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Finite Kripke Models of HA are Locally PA

D. van DALEN, H. MULDER, E. C. W. KRABBE, and A. VISSE

Introduction In a Kripke model of Heyting's Arithmetic, HA, the nodes, when viewed as classical structures, are models of classical arithmetic with (at least) $\Delta_0^1$-induction. In general, it is an open problem which form of induction holds in the classical structures at the nodes of Kripke models. However, in the case of finite Kripke models (i.e., those containing a finite number of nodes) one can show that all these structures satisfy full induction, and consequently are models of full Peano Arithmetic, PA. It can also be shown that any Kripke model with an underlying model structure of type $\omega$ must contain an infinite number of such Peano models. These results were established in a workshop in Utrecht (1983).

I Preliminaries Let $L$ be a first-order language with logical constants: $\bot$, $\land$, $\lor$, $\rightarrow$, $\forall$, $\exists$, $\equiv$. Let $\neg \phi$ be short for $\phi \rightarrow \bot$ and let $\phi \leftrightarrow \psi$ be short for $(\phi \rightarrow \psi) \land (\psi \rightarrow \phi)$. An extension $L_D$ of $L$ is obtained by adding an individual constant $c$ for each element $c$ of $D$. In practice, $D$ shall always be the local domain $D_\alpha$ of some node $\alpha$ in a Kripke model, and we shall write $L_\alpha$ instead of $L_D$. A Kripke model $K = (K, \leq, D, I)$ consists of a nonempty set $K$ of nodes, partially ordered by $\leq$, a function $D$ that assigns a nonempty local domain of individuals to each $\alpha \in K$, and a function $I$ that assigns an interpretation function $I_\alpha$ to each $\alpha \in K$. Each $I_\alpha$ assigns values to the individual constants, the function symbols, and the predicate symbols of $L_\alpha$, so as to provide for a local model $M_\alpha = (D_\alpha, I_\alpha)$. The different $I_\alpha$ agree on the values assigned to individual constants that belong to $L$. Moreover, $D$ and $I$ are to be cumulative in the following sense: if $\alpha \leq \beta$ then $D_\alpha \subseteq D_\beta$, and, for each function symbol or predicate symbol $X$, $I_\alpha(X) \subseteq I_\beta(X)$. $K$ is called finite if $K$ is finite.

Since we are interested in a theory with decidable equality it is no restriction to assume that '$\equiv$' is interpreted by the actual identity in each node (cf. [1], p. 184).

Semantic evaluations proceed as usual. We write $\alpha \vdash \phi$ if $\phi$ is true in the (classical) model $M_\alpha$, and $\alpha \models \phi$ if $\alpha$ forces $\phi$. Further, we write $\alpha \models \Gamma$ if for each $\phi \in \Gamma$, $\alpha \models \phi$. The symbol '$\vdash$' shall denote derivability on the strength of intuitionistic logic.

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It is well-known that \( \models \) is cumulative, i.e., if \( \alpha \models \phi \) then, for all \( \beta \) such that \( \alpha \leq \beta, \beta \models \phi \).

Deletion of some (but not all) nodes from a Kripke model \( K \) again yields a Kripke model. It suffices to restrict \( \leq, D, \) and \( I \) to the remaining set of nodes. If \( \alpha \in K \) then the model obtained by deleting all \( \beta \) such that not \( \alpha \leq \beta \) will be denoted as \( K^\alpha = (K^\alpha, \leq^\alpha, D^\alpha, I^\alpha) \), its relation of forcing as \( \models^\alpha \). Obviously, for all \( \beta \in K^\alpha \) and for all \( \phi \in L_\beta : \beta \models^\alpha \phi \iff \beta \models \phi \).

A classical node in a Kripke model \( K \) is to be a node \( \alpha \) of \( AT \) that forces all sentences \( \forall \alpha_1 \ldots \forall \alpha_n (\phi \lor \neg \phi) \in L_\alpha \). We note the following properties of classical nodes:

1. The following conditions are equivalent:
   (i) \( \alpha \) is a classical node
   (ii) \( \alpha \) forces all sentences \( \forall \alpha_1 \ldots \forall \alpha_n (\phi \lor \neg \phi) \in L \)
   (iii) For all \( \phi \in L_\alpha \) \( \alpha \models \phi \iff \alpha \models \phi \).

