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Nonlinear network dynamics for interconnected micro-grids

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Abstract

This paper deals with transient stability in interconnected micro-grids. The main challenge is to understand the impact of the connectivity of the graph and model nonlinearities on transient and steady-state behavior of the system as a whole. The contribution of this paper is three-fold. First, we provide a robust classification of transient dynamics for different intervals of the parameters for a single micro-grid. We prove that underdamped dynamics and oscillations arise when the damping coefficient is below a certain threshold which we calculate explicitly as function of the product between the inertia coefficient and the synchronization parameter. Second, for interconnected micro-grids, under the hypothesis of homogeneity, we prove that the transient dynamics mimics a consensus dynamics. We provide bounds on the damping coefficient characterizing underdamped and overdamped consensus. Third, we extend the study to the case of disturbed measurements due to hacking or parameter uncertainties. We introduce model nonlinearities and prove absolute stability.

Keywords: Synchronization, Consensus, Transient stability

1. Introduction

This paper investigates transient stability of interconnected micro-grids. First we develop a model for a single micro-grid combining swing dynamics.
and synchronization, inertia and damping parameters. We focus on the main characteristics of the transient dynamics especially the insurgence of oscillations in underdamped transients. The analysis of the transient dynamics is then extended to multiple interconnected micro-grids. By doing this we relate the transient characteristics to the connectivity of the graph. We also investigate the impact of the disturbed measurements (due to hacking or parameter uncertainties) on the transient.

1.1. Main theoretical findings

The contribution of this paper is three-fold. First, for the single micro-grid we identify intervals for the parameters within which the behavior of the transient stability has similar characteristics. This shows robustness of the results and extends the analysis to cases where the inertia, damping and synchronization parameters are uncertain. In particular we prove that underdamped dynamics and oscillations arise when the damping coefficient is below a certain threshold which we calculate explicitly. The threshold is obtained as function of the product between the inertia coefficient and the synchronization parameter.

Second, for interconnected micro-grids, under the hypothesis of homogeneity, we prove that the transient stability mimics a consensus dynamics and provide bounds on the damping coefficient for the consensus value to be overdamped or underdamped. This result is meaningful as it sheds light on the insurgence of topology-induced oscillations. These bounds depend on the topology of the grid and in particular on its maximum connectivity, namely, the maximum number of links over all the nodes of the network. We also observe that the consensus value changes dramatically with increasing damping coefficient. This implies that the micro-grid, if working in islanding mode, can synchronize to a frequency which deviates from the nominal one of 50 Hz. This finding extends to smart-grids with different inertia but same ratio between damping and inertia coefficient.

Third, we extend the analysis to the case where both frequency and power flow measurements are subject to disturbances. Using a traditional technique in nonlinear analysis and control we isolate the nonlinearities in the feedback loop, and analyze stability under some mild assumptions on the nonlinear parameters. The obtained result extends also to the case where the model parameters like synchronization coefficient, inertia and damping coefficients are uncertain. This adds robustness to our findings and proves validity of the results even under modeling errors. The nonlinear framework accommodates
also output limits assuming that they can be modeled using first and third quadrant sector nonlinearities.

To corroborate our theoretical findings a case study from the Nigerian distribution network is discussed.

1.2. Related literature

This study leverages on previous contributions of the authors in [2] and [3]. In [2] the author studies flexible demand in terms of a population of smart thermostatically controlled loads and shows that the transient dynamics can be accommodated within the mean-field game theory. In [3] the author extends the analysis to uncertain models involving both stochastic and deterministic (worst-case) analysis approaches. The analysis of interconnected micro-grids builds on previous studies provided in [6]. Here the authors link transient stability in multiple electrical generators to synchronization in a set of coupled Kuramoto oscillators. We refer the reader to the survey [7]. The connection between Kuramoto oscillators and consensus dynamics is addressed in [10]. A game perspective on Kuramoto oscillators is in [11], where it is shown that the synchronization dynamics admits an interpretation as game dynamics with equilibrium points corresponding to Nash equilibria. The observed deviation of the consensus value from the nominal mains frequency in the case of highly overdamped dynamics can be linked to inefficiency of equilibria as discussed in [12]. This study has benefited from some graph theory tools and analysis efficiently and concisely exposed in [5]. The model used in this paper, which combines swing dynamics with synchronization, inertia and damping parameters has been inspired by [9]. The numerical analysis has been conducted using data provided in [1].

