda^2 = \frac{1}{2} b(\tau)A_0^2(\lambda) \{a(\tau) + b(\tau)A_0^2(\lambda)\}^{-1}.
\end{align*}
As \( a \) and \( b \) are positive functions it follows that \( 0 < \lambda^2 < \frac{1}{2} \). This implies that
\( \Phi_0 \) is a \( 2\pi \)-periodic function of \( p \) indeed. The odd and even \( 2\pi \)-periodic solutions
\( z_1^* \), respectively \( z_2^* \), of the first variational equation of (5.2) with respect to \( A_0 \),
\[ \omega^2(\tau)z_{pp} + \left\{a(\tau) + 3b(\tau)A_0^2(\tau) \text{cn} \left[ \frac{2K(\lambda)}{\pi} p; \lambda \right] \right\} z^* = 0, \]
can also be expressed in terms of the Jacobian elliptic functions:
\[ z_1^*(p, \tau) = A_0(\tau) \left[ \frac{2K(\lambda)}{\pi} \text{cn} \left[ \frac{2K(\lambda)}{\pi} p; \lambda \right] \right], \]
and
\[ z_2^*(p, \tau) = H\left\{A_0(\tau) \text{cn} \left[ \frac{2K(\lambda)}{\pi} p; \lambda \right], A_0(\tau), \tau\right\}, \]
where (cf. Appendix B)
\[ H(\xi, A_0, \tau) = \frac{1}{2} \tilde{\omega}^2(A_0, \tau)\left\{\frac{1}{2} a(\tau)(A_0^2 - \xi^2) + \frac{1}{4} b(\tau)(A_0^4 - \xi^4)\right\}^{1/2} \]
\[ \cdot \int_0^\xi \left\{\frac{1}{2} a(\tau)(A_0^2 - \xi^2) + \frac{1}{4} b(\tau)(A_0^4 - \xi^4)\right\}^{-3/2} d\xi. \]
We substitute (4.16) in the left-hand side of (5.1a), expand the expression
considered as a function of \( p, \tau \) and \( \epsilon \) in an asymptotic power series with respect to
and equate to zero the $O(\epsilon)$-contribution. This leads to the equation for $U_1$:

$$\omega^2(\tau) U_{1,pp} + \left\{ a(\tau) + 3b(\tau)A_0^2(\tau) \right\} \frac{2K}{\pi} \frac{p}{\lambda} U_1 = -\gamma_1(p, \tau),$$

where

$$\gamma_1 = 2\omega S_0, r A_0 \frac{4K^2}{\pi^2} \frac{p}{\lambda} + \omega \frac{2K}{\pi} \frac{p}{\lambda}$$

and

The secular terms in the equation for $U_1$ are suppressed if $A_0$ and $S_0$ satisfy the first order differential equations:

$$\frac{d}{d\tau} [A_0(\tau) \tilde{o} A_0(\tau) A_0(\tau) \tilde{A} A_0(\tau) \tilde{A}] = -d(\tau) A_0(\tau) \tilde{o} A_0(\tau) \tilde{A} A_0(\tau) \tilde{A} A_0(\tau) \tilde{A}$$

and

$$2\pi \tilde{o} A_0(\tau) S_0, r(\tau) = 4c(\tau) A_0^2(\tau) \frac{\pi}{2K(\lambda)} \int_0^{K(\lambda)} \frac{\pi}{c} [u; \lambda] \cdot H[A_0(\tau) [u; \lambda], A_0(\tau), \tau] d\tau,$$

where

$$\tilde{A}(A_0, \tau) = 4 \frac{2K(\lambda)}{\pi} A_0 \int_0^{K(\lambda)} \frac{1}{\{c[n[u]; \lambda]\}^2} du.$$

The initial values for $A_0$ and $S_0$ follow from the asymptotic power series expansion of the expressions:

$$\alpha - \phi(0, \epsilon) \quad \text{and} \quad \epsilon \phi'(0, \epsilon)$$

where $\phi$ is given by (4.16). At this stage these expansions are known up to $O(\epsilon)$. Equating to zero the $O(1)$-contribution we get

$$A_0(0) = \alpha \quad \text{and} \quad S_0(0) = 0.$$

Solving the nonlinear equation (5.6a) we arrive at the nonlinear expression for $A_0$:

$$A_0^2 \tilde{o}(A_0, \tau) K(\lambda) L_1(\lambda) = \alpha^2 \tilde{o}(\alpha, 0) K(\lambda_0) L_1(\lambda_0) \exp \left[ -\int_0^\tau d(\sigma) d\sigma \right],$$

where

$$\lambda_0^2 = \frac{1}{2} b(0) \alpha^2 \right[ a(0) + b(0) \alpha^2 \right]^{-1}$$

and

$$L_1(\lambda) = \int_0^{K(\lambda)} \{c[n[u]; \lambda]\}^2 du.$$
Integrating equation (5.6b) we get

\[(5.7b)\]

\[S_0(\tau) = \int_0^\tau \frac{c(\sigma)A_0^2(\sigma) d\sigma}{\omega[A_0(\sigma), \sigma]K(\lambda)} \int_0^{K(\lambda)} \text{cn}^2[u; \lambda]H[A_0(\sigma)\text{cn}[u; \lambda], A_0(\sigma), \sigma] du.\]