2. All final nodes (i.e., nodes such that for no \( \beta \neq \alpha : \alpha \leq \beta \)) are classical.

3. If \( \alpha \) is classical, so are all \( \beta \) such that \( \alpha \leq \beta \).

4. Let \( L \) be the language of arithmetic. If \( \alpha \) is classical and \( \alpha \models HA \) then \( \alpha \models PA \). Moreover \( M_\alpha \) will be a Peano model.

Let \( \rho \) be any sentence of \( L \). For each formula \( \phi \) of \( L \) we can construct another formula, \( \phi^\rho \), by substituting \( \phi_0 \lor \rho \) for each atomic component \( \phi_0 \) of \( \phi \). The result, \( \phi^\rho \), is called the Friedman translation of \( \phi \) by \( \rho \) in \( L \). We write \( \Gamma^\rho \) for \( \{ \phi^\rho | \phi \in \Gamma \} \). We shall exploit the following facts about Friedman translations (cf. [2]):

(A) \( \rho \models \phi^\rho \).
(B) If \( \Gamma \models \phi \) then \( \Gamma^\rho \models \phi^\rho \).
(C) Let \( L \) be the language of arithmetic: if \( HA \models \phi \) then \( HA \models \phi^\rho \).
(D) Let \( L \) be the language of arithmetic, \( \phi \in \Sigma^0_1 \), then \( HA \models \phi^\rho \iff (\phi \lor \rho) \).

2 Pruning

Definition 1 Let \( K \) be a Kripke model, \( \rho \) a sentence such that, for at least one node \( \alpha \in K, \rho \in L_\alpha \) and \( \alpha \not\models \rho \). Then the model obtained by pruning \( \rho \)-nodes from \( K \) shall be the model obtained from \( K \) by deleting all nodes that force \( \rho \). This model will be denoted as \( K^\rho (= (K^\rho, \leq^\rho, D^\rho, I^\rho)) \), its forcing relation by \( \models^\rho \).

First Pruning Lemma If \( \beta \in K^\rho \) and \( \phi, \rho \in L_\beta \) then: \( \beta \models^\rho \phi \iff \beta \models \phi^\rho \).

Proof: This is proved by induction on \( \phi \), the two relatively complex cases being \( \to \) and \( \lor \).

Case \( \phi = \phi_1 \to \phi_2 \). (\( \Rightarrow \) : Assume \( \beta \not\models^\rho \phi_1 \to \phi_2 \). Then, for some \( \beta' \) such that \( \beta \leq^\rho \beta' \), \( \beta' \not\models^\rho \phi_1 \) and \( \beta' \not\models^\rho \phi_2 \). Obviously, \( \beta \leq \beta' \) and \( L_\beta \subseteq L_{\beta'} \), so \( \phi_1, \phi_2, \rho \in L_{\beta'} \). By the induction hypothesis \( \beta' \not\models^\rho \phi_1 \) and \( \beta' \not\models^\rho \phi_2 \), whence it follows that \( \beta' \not\models^\rho \phi_1 \to \phi_2 \), i.e., \( \beta' \not\models (\phi_1 \to \phi_2)^\rho \).

(\( \Leftarrow \) : Assume \( \beta \not\models (\phi_1 \to \phi_2)^\rho \), i.e., \( \beta \not\models \phi_1 \to \phi_2 \). Then, for some \( \beta' \) such that \( \beta \leq \beta' \), \( \beta' \not\models \phi_1 \) and \( \beta' \not\models \phi_2 \). Since \( \rho \models \phi^\rho \) (fact A, Section 1), it follows
that $\beta' \not\vDash \rho$. Hence $\beta' \in K^\rho$ and $\beta \leq^\rho \beta'$. Obviously $\phi_1, \phi_2, \rho \in L_{\beta'}$, so we can apply the induction hypothesis to obtain $\beta' \not\vDash^\rho \phi_1$ and $\beta' \not\vDash^\rho \phi_2$, whence it follows that $\beta' \not\vDash^\rho \phi_1 \rightarrow \phi_2$.