This paper is organized as follows. In Section 2, we model a single micro-grid. In Section 3, we turn to multiple interconnected micro-grids. In Section 4, we analyze the impact of measurement disturbances. In Section 5, we provide numerical studies on the Nigerian grid. Finally, in Section 6, we provide conclusions.

2. Model of a single micro-grid

Consider a single micro-grid connected to the network, refer to it as the $i$th micro-grid. Let us denote by $P_i$ the power flow into the $i$th micro-grid. Also let $f_i$ be the frequency deviation of micro-grid $i$ and $\phi$ a virtual signal
representing the frequency of the mains. From [4, Chapter 3] and [8, Chapter 3.9.2], by applying dc approximation, the power $P_i$ evolves according to

$$\dot{P}_i = T_{ij}(\phi - f_i) = T_{ij}e_{ij},$$

where $T_{ij}$ is the synchronizing coefficient. This coefficient is obtained as the inverse of the transmission reactance between micro-grid $i$ and the mains. In other words, the power $P_i$ depends on the frequency error $e_{ij} = \phi - f_i$. The physical intuition of this is that in response to a positive error we have power injected into the $i$th micro-grid. Vice versa, a negative error induces power out from micro-grid $i$.

The dynamics for $f_i$ follows a traditional swing equation

$$\dot{f}_i = -\frac{D_i}{M_i}f_i + \frac{P_i}{M_i},$$

where $M_i$ and $D_i$ are the inertia and damping constants of the $i$th micro-grid, respectively. By denoting $f_i = x^{(i)}_1$, $P_i = x^{(i)}_2$, $\phi = x^{(j)}_1$, and by considering $\phi$ as an exogenous input to the $i$th micro-grid, the dynamics of the $i$th micro-grid reduces to the following second-order system

$$\begin{bmatrix}
\dot{x}^{(i)}_1 \\
\dot{x}^{(i)}_2
\end{bmatrix} = \begin{bmatrix}
-\frac{D_i}{M_i} & \frac{1}{M_i} \\
-T_{ij} & 0
\end{bmatrix} \begin{bmatrix}
x^{(i)}_1 \\
x^{(i)}_2
\end{bmatrix} + \begin{bmatrix}
0 \\
T_{ij}
\end{bmatrix} x^{(j)}_1.
$$

Theorem 1. Dynamics (3) is asymptotically stable. Furthermore, let $D_i > 2\sqrt{T_{ij}M_i}$ then the origin is an asymptotically stable node. Vice versa, if $D_i < 2\sqrt{T_{ij}M_i}$ then the origin is an asymptotically stable spiral.

Proof. For the first part, stability derives from $Tr(A) = -\frac{D_i}{M_i}$, where $Tr(A)$ is the trace of matrix $A$ and from $\Delta(A) = \frac{T_{ij}}{M_i} > 0$, where $\Delta(A)$ is the determinant of matrix $A$. Let us recall that stability depends on the eigenvalues of $A$ and that the expression of the eigenvalues is given by

$$\lambda_{1,2} = \frac{Tr(A)\pm \sqrt{Tr(A)^2 - 4\Delta(A)}}{2} = \frac{1}{2} \left( -\frac{D_i}{M_i} \mp \sqrt{\left(\frac{D_i}{M_i}\right)^2 - 4\frac{T_{ij}}{M_i}} \right).$$

As for the rest of the proof, we know that if $D_i > 2\sqrt{T_{ij}M_i}$ then $Tr(A)^2 > 4\Delta(A)$ and the origin is an asymptotically stable node.

The last case is when $D_i < 2\sqrt{T_{ij}M_i}$ which implies $Tr(A)^2 < 4\Delta(A)$ and therefore the origin is an asymptotically stable spiral. □

The above theorem identifies intervals for the parameters within which the behavior of the transient stability is unchanged. This provides robustness to our results and extend the analysis to cases where the inertia, damping and synchronization parameters are uncertain.
3. Multiple interconnected micro-grids

Let us now consider a network \( G = (V, E) \) of interconnected smart-grids, where \( V \) is the set of nodes, and \( E \) is the set of arcs. Figure 1 displays an example of interconnection topology. Nodes represent smart-grids units and arcs represent power lines interconnections. We use shades of gray to emphasize different levels of connectivity of the smart-grids. The connectivity of a grid is indicated by the degree of the node. We recall that for undirected graphs the degree of a node is number of links with an extreme in node \( i \). We denote by \( d_i \) the degree of node \( i \).

