A new subgrid characteristic length for turbulence simulations on anisotropic grids
Trias, Francesc Xavier; Gorobets, Andrey; Silvis, Maurits H.; Verstappen, R.W.C.P.; Oliva, Assensi

Published in:
Physics of Fluids

DOI:
10.1063/1.5012546

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2017

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

Copyright
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

Take-down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.
A new subgrid characteristic length for turbulence simulations on anisotropic grids
F. X. Trias, A. Gorobets, M. H. Silvis, R. W. C. P. Verstappen, and A. Oliva

Citation: Physics of Fluids 29, 115109 (2017); doi: 10.1063/1.5012546
View online: https://doi.org/10.1063/1.5012546
View Table of Contents: http://aip.scitation.org/toc/phf/29/11
Published by the American Institute of Physics

Articles you may be interested in
Physical consistency of subgrid-scale models for large-eddy simulation of incompressible turbulent flows
Physics of Fluids 29, 015105 (2017); 10.1063/1.4974093

Building proper invariants for eddy-viscosity subgrid-scale models
Physics of Fluids 27, 065103 (2015); 10.1063/1.4921817

A priori study of subgrid-scale features in turbulent Rayleigh-Bénard convection
Physics of Fluids 29, 105103 (2017); 10.1063/1.5005842

Turbulent kinetic energy budgets in wall bounded flows with pressure gradients and separation
Physics of Fluids 29, 115108 (2017); 10.1063/1.4992793

Dynamic large eddy simulation: Stability via realizability
Physics of Fluids 29, 105104 (2017); 10.1063/1.4986890

A dynamic subgrid-scale eddy viscosity model
A new subgrid characteristic length for turbulence simulations on anisotropic grids

F. X. Trias,1,a) A. Gorobets,1,b) M. H. Silvis,3,c) R. W. C. P. Verstappen,3,d) and A. Oliva1,e)
1Heat and Mass Transfer Technological Center, Technical University of Catalonia, C/Colom 11, 08222 Terrassa, Spain
2Keldysh Institute of Applied Mathematics, 4A, Miusskaya Sq., Moscow 125047, Russia
3Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands

(Received 12 May 2017; accepted 6 November 2017; published online 29 November 2017)

Direct numerical simulations of the incompressible Navier-Stokes equations are not feasible yet for most practical turbulent flows. Therefore, dynamically less complex mathematical formulations are necessary for coarse-grained simulations. In this regard, eddy-viscosity models for Large-Eddy Simulation (LES) are probably the most popular example thereof. This type of models requires the calculation of a subgrid characteristic length which is usually associated with the local grid size. For isotropic grids, this is equal to the mesh step. However, for anisotropic or unstructured grids, such as the pancake-like meshes that are often used to resolve near-wall turbulence or shear layers, a consensus on defining the subgrid characteristic length has not been reached yet despite the fact that it can strongly affect the performance of LES models. In this context, a new definition of the subgrid characteristic length is presented in this work. This flow-dependent length scale is based on the turbulent, or subgrid stress, tensor