Case $\phi = \forall x \phi_1$. ($\Rightarrow$) Assume $\beta \not\vDash^\rho \forall x \phi_1(x)$ (writing $'\phi_1(x)$' for $'\phi_1'$). Then, for some $\beta'$ such that $\beta \leq^\rho \beta'$, and for some $c \in D_{\beta'}$, $\beta' \not\vDash^\rho \phi_1(c)$. Obviously, $\beta \leq \beta'$ and $L_{\beta'} \subseteq L_{\beta'}$, so $\forall x \phi_1, \rho \in L_{\beta'}$. Moreover $D_{\beta'} \subseteq D_{\beta'}$, so $c \in D_{\beta'}$ and $\phi_1(c) \in L_{\beta'}$. By the induction hypothesis $\beta' \not\vDash^\rho (\phi_1(c))^\rho$. Since $\rho$ is a sentence, $(\phi_1(c))^\rho = (\phi_1)^{[c/x]}$. It follows that $\beta \not\vDash \forall x (\phi_1)^{[c/x]}$, i.e., $\beta \not\vDash (\forall x \phi_1)^\rho$.

($\Leftarrow$) Assume $\beta \not\vDash (\forall x \phi_1)$, i.e., $\beta \not\vDash \forall x (\phi_1)^\rho$. Then, for some $\beta'$ such that $\beta \leq \beta'$, and for some $c \in D_{\beta'}$, $\beta' \not\vDash (\phi_1(c))^\rho$, i.e., $\beta' \not\vDash (\phi_1(c))^\rho$. Since $\rho \vdash (\phi_1(c))^\rho$ (fact A), it follows that $\beta \not\vDash^\rho (\phi_1(c))^\rho$. Since $\rho \vdash (\phi_1(c))^\rho$ (fact A), it follows that $\beta \not\vDash^\rho (\phi_1(c))^\rho$. Hence $\beta \not\vDash^\rho (\forall x \phi_1)$.

Second Pruning Lemma Let $L$ be the language of arithmetic. If $\beta \in K^\rho$ and $\rho \in L_{\beta'}$ and $\beta \not\vdash HA$ then $\beta \not\vDash^\rho HA$.

Proof: Assume $\beta \in K^\rho$, $\rho \in L_{\beta'}$, $\beta \vdash HA$. Let $\phi$ be any theorem of HA. Since $HA \vdash \phi^\rho$ (fact C), it follows that $\beta \not\vDash \phi^\rho$. According to the first pruning lemma and $\phi \in L_{\beta'}$, $\beta \not\vDash \phi$. Hence $\beta \not\vDash^\rho \phi$. Hence $\beta \not\vDash^\rho HA$.

3 Spotting Peano models From now on we shall assume that $L$ is (any suitable variant or extension of) the language of arithmetic.

Theorem 1 The local models in finite Kripke models of Heyting arithmetic are Peano models.

Proof: Let $K$ be a finite Kripke model, $\alpha \in K$, $\alpha \vdash HA$. Avoiding $\alpha$, we shall apply several prunings to $K$. Construct a sequence of models $K^{(0)}, \ldots, K^{(n)}$ as follows. Let $K^{(0)}$ be $K$. Let $K^{(i)}$ be given and assume $\alpha \in K^{(i)}$. If there is a sentence $\rho \in L_{\alpha}^{(i)}$ such that $\alpha \not\vDash^{(i)} \rho$ whereas some $\beta \in K^{(i)}$ can be found such that $\beta \vdash^{(i)} \rho$, take any such $\beta$ and let $K^{(i+1)}$ be the model obtained by pruning $\rho$-nodes from $K^{(i)}$. Otherwise, if there is no such $\rho$, the construction will halt. Let $n$ be the stage where the process halts.