![Figure 1: Graph topology indicating smart-grids and interconnections.](image)

Building on model (3) developed for the single grid, we derive the following macroscopic dynamics for the whole grid:

\[
\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2
\end{bmatrix} =
\begin{bmatrix}
-Diag \left( \frac{D_i}{M_i} \right) & Diag \left( \frac{1}{M_i} \right) \\
-L & 0
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}.
\]

(5)

In the above set of equations, the block matrix \( L = [l_{ij}]_{i,j \in \{1,...,n\}} \) is the graph-Laplacian matrix:

\[
L :=
\begin{bmatrix}
T_{11} & \ldots & -T_{1n} \\
\vdots & \ddots & \vdots \\
-T_{n1} & \ldots & T_{nn}
\end{bmatrix},
\]

\[
l_{ij} = \begin{cases} 
-T_{ij} & \text{if } i \neq j, \\
\sum_{h=1, h \neq i} T_{ih} & \text{if } i = j.
\end{cases}
\]

(6)

Its row-sums are zero, its diagonal entries are nonnegative, and its non-diagonal entries are nonpositive. We also recall that \( L = [l_{ij}]_{i,j \in \{1,...,n\}} \) where for an unweighted and undirected graph we have

\[
l_{ij} = \begin{cases} 
-1 & \text{if } (i, j) \text{ is an edge and not self-loop}, \\
d(i) & \text{if } i = j, \\
0 & \text{otherwise}.
\end{cases}
\]

(7)
We are ready to establish the next result. Let us denote by \( \text{span}\{1\} = \{\xi \in \mathbb{R}^n : \exists \eta \in \mathbb{R} \text{ s.t. } \xi = \eta 1\} \).

**Theorem 2.** Let a network of homogeneous micro-grids be given, and set \( D_i = D \) for all \( i \). Let \( M_i = 1 \) for all \( i \), and \( T_{ij} = 1 \) for any \((i, j) \in E\). Then dynamics (5) describes a consensus dynamics, i.e.,

\[
\lim_{t \to \infty} X_i(t) = x_i^* \in \text{span}\{1\}, \quad i = 1, 2.
\]

Furthermore, let \( \mu_i \) be the \( i \)th eigenvalue of \(-L\) and let \( D > \sqrt{-4\mu_i} \) for all \( i = 1, \ldots, n \), then the consensus value vector \((x_1^*, x_2^*)^T\) is an asymptotically stable node. Vice versa, if \( D < \sqrt{-4\mu_i} \) for some \( i = 1, \ldots, n \), then \((x_1^*, x_2^*)^T\) is an asymptotically stable spiral.

**Proof.** Let us start by finding the roots of \( \det(\lambda I - A) \), where \( \det(.) \) denotes the determinant. We recall that for any generic block matrix it holds

\[
\det\left[\begin{array}{cc}
A & B \\
C & D
\end{array}\right] = \det(DA - BC), \quad \text{if } BD = DB. \tag{8}
\]

Let us denote \( I \) the identity matrix. Then from the above we have

\[
\det(\lambda I - A) = \det\left(\begin{array}{cc}
\lambda I + \text{Diag}(\frac{D_i}{M_i}) & -\text{Diag}(\frac{1}{M_i}) \\
L & \lambda I
\end{array}\right) \tag{9}
\]

Under the homogeneity assumption \( D_i = D \) for all \( i \), we have

\[
\det\left(\lambda^2 I + \lambda I \cdot \text{Diag}(\frac{D_i}{M_i}) + \text{Diag}(\frac{1}{M_i}) \cdot L\right) = \det((\lambda^2 + \lambda D)I + L) = \prod_{i=1}^{n}(\lambda^2 + \lambda D - \mu_i). \tag{10}
\]

The roots of (10) can be obtained by solving \( \lambda^2 + \lambda D - \mu_i = 0 \) from which we have

\[
\lambda_i^+ = \frac{-D + \sqrt{D^2 + 4\mu_i}}{2}, \quad \lambda_i^- = \frac{-D - \sqrt{D^2 + 4\mu_i}}{2}. \tag{11}
\]

From the above, after noting that the real part of the eigenvalues is negative, we can conclude that system (5) is asymptotically stable. Now, if \( D > \sqrt{-4\mu_i} \) for all \( i = 1, \ldots, n \), then all eigenvalues are real and the consensus value vector \((x_1^*, x_2^*)^T\) is an asymptotically stable node. Differently, if
$D < \sqrt{-4\mu_i}$ for some $i = 1, \ldots, n$, then the corresponding eigenvalues have imaginary parts and therefore $(x_1^*, x_2^*)^T$ is an asymptotically stable spiral.