For a zeroth order asymptotic solution \(\phi_0\) the procedure should be terminated after the determination of \(A_0\) and \(S_0\). Then, \(\phi_0\) is obtained from the formal series (4.16) by truncating after \(A_0\) and \(S_0\):

\[\phi_0(\tau, \varepsilon) = A_0(\tau)\text{cn}\left[\frac{2K(\lambda)}{\pi} p; \lambda\right], \quad p = \frac{1}{\varepsilon} S(\tau, \varepsilon),\]

\[S(\tau, \varepsilon) = \int_0^\tau \omega(\sigma) d\sigma + \varepsilon S_0(\tau),\]

with \(K, \omega, \lambda, A_0, S_0\) are given by (5.4), (5.5) and (5.7) respectively.

In order to examine the solvability of expression (5.7a) for \(A_0\) we rewrite it as an expression involving \(\lambda^2\). With the help of (5.5) we find after some simple transformations

\[(5.8) \quad p_1(\lambda) = \beta(\tau)p_1(\lambda_0), \quad \tau \in I,\]

where

\[p_1(\lambda) = \frac{\lambda^2 L_1(\lambda)}{(1 - 2\lambda^2)^{3/2}}\]

and

\[\beta(\tau) = \frac{b(\tau)}{b(0)} \left[\frac{a(0)}{a(\tau)}\right]^{3/2} \exp\left[-\int_0^\tau d(\sigma) d\sigma\right].\]

For \(a(\tau) > 0\) and \(b(\tau) > 0\), \(\tau \in I\), we have \(0 < \lambda^2 < 1/2\) for all values of the amplitude \(A_0(\tau)\). Hence, we need to examine the solvability of (5.8) on \((0, \frac{1}{2})\) as \(\tau\) takes on values on \(I\).

In order to calculate the derivative of \(p_1\) with respect to \(\lambda\) we use the equality

\[L_1(\lambda) = \int_0^{K(\lambda)} \text{cn}^2[u; \lambda] du = \int_0^1 t^2 \frac{(1 - \lambda^2 t^2)^{1/2}}{(1 - t^2)^{1/2}} \frac{dt}{\sqrt{1 - \lambda^2 t^2}}.\]

Then,

\[\frac{dp_1}{d\lambda} = \frac{\lambda^2}{(1 - 2\lambda^2)^{3/2}} \int_0^1 \frac{t^2}{(1 - t^2)^{1/2}} \left[2(1 + \lambda^2) - t^2(3\lambda^2 - \lambda^4)\right] \frac{dt}{\sqrt{1 - \lambda^2 t^2}}.\]

The expression between square brackets could be estimated on \(0 \leq t \leq 1\) in the following way:

\[2(1 + \lambda^2) - t^2(3\lambda^2 - \lambda^4) \geq \lambda^4 - \lambda^2 + 2 \geq 1.\]

It follows that

\[\frac{d}{d\lambda} p_1 > 0 \quad \text{for} \quad 0 < \lambda^2 < 1/2.\]
Thus, $p_1$ is a monotonically increasing function from zero to infinity as $\lambda^2$ increases from 0 to $1/2$.

Note that $\beta(0) = 1$ and that $\beta(\tau) > 0$ for $\tau \in I$. Then, for each positive value of $A_0(0) = \alpha, \lambda^2$ could be solved as a function of $\tau$ on $I$ from equation (5.8) with

$$0 < \lambda^2 < 1/2.$$  

From (5.5b) we have

$$A_0^2(\tau) = \frac{2a(\tau)\lambda^2(\tau)}{b(\tau)} \{1 - 2\lambda^2(\tau)\}^{-1}, \quad \tau \in I.$$  