Claim $\alpha$ is a classical node in $K^{(n)}$. For, let $\rho$ be any sentence $\forall x_1 \ldots \forall x_n (\phi \vee \neg \phi) \in L_{\alpha}^{(n)}$. Let $\beta$ be some final node such that $\alpha \leq \beta$. $\beta$ is classical (fact 2, Section 1) and $L_{\alpha}^{(n)} \subseteq L_{\beta}^{(n)}$. Hence $\beta \vdash^{(n)} \rho$, and by definition of $n \alpha \vdash^{(n)} \rho$. Further, it follows from $\alpha \vdash HA$, by the second pruning lemma, that $\alpha \vdash^{(i)} HA$ (for all $1 \leq i \leq n$). Hence $M_{\alpha}^{(n)}$ will be a Peano model (fact 4). But $M_{\alpha}^{(n)} = M_{\alpha}$.

Corollary Let $\alpha$ be a node in a Kripke model $K$ such that $\alpha \vdash HA$. Let $K^{\alpha}$ be finite. Then $M_{\alpha}$ is a Peano model.

There seem to be no straightforward extensions of this result to infinite Kripke models. However, if the underlying structure is of type $\omega$, we have:

Theorem 2 A Kripke model of HA over $\omega$ (with its natural order) contains infinitely many local Peano models.

Proof: Let $K = \langle \omega, \leq, D, I \rangle$ be a Kripke model of HA (i.e., for each $n \in \omega$, $n \vdash HA$), where $\leq$ is the natural ordering on $\omega$. 
Case 1. Let $K$ contain a classical node $n$. Then all $m > n$ will be classical as well (fact 3, Section 1). For each such $m$, since $m \Vdash \text{HA}$, $M_n$ will be a Peano model (fact 4).

Case 2. Let $K$ contain no classical nodes. Consider the set $A = \{ n \mid n \in \omega \text{ and for all } \phi \in L_n: \text{ if } n + 1 \Vdash \phi \text{ then } n \Vdash \phi \}$. We shall first show that

(i) $\omega \sim A$ is infinite.

Suppose $\omega \sim A$ were finite. Let $n$ be such that, for all $m \geq n$, $m \in A$. Since $n$ is not classical, there is a sentence $\forall x_1 \ldots \forall x_r (\phi \lor \neg \phi) \in L_n$ such that $n \not\Vdash \forall x_1 \ldots \forall x_r (\phi \lor \neg \phi)$. Hence, for some $m \geq n$ and for certain $c_1, \ldots, c_r \in D_m$, $m \not\Vdash \phi(c_1 \ldots c_r) \lor \neg \phi(c_1 \ldots c_r)$. Let $\phi' = \phi(c_1, \ldots, c_r)$. Then $m \not\Vdash \phi'$ and $m \Vdash \neg \phi'$. Hence, for some $k > m$, $k \Vdash \phi'$. Let $k^*$ be minimal with the property: $k^* > m$, $k^* \Vdash \phi'$. Then $k^* - 1 \not\Vdash \phi'$ and $k^* - 1 \geq m \geq n$. Since $\phi' \in L_{k^*-1}$, it follows that $k^* - 1 \in \omega \sim A$, contradicting the choice of $n$. Therefore (i) holds.

Let $K^-(= (K^-, \leq, D^-, I^-))$ be the model obtained from $K$ by deleting all nodes in $A$. Forcing in $K^-$ will be denoted by $\Vdash^-$. It can be shown, by a simultaneous induction on $\phi$ for all $n \in K^-$, that the following holds:

(ii) For all $n \in K^-$, $\phi \in L_n$, $n \Vdash \phi$ iff $n \Vdash^-= \phi$.

We consider the case of the implication.