To prove that $\lim_{t \to \infty} X_i(t) = x_i^* \in \mathcal{C}$, $i = 1, 2$, note that the Laplacian matrix has one zero eigenvalue, say $\mu_1 = 0$, which yields $\lambda_1^+ = 0$ and $\lambda_1^- = -D$. Thus $A$ has also a zero eigenvalue. As every row-sum of $L$ is zero, the corresponding eigenvector is in $\{\text{span}\{1\}, \text{span}\{1\}\}$ and this concludes our proof. □

**Remark 1.** The result stated in the above theorem applies also to the case where the micro-grids have different inertia but the same ratio $D = \frac{D_i}{M_i}$ for all $i \in V$. In this case we need to consider the Laplacian matrix of the corresponding weighted graph $\tilde{L} = \text{Diag}(\frac{1}{M_i})L$ and the associated eigenvalues.

We next recall some properties of the Laplacian spectrum and use such properties to investigate the insurgence of topology-induced oscillations. The maximal eigenvalue $\tilde{\mu}_n$ of a symmetric Laplacian matrix $L = L^T$ in $\mathbb{R}^{n \times n}$ satisfies the following lower and upper bounds which are degree-dependent:

$$d_{\text{max}} \leq \tilde{\mu}_n \leq 2d_{\text{max}}, \tag{12}$$

where the maximum degree is $d_{\text{max}} = \max_{i \in 1, \ldots, n} d_i$ [5, Chapter 6]. We also observe that the eigenvalues appearing in (11) refer to the negative Laplacian, and therefore we have $\mu_i = -\tilde{\mu}_i$ for every eigenvalue $\mu_i$ of the negative Laplacian $-L$ and $\tilde{\mu}_i$ of the Laplacian $L$.

**Corollary 1.** The following properties hold:

- Eigenvalues $\lambda_i^+, \lambda_i^-$, $i = 1, \ldots, n$ are real and negative if $D \geq \sqrt{8d_{\text{max}}}$;
- There exists at least one complex eigenvalue/eigenmode if $D \leq \sqrt{4d_{\text{max}}}$.

**Corollary 2.** Given a chain topology of $n \geq 3$ nodes, for which $d_{\text{max}} = 2$ the following properties hold:

- All eigenvalues $\lambda_i^+, \lambda_i^-$ for $i = 1, \ldots, n$ are real and negative if $D \geq 4$,
- There exists at least one complex eigenvalue/eigenmode if $D \leq 2\sqrt{2}$. 

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3.1. Example of two interconnected micro-grids

In this section, we specialize the above results to the case of two interconnected micro-grids. The interconnection topology is a chain one with two nodes, and the maximal degree is $d_{\text{max}} = 1$. A graph representation is displayed in Fig. 2.

Dynamics (5) can be rewritten as

$$
\begin{bmatrix}
\dot{x}^{(i)}_1 \\
\dot{x}^{(j)}_1 \\
\dot{x}^{(i)}_2 \\
\dot{x}^{(j)}_2 \\
\end{bmatrix} =
\begin{bmatrix}
-\frac{D_1}{M_1} & 0 & \frac{1}{M_1} & 0 \\
0 & -\frac{D_2}{M_2} & 0 & \frac{1}{M_2} \\
-T_{11} & T_{12} & 0 & 0 \\
T_{12} & -T_{11} & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x^{(i)}_1 \\
x^{(j)}_1 \\
x^{(i)}_2 \\
x^{(j)}_2 \\
\end{bmatrix}
$$

(13)

The Laplacian of the weighted graph is given by

$$
\tilde{L} = \text{Diag}(\frac{1}{M_i})L = \begin{bmatrix}
\frac{1}{M_1} & 0 & 0 & -1 \\
0 & \frac{1}{M_2} & -1 & 1 \\
\end{bmatrix}
$$

(14)

Assuming $D = \frac{D_1}{M_1} = \frac{D_2}{M_2}$, from Corollary 1 we infer that

- All eigenvalues $\lambda_i^+, \lambda_i^-$ for $i = 1, \ldots, n$ are real and negative if $D \geq \sqrt{8}$;

- There exists at least one complex eigenvalue/eigenmode if $D \leq 2$.

In other words, if the ratio between the damping coefficient and the inertia of each micro-grid is greater than $\sqrt{8}$ then we certainly have an overdamped dynamics, and observe no overshoots and no oscillations. Differently, if the ratio between the damping coefficient and the inertia of each micro-grid is less than 2 then we certainly have an underdamped dynamics, and observe overshoots and oscillations.
4. Absolute stability

In this section we extend the analysis to the case where both frequency and power flow measurements are subject to disturbances. Using a traditional technique in nonlinear analysis and control we isolate the nonlinearities in the feedback loop, and analyze stability under some mild assumptions on the nonlinear parameters.