Since the asymptotic correctness of a zeroth order asymptotic solution $\phi_0$ is obtained by comparison with a first order asymptotic solution $\phi_1$, we ought to show the existence of $\phi_1$ first. As indicated in § 4 this implies the examination of the solvability of a first order ordinary differential equation for $A_1$, which is given by (4.14a) for the general nonlinear case. For problem (5.1) the equation for $A_1$ is given by

$$2\pi \omega A_{1,\tau} + \left(\pi \omega + 2\omega \int_0^{2\pi} z_{2,p}^* z_1^* dp\right) A_1 = -\frac{d}{d\tau} \left(A_0 S_0, L\right) - \Lambda_1^{(2)} - \Lambda_1^{(2)},$$

where

$$\Lambda_1^{(2)} = \int_0^{2\pi} 3bA_0 \Phi_0 (A_1 z_2^* + \Phi_1)^2 z_1^* dp,$$

and

$$\Lambda_1^{(2)} = \int_0^{2\pi} 2cA_0 \Phi_0 (A_1 z_2^* + \Phi_1) z_1^* dp + \int_0^{2\pi} \{(A_0 \Phi_0)_{\tau} + S_{0,p} A_0 \Phi_{0,p} + \omega A_1 z_{2,p}^* + \omega \Phi_{1,p}\} z_1^* dp.$$  

$\Phi_0$ is the even, $2\pi$-periodic Jacobian elliptic function $cn$; $z_1^*$ and $z_2^*$ represent the odd, respectively even, homogeneous solutions of the first variational equation of (5.2) with respect to $A_0 \Phi_0$. As the odd contributions to the integrands of $\Lambda_1^{(2)}$ and $\Lambda_1^{(2)}$ vanish when we integrate over one period $2\pi$, the quadratic dependence of $\Lambda_1^{(2)}$ on $A_1$ vanishes. Hence, the equation for $A_1$ is a linear first order differential equation with smooth coefficient functions on $I$. The existence of a solution $A_1$ on $I$ implies the existence of $\phi_1$.

Appendix A.

**Lemma 2.** Let $J$ be a function of $\tau$ and $\epsilon$ which belongs to $P^\infty$. If

$$j(\tau, \epsilon) = \int_0^{2\pi} J^*(p, \tau, \epsilon) dp = O(\epsilon),$$  

then

$$\int_0^\tau J(s, \epsilon) ds = O(\epsilon).$$
Proof. The following identity holds:

\[
\int_0^\tau J^*_{\{e^{-1}S(s, \varepsilon), \tau, \varepsilon}\} \, ds = \int_0^\tau J^*_{\{e^{-1}S(s, \varepsilon), s, \varepsilon\} \, ds
\]  
(A.3)

\[-\int_0^\tau d\xi \int_0^\xi J^*_{\{e^{-1}S(s, \varepsilon), \xi, \varepsilon\} \, ds.
\]

We introduce the new integration variable \( \eta = e^{-1}S(s, \varepsilon) \) which has an inverse transform for small \( \varepsilon > 0 \) by virtue of the properties of \( S \):

\[ s = \xi(\varepsilon \eta, \varepsilon), \quad \xi \in C^\infty(\mathbb{R} \times [0, \varepsilon_0], \mathbb{R}). \]

For the first integral in the right-hand side of (A.3) we find

\[
\int_0^\tau J^*_{\{e^{-1}S(s, \varepsilon), \tau, \varepsilon\} \, ds = e \int_0^\tau J^*_{\{e^{-1}S(\tau, \varepsilon), \tau, \varepsilon\} \xi'(\varepsilon \eta, \varepsilon) \, d\eta
\]

\[= e \xi'_{\{S(\tau, \varepsilon), \varepsilon\} \int_0^{e^{-1}S(\tau, \varepsilon)} J^*_{\{\eta, \tau, \varepsilon\} \, d\eta
\]

\[= -e^2 \int_0^{e^{-1}S(\tau, \varepsilon)} \xi''(\varepsilon \eta, \varepsilon) \, d\xi \int_0^\xi J^*_{\{\eta, \tau, \varepsilon\} \, d\eta,
\]

where the prime indicates differentiation with respect to the first argument. By virtue of the given property of \( j \) and the periodicity of \( J^* \) as a function of \( \rho \), we know that

\[\int_0^\xi J^*_{\{\eta, \tau, \varepsilon\} \, d\eta = O(1) \quad \text{uniformly for} \quad (\xi, \tau) \in [0, e^{-1}S(\tau, \varepsilon)] \times I.\]

Since \( \xi \in C^\infty(\mathbb{R} \times [0, \varepsilon_0]) \) we may conclude that the first integral in the right-hand side of (A.3) is \( O(\varepsilon) \).