$\phi = \phi_1 \rightarrow \phi_2$. (\(\Rightarrow\) : ) Assume $n \Vdash^- \phi_1 \rightarrow \phi_2$. Then, for some $m$ such that $n \leq m$, $m \Vdash \phi_1$ and $m \Vdash^- \phi_2$. Obviously $n \leq m$ and $\phi_1, \phi_2 \in L_m$. According to the induction hypothesis $m \Vdash \phi_1$ and $m \not\Vdash \phi_2$. Hence $n \not\Vdash \phi_1 \rightarrow \phi_2$.

(\(\Rightarrow\) : ) Assume $n \not\Vdash \phi_1 \rightarrow \phi_2$. Then, for some $m$, such that $n \leq m$, $m \not\Vdash \phi_1$ and $m \Vdash \phi_2$. Suppose first that $m \in K^-$. Since $\phi_1, \phi_2 \in L_m$, it follows by the induction hypothesis that $m \Vdash^- \phi_1$ and $m \not\Vdash \phi_2$. Obviously $n \leq m$, so $n \not\Vdash \phi_1 \rightarrow \phi_2$. Now suppose that $m \in K^-$. Since (i) holds there is a $k > m$ such that $k \in K^-$. Let $k^*$ be minimal with that property: $k^* > m$, $k^* \in K^-$. Then, for all $k$ such that $m \leq k < k^*$, $k \in A$ and also $\phi_2 \in L_k$. By definition of $A$ the following holds: if $k \not\Vdash \phi_2$ then $k + 1 \not\Vdash \phi_2$. Hence, since $m \not\Vdash \phi_2$, $k^* \not\Vdash \phi_2$. On the other hand $k^* \Vdash \phi_1$ (cumulation). Since $\phi_1, \phi_2 \in L_{k^*}$, it follows by the induction hypothesis that $k^* \Vdash^- \phi_1$ and $k^* \not\Vdash^- \phi_2$. Since obviously $n \leq k^*$ we may conclude that $n \not\Vdash^-= \phi_1 \rightarrow \phi_2$. The case $\phi = \forall x \phi_1$ can be treated similarly, whereas the other cases are even simpler. So (ii) holds.

An immediate consequence of (ii) is that for each node $n \in K^- n \not\Vdash \text{HA}$. We shall now show that $M_n$ is a Peano model. Since $n \not\in A$, there is a sentence $\rho \in L_n (= L_n)$ such that $n \not\Vdash \rho$ and $n + 1 \Vdash \rho$. According to (ii) $n \not\Vdash \rho$, hence the model $K^- \rho$ exists and contains $n$. By the second pruning lemma it follows that $n \not\Vdash^-= \rho \text{ HA}$. Moreover $n$ is a final node of $K^- \rho$. For if $n < m \geq n$, it follows that $n + 1 \leq m$, therefore $m \Vdash \rho$ (cumulation) and by (ii) $m \not\Vdash \rho$. Hence $m$ will be pruned away. Since $n$ is final it is classical in $K^- \rho$ (fact 2, Section 1) and so $M_n^\rho$ is a Peano model (fact 4). But $M_n = M_n^\rho$, hence each of the infinitely many $M_n$ such that $n \in K^-$ is a Peano model.

4 Other applications of pruning Friedman's proof of Markov's rule (MR) (cf. Friedman, [2]) has a model theoretic version.
MR Let $\phi \in \Sigma^0_1$. Then $\text{HA} \vdash \forall x_1 \ldots \forall x_n (\neg \neg \phi \rightarrow \text{HA} \vdash \forall x_1 \ldots \forall x_n \phi)$.