We denote $f_i = x_1^{(i)}$, $P_i = x_2^{(i)}$, $f_j = x_1^{(j)}$. Consider $f_j$ and $\omega$ as exogenous inputs to micro-grid $i$ and let us take them null. Consider the following dynamics of micro-grid $i$ (we drop index $i$ to simplify notation)

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{D}{M} & \frac{1}{M} \\ -T & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \psi_1(x_1) \\ \psi_2(x_2) \end{bmatrix},$$

(15)

where functions $\psi_i$ satisfies the following first and third quadrant sector condition. Let $K = kI$.

**Assumption 1.** The function $\psi(.) = [\psi_1 \psi_2]^T$ satisfies the sector condition

$$\psi^T[\psi - Ky] \leq 0, \quad \forall t \geq 0 \quad \forall y \in \mathbb{R}^2.$$

The block system of the $i$th micro-grid, which admits the state space representation (15), is displayed in Fig. 3. In accordance to the above, the evolution of power $P_i$ depends on a disturbed measure of the frequency error $\tilde{e} := f_j - \psi_2(P) - f$:

$$\dot{P} = T(f_j - \psi_2(P) - f) = T\tilde{e},$$

(16)

where $T$ is the synchronizing coefficient as in the previous section. The dynamics for $f$ still follows a traditional swing equation, which now involves disturbed measurements of the frequency $\psi_1(f)$:

$$\dot{f} = -\frac{D}{M} f + \frac{P}{M} + \omega - \psi_1(f).$$

(17)
Building on the Kalman-Yakubovich-Popov lemma, absolute stability is linked to strictly positive realness of \( Z(s) = \mathbb{I} + KG(s) \) and \( G(s) \) is the transfer function of linear part of system (15) which is obtained as \( G(s) = C[s\mathbb{I} - A]^{-1}B \) where we set

\[
A = \begin{bmatrix} -\frac{D}{M} & \frac{1}{M} \\ -T & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

We recall from Theorem 1 that matrix \( A \) is Hurwitz. The idea now is to isolate the nonlinearities in the feedback loop and introduce a new variable for them, say \( \psi \). Let us first obtain the transfer function associated to the dynamical system (15):

\[
G(s) = C[s\mathbb{I} - A]^{-1}B = \frac{1}{s(s + \frac{D}{M}) + \frac{T}{M}} \begin{bmatrix} s & \frac{1}{M} \\ -T & s + \frac{D}{M} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix},
\]

where \( \Delta(s\mathbb{I} - A) = s(s + \frac{D}{M}) + \frac{T}{M} \). Then, for \( Z(s) \) we obtain

\[
Z(s) = \mathbb{I} + KG(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{\Delta(s\mathbb{I} - A)} \begin{bmatrix} s & \frac{T}{M} \\ -T & s + \frac{D}{M} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix},
\]

Note that matrix \( A \) is Hurwitz. This implies that also \( Z(s) \) is Hurwitz as the poles of \( Z(s) \) coincide with the eigenvalues of \( A \). We use this in the proof of absolute stability of the dynamical system (15) established next.

**Theorem 3.** Let the dynamical system (15) be given where \( A \) is Hurwitz and \( M_i = 1 \). Let us consider the sector nonlinearities as in Assumption 1. Then, \( Z(s) \) is strictly positive real and system (15) is absolutely stable.

**Proof.** We first prove that \( Z(s) \) is strictly positive real. For this to be true, the following conditions must hold true:

- \( Z(s) \) is Hurwitz, namely the poles of all entries of the matrix \( Z(s) \) have negative real parts;
\begin{itemize}
  \item $Z(j\omega) + Z(-j\omega) > 0, \ \forall \omega \in \mathbb{R}$;
  \item $Z(\infty) + Z^T(\infty) > 0$.
\end{itemize}