Similarly,

\[\int_0^\xi J^*_{\{e^{-1}S(s, \varepsilon), \xi, \varepsilon\} \, ds = e \int_0^{e^{-1}S(\xi, \varepsilon)} J^*_{\{\eta, \xi, \varepsilon\} \xi'(\varepsilon \eta, \varepsilon) \, d\eta
\]

is uniformly \( O(\varepsilon) \) for \( \xi \in I \) if

\[(A.4) \quad j'(\tau, \varepsilon) = \int_0^{2\pi} J^*_{\{p, \tau, \varepsilon\} \, dp = O(\varepsilon).\]

To show (A.4) we note that \( j \) is infinitely differentiable for \( (\tau, \varepsilon) \in I \times [0, \varepsilon_0] \) and \( j = O(\varepsilon) \). Then \( j(\tau, 0) = 0 \) for \( \tau \in I \) and (A.4) follows from the fact that

\[\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \frac{\partial j}{\partial \varepsilon}(\tau, \varepsilon) = \left[ \frac{\partial}{\partial \varepsilon} \left( \frac{\partial j}{\partial \tau} \right) \right]_{\varepsilon = 0}
\]

is bounded on \( I \). Hence both integrals with respect to \( s \) in the right-hand side of (A.3) are uniformly \( O(\varepsilon) \) for \( \tau \in I \), respectively \( \xi \in I \), from which (A.2) follows.
Appendix B. In this Appendix we determine the functions \( z_1^* \) and \( z_2^* \) which occur in §§ 3 and 4 as solutions of the differential equation

\[
S_{1,2}^{-1} z_{1,2,pp} + F_{yy}(U_0(p, \tau), \tau)z_{1,2}^* = 0, \quad (p, \tau) \in \mathbb{R} \times I,
\]

where \( F(y, \tau) \) has an absolute minimum at \( y = 0 \) and where the total mechanical energy \( E_0 \) of the system (4.3) is assumed to be noncritical uniformly for \( \tau \in I \). Both \( z_1^* \) and \( z_2^* \) should belong to \( P^\infty \) and their Wronskian \( D(z_1^*, z_2^*) = z_1^* z_2^* - z_1^* z_2^* \) should be equal to 1.

The first solution \( z_1^* \) of (B.1) is obtained by noting that \( U_0(p, \tau) \) is a solution of

\[
S_{1,2}^{-1} \Phi_{0,pp}(p, \tau) = \Phi_0(p, \tau) \Phi_{0,p}(p, \tau), \quad (p, \tau) \in I,
\]

where we used the representations (4.4) for \( U_0 \) and (4.8a) for \( \Phi_0 \). Clearly, \( z_1^* \) belongs to \( P^\infty \) and since \( U_0 \) is an even, \( 2\pi \)-periodic function of \( p \), \( z_1^* \) is an odd, \( 2\pi \)-periodic function of \( p \).

The second solution \( z_2^* \) of (B.1) is obtained by putting \( z_2^* = wU_0(p, \tau) \). Substituting this in equation (B.1), integrating once with respect to \( p \) and choosing the constant of integration equal to 1, we get \( w = U_0^{-2} \). This holds for any \( p \)-interval where \( U_0 \neq 0 \). So we have

\[
z_2^* = U_0 \int_{p_0}^{p} U_0^{-2} dp, \quad 0 < p < \pi,
\]

where \( p_0 \) denotes the zero of \( U_0 \) on the interval \((0, \pi)\). The choice of \( p_0 \) as the lower bound of integration in (B.3) is a matter of convenience. With the help of results from § 4 on \( \Phi_0 \) we deduce

\[
U_0 = -\frac{\omega^{-1}(A_0, \tau)}{2}(F(\eta + A_0, \tau) - F(U_0, \tau))^{1/2}, \quad 0 < p < \pi.
\]

We see that \( U_0 \) is a monotonic function of \( p \) on \([0, \pi]\). Then, introducing \( U_0 \) as a new integration variable in (B.3), we get for all \((p, \tau) \in (0, \pi) \times I, \)

\[
z_2^*(p, \tau) = U_0(p, \tau) \int_0^{U_0} U_0^{-3} dU_0
\]

As the total mechanical energy \( E_0 \) is noncritical, that is,

\[
F_j(\eta \pm A_0, \tau) \neq 0 \quad \text{for all} \quad \tau \in I,
\]

the function \( F(\eta + A_0, \tau) - F(U_0, \tau) \) has simple zeros at \( U_0 = \eta \pm A_0 \) and hence the limit values of \( H \) at \( U_0 = \eta \pm A_0 \) exist. Therefore (B.4) defines a continuous function \( z_2^* \) on \([0, \pi] \times I \). We now extend \( z_2^* \) onto the next half period \( \pi \) by using symmetry considerations

\[
z_2^*(\pi + p, \tau) = z_2^*(\pi - p, \tau), \quad 0 \leq p \leq \pi.
\]
and further on by the periodicity condition

$$z_2^*(p + 2\pi, \tau) = z_2^*(p, \tau).$$

The resulting function, defined on the whole $p$-axis satisfies equation (B.1) everywhere. From similar considerations for the derivative of $z_2^*$ with respect to $p$ it is seen that $z_2^*$ is an even, $2\pi$-periodic function which belongs to $P$. A simple calculation shows that the Wronskian of $z_1^*$ and $z_2^*$ is equal to 1.

REFERENCES