Proof: Assume $\phi_0 \in \Sigma^0_1$, $\text{HA} \vdash \forall x_1 \ldots \forall x_n (\neg \phi_0$, but $\text{HA} \nvdash \forall x_1 \ldots \forall x_n \phi_0$. By the completeness theorem there is a Kripke model $K$ of $\text{HA}$ with a node $\alpha$ such that $\alpha \not\vDash \forall x_1 \ldots \forall x_n \phi_0$. Therefore, $K$ contains a node $\beta$ such that, for certain $c_1, \ldots, c_n \in D_\beta$, $\beta \vDash \phi_0(c_1, \ldots, c_n)$. Put $\phi = \phi_0(c_1, \ldots, c_n)$, then $\phi \in L_\beta$ and $\beta \vDash \phi$. Hence $K^\phi$ exists and $\beta \in K^\phi$. According to the second pruning lemma, $\beta \vDash ^\phi \text{HA}$, so $\beta \vDash ^\phi \neg \neg \phi$. Consequently $\beta \vDash ^\phi \neg \phi$ and there is some $\gamma \in K^\phi$ such that $\gamma \vDash ^\phi \phi$. By the first pruning lemma $\gamma \vDash \phi$. Since $\phi \in \Sigma^0_1$, $\phi^\phi$ is equivalent to $\phi \lor \phi$ in $\text{HA}$ (fact D, Section 1). Since $\gamma \vDash \text{HA}$, $\gamma \vDash \phi \lor \phi$. Therefore $\gamma \vDash \phi$. This means that $\gamma$ must have been pruned away, contradicting $\gamma \in K^\phi$.

In the same way we can formulate a model-theoretic version of Visser’s proof of the following (cf. [4]):

VR Let $\phi \in \Sigma^0_1$. Then $\text{HA} \vdash \forall x_1 \ldots \forall x_n (\neg \neg \phi \rightarrow \phi)$ implies $\text{HA} \vdash \forall x_1 \ldots \forall x_n (\phi \lor \neg \phi)$.

Proof: Assume $\phi_0 \in \Sigma^0_1$, $\text{HA} \vdash \forall x_1 \ldots \forall x_n (\neg \phi_0 \rightarrow \phi_0)$, but $\text{HA} \nvdash \forall x_1 \ldots \forall x_n (\phi_0 \lor \neg \phi_0)$. By the completeness theorem there is a Kripke model $K$ of $\text{HA}$ with a node $\alpha$ such that $\alpha \not\vDash \forall x_1 \ldots \forall x_n \phi_0 \lor \neg \phi_0$. Therefore, $K$ contains a node $\beta$ such that for certain $c_1, \ldots, c_n \in D_\beta$, $\beta \vDash \phi_0 \lor \neg \phi_0$, where $\phi = \phi_0(c_1, \ldots, c_n)$. Certainly, $\neg \phi \in L_\beta$ and $\beta \vDash \neg \phi$, hence $K^{\neg \phi}$ exists and $\beta \in K^{\neg \phi}$. According to the second pruning lemma $\beta \vDash ^{\neg \phi} \text{HA}$, so $\beta \vDash ^{\neg \phi} \neg \neg \phi \rightarrow \phi$. Consider any $\gamma \in K^{\neg \phi}$ such that $\beta \leq ^{\neg \phi} \gamma$. For such $\gamma$: $\gamma \vDash \neg \phi$, whereas $\neg \phi \in L_\gamma$, therefore there is some $\gamma'$ such that $\gamma \leq ^{\neg \phi} \gamma'$ and $\gamma' \vDash \phi$. Since $\gamma' \vDash \neg \phi$ it follows that $\gamma' \in K^{\neg \phi}$ and $\gamma \leq ^{\neg \phi} \gamma'$. Obviously, $\gamma' \vDash \phi \lor \neg \phi$. Since $\phi \in \Sigma^0_1$, $\phi \lor \neg \phi$ is equivalent to $\phi^{\neg \phi}$ in $\text{HA}$ (fact D). By the first pruning lemma $\gamma' \vDash ^{\neg \phi} \phi$. Therefore $\gamma \vDash ^{\neg \phi} \neg \phi$. Since this holds for any $\gamma$ such that $\beta \leq ^{\neg \phi} \gamma$ we can conclude that $\beta \vDash ^{\neg \phi} \neg \phi$ and therefore $\beta \vDash ^{\neg \phi} \phi$. Applying the first pruning lemma once more we get $\beta \vDash ^{\neg \phi} \phi^\phi$, and, again by fact D, $\beta \vDash \phi \lor \neg \phi$, a contradiction.

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