For the first condition note that $Z(s)$ is Hurwitz as its poles are the roots of $s(s + \frac{D}{M}) + \frac{T}{M} = 0$, which coincide with the values obtained in [4] and which we rewrite here for convenience: $\lambda_{1,2} = \frac{1}{2} \left( -\frac{D}{M} \pm \sqrt{\left(\frac{D}{M}\right)^2 - 4 \frac{T}{M}} \right)$. As for the second condition, $Z(j\omega) + Z(-j\omega) > 0, \ \forall \omega \in \mathbb{R}$, let us obtain for $Z(j\omega)$ and $Z(-j\omega)$ the following expressions:

\begin{align*}
  Z(j\omega) &= \frac{1}{\frac{T}{M} - \omega^2 + \frac{k}{M}j\omega} \left[ \frac{T}{M} - \omega^2 + \frac{k}{M}j\omega \ \frac{k}{M}\left( \frac{T}{M} + kT \right) - \omega^2 + \frac{k}{M}j\omega \right], \\
  Z(-j\omega) &= \frac{1}{\frac{T}{M} - \omega^2 - \frac{k}{M}j\omega} \left[ \frac{T}{M} - \omega^2 - \frac{k}{M}j\omega \ \frac{k}{M}\left( \frac{T}{M} + kT \right) - \omega^2 - \frac{k}{M}j\omega \right].
\end{align*}

By combining the expressions above for $Z(j\omega)$ and $Z(-j\omega)$ we then obtain

\begin{align*}
  Z(j\omega) + Z^T(-j\omega) &= \frac{1}{\Delta(j\omega I - A)\Delta(-j\omega I - A)} \left( \Delta(-j\omega I - A) \right) \left( \frac{T}{M} - \omega^2 + \frac{k}{M}j\omega \right) \left( \frac{k}{M}\left( \frac{T}{M} + kT \right) - \omega^2 + \frac{k}{M}j\omega \right) \\
  &\quad + \Delta(j\omega I - A) \left( \frac{T}{M} - \omega^2 + \frac{k}{M}j\omega \right) \left( \frac{k}{M}\left( \frac{T}{M} + kT \right) - \omega^2 + \frac{k}{M}j\omega \right) \\
  &= \frac{1}{\frac{T}{M} - \omega^2 - \frac{k}{M}j\omega} \left[ \frac{T}{M} - \omega^2 - \frac{k}{M}j\omega \right] \left[ \frac{k}{M}\left( \frac{T}{M} + kT \right) - \omega^2 - \frac{k}{M}j\omega \right] \\
  &\quad \left( \frac{T}{M} - \omega^2 - \frac{k}{M}j\omega \right) \left( \frac{k}{M}\left( \frac{T}{M} + kT \right) - \omega^2 - \frac{k}{M}j\omega \right) \\
  &> 0, \text{ for all } \omega \text{ and for } M_i = 1. \tag{19}
\end{align*}

The last inequality follows from $z_{21}(\omega) = 2kTDj\omega = -z_{12}(\omega)$ when $M = 1$ as well as $z_{11}(\omega), z_{22}(\omega) > 0$, where $z_{ij}(\omega)$ is the $ij$th entry of matrix as defined below

\begin{equation}
  Z(j\omega) + Z^T(-j\omega) := \frac{1}{\left( \frac{T}{M} - \omega^2 \right) - \left( \frac{k}{M}j\omega \right)^2} \left[ \begin{array}{cc}
  z_{11}(\omega) & z_{12}(\omega) \\
  z_{21}(\omega) & z_{22}(\omega)
  \end{array} \right]. \tag{20}
\end{equation}

This implies that for every vector $\xi = [\xi_1 \ \xi_2]^T \in \mathbb{R}^2$ we have

\begin{equation}
  \xi^T \left( Z(j\omega) + Z^T(-j\omega) \right) \xi = z_{11}(\omega)\xi_1^2 + z_{22}(\omega)\xi_2^2 > 0, \quad \text{for all } \omega.
\end{equation}
As for the third condition, namely \( Z(\infty) + Z^T(\infty) > 0 \), we have that
\[
\lim_{\omega \to \infty} \frac{1}{(T - \omega^2) - (D_j \omega)^2} z_{12} = \lim_{\omega \to \infty} \frac{1}{(T - \omega^2) - (D_j \omega)^2} z_{21} = 0,
\]
\[
\lim_{\omega \to \infty} \frac{1}{(T - \omega^2) - (D_j \omega)^2} z_{11} = \lim_{\omega \to \infty} \frac{1}{(T - \omega^2) - (D_j \omega)^2} z_{22} = 2.
\]

Then we obtain that \( Z(\infty) + Z^T(\infty) = 2 \mathbb{I} > 0 \). We can conclude that also the third condition is verified.

Now we wish to show that there exists a Lyapunov function \( V(x) = x^T \Phi x \), where \( \Phi = [\Phi_{ij}] \in \mathbb{R}^{2 \times 2} \) is symmetric. After differentiation with respect to time and using (15) we obtain
\[
\dot{V}(t, x) = \dot{x}^T \Phi x + x^T \Phi \dot{x} = x^T A^T \Phi x + x^T \Phi A x - \psi^T B^T \Phi x - x^T \Phi B \psi,
\]
where we denote \( \psi(t, y) = [\psi_1 \psi_2]^T \). From Assumption 1 and the property of first and third sector nonlinearities we have \(-2 \psi^T (\psi - Ky) \geq 0\). Furthermore, from symmetry of matrices \( P \) and \( K = \tilde{k} \mathbb{I} \), the time derivative of the candidate Lyapunov function can be rewritten as
\[
\dot{V}(t, x) \leq -\epsilon x^T A^T \Phi x - x^T P \Pi x + 2\sqrt{2} x^T P \Pi \psi - 2 \psi^T (\psi - Ky) \]
\[
= -\epsilon x^T A^T \Phi x - x^T A^T \Phi x - \epsilon x^T A^T \Phi x - 2 \psi^T (\psi - Ky) \]
\[
= -\epsilon x^T A^T \Phi x - 2 x^T (C^T K - \Phi B) \psi - 2 \psi^T \psi.
\]
The right-hand side of the above inequality is negative if there exist matrices \( \Pi \in \mathbb{R}^{2 \times 2} \) and a positive scalar \( \epsilon \) such that
\[
\begin{cases}
A^T \Phi + \Phi A = -\Pi^T \Pi - \epsilon \Phi, \\
\Phi B = C^T K - \sqrt{2} \Pi^T,
\end{cases}
\]
(22)

By introducing the solutions of the above in terms of \( \Phi, \Pi \) and \( \epsilon \), the time derivative of the candidate Lyapunov function can be rewritten as
\[
\dot{V}(t, x) \leq -\epsilon x^T A^T \Phi x - x^T \Pi^T \Pi x + 2\sqrt{2} x^T \Pi \psi - 2 \psi^T \psi
\]
\[
= -\epsilon x^T A^T \Phi x - [\Pi x - \sqrt{2} \psi]^T [\Pi x - \sqrt{2} \psi] \leq -\epsilon x^T A^T \Phi x
\]
\[
-\epsilon [x_1 \ x_2] \begin{bmatrix} \Phi_{11} & \Phi_{21} \\ \Phi_{12} & \Phi_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]

It is well known that from the Kalman-Yakubovich-Popov lemma, there exist solutions in terms of \( \Phi, \Pi, \) and \( \epsilon \) satisfying the above set of matrix equalities, as the transfer function \( Z(s) \) is positive real. This concludes our proof. \( \square \)

**Remark 2.** The above theorem has been obtained under the hypothesis that both frequency and power flow measurements are subject to disturbances. The same result extend straightforwardly also to the case where the model parameters \( T_{ij}, M_i \) and \( D_i \) are uncertain.
5. Simulations

This section provides simulation studies to extend the theoretical results developed in the previous sections in the context of micro-grids also to transmission networks. The analysis is based on open source data relating to a part of the Nigerian grid obtained from [1]. The data set shows the one-line diagram of part of the transmission network including the geographical location of generators and load buses. Figure 4 displays the one-line diagram with the geographical names.

From the one-line diagram we obtain the graph representation showing the interconnection between bus loads as in Fig. 5. The graph is characterized by 11 nodes and 10 arcs. Most nodes have degree 1 or 2 except for Gombe, and Kano which have degree 4, and 3, respectively. The graph is undirected, i.e. the influence of smart-grid $i$ on $j$ is bidirectional.

The numerical studies involve two sets of simulations. The first set of simulations considers the following parameters: number of smart-grids $n = 11$, damping constant $D = 1, 3, 6$ for three consecutive runs of simulations; Inertial constant $M = 1$; Synchronizing coefficient $T = 1$; Horizon window involving $N = 500$ iterations; Step size $dt = .01$. The parameters and the dynamics are normalized in an interval $[0, 1]$. For instance the initial state of each grid is a randomized bidimensional vector in the interval $[0, 1]$. Random initial values capture positive or negative peaks of demand. To simulate periodic disturbances, the initial state is reinitialized every 10 sec. We rescale the state variable around 50 Hz for the frequency and 30 MW for the power.
Figure 5: Graph representation of part of Nigerian grid [1].

oscillations remain within 1% of 50 Hz, i.e., in the interval [49.95, 50.05]. From top to bottom we consider an increasing damping constant $D = 1, 3, 6$ which reflects in damped oscillations and smaller time constants. Figure 6 displays the evolution of the frequency of each bus. Frequencies are measured in Hz and are centered around the nominal value of 50 Hz. From top to bottom we observe that oscillations remain within 3.3% of the nominal value, i.e., in the interval [29.00, 31.00] MW. From the plots we observe that oscillations remain within 3.3% of the nominal value, i.e., in the interval [29.00, 31.00] MW. From top to bottom the damping constant is increasing and equal to $D = 1, 3, 6$. Note that the maximal degree of the network is $d_{\text{max}} = 4$ and therefore for $D = 1, 3$ we have $D < \sqrt{4d_{\text{max}}} = 4$ and oscillations emerge as we have complex eigenvalues for $\lambda_+^-, \lambda_+^-$, for $A$. Unlikewise for $D = 6$ it holds $D > \sqrt{8d_{\text{max}}} = \sqrt{32}$ and therefore no oscillations and no complex eigenvalues emerge. The maximal eigenvalue of the Laplacian is $\tilde{\mu}_n = 5.1748$. Also note that the interconnected buses work in islanding modes, thus the consensus value depends on the initial values and may deviate from 50 Hz.

In a second set of simulations, we isolate one bus from the rest of the power network and investigate the transient response under disturbances in the measurement of frequency and power. Such disturbances are modeled using the paradigm developed in Section 4. We consider sector nonlinearity in the feedback loop. The function is periodic and we take for it the expression $\psi(t) = \tilde{\eta}_1 + \tilde{\eta}_2 \sin(\xi ft)$ in $[\tilde{\eta}_1 - \tilde{\eta}_2, \tilde{\eta}_1 + \tilde{\eta}_2]$, where $\tilde{\eta}_1, \tilde{\eta}_2$ are positive scalars, $f$ is the frequency, $t$ is time, and $\xi$ is a factor increasing the periodicity of the oscillation. Note that when $\tilde{\eta}_1 = \tilde{\eta}_2$ we have a first and third quadrant
nonlinearity, namely $\psi(t)$ in $[0, 2\tilde{\eta}_1]$. We consider the following normalized parameters: number of smart-grids $n = 1$, damping constant $D = 1$; Inertial constant $M = 1$; Synchronizing coefficient $T = 1$; periodicity factor $\xi = 5, 10, 15$ for three consecutive runs of simulations; Horizon window involving $N = 200$ iterations; Step size $dt = .01$; The initial state of each grid is a randomized bidimensional vector in the interval $[0, 1]$. Both variables are rescaled around 50 Hz for the frequency and 30 MW for the power flow. Figure 8 is obtained for $\tilde{\eta}_1 = 1$ and $\tilde{\eta}_2 = 15$ and displays the time evolution of the frequency of each bus (left) and power flow (right). As in the previous simulation example, frequencies are measured in Hz and are centered around 50 Hz which is the nominal value. We observe that oscillations remain within 1\% of the nominal value, i.e., in the interval $[49.95, 50.05]$. From top to bottom the damping constant is $D = 1, 3, 5$ and this implies a higher damping, smaller time constants, and faster convergence. Power flows are measured in MW and are centered around the nominal value of 30 MW. The plots show that oscillations remain within 3.3\% of the nominal value, i.e., in the interval.
[29.00, 31.00] MW. From top to bottom the damping constant is equal to $D = 1, 3, 5$. Simulations in the case of first and third quadrant nonlinearity essentially show same qualitative behavior.

6. Discussion and conclusions

We have showed that transient dynamics can be robustly classified depending on specific intervals for the micro-grid parameters, such as synchronization, inertia, and damping parameters. We have then obtained bounds on the damping coefficient which determine whether the network dynamics is an underdamped or overdamped consensus dynamics. The bounds are linked to the connectivity of the network. We have also extended the stability analysis to the case of disturbed measurements due to hackering or parameter uncertainties. Using traditional nonlinear analysis and the Kalman-Yakubovich-Popov lemma we have first isolated the nonlinear terms in the feedback loop and have showed that nonlinearities do not compromise the stability of the system.
Figure 8: Time series of smart-grids power flows in MW.

There are three key directions for future work. First we wish to relax constraints on the nature of the disturbances. Indeed here we have assumed that such disturbances can be modeled using first and third quadrant nonlinearities. A second direction involves the analysis of the impact of stochastic disturbances on the transient stability. Concepts like stochastic stability, stability of moments, and almost sure stability will be used to classify the resulting stochastic transient dynamics. Finally, a third direction involves the extension to the case of a single or multiple heterogeneous populations of micro-grids. In this context we will try to gain a better insights on scalability properties and emergent behaviors.

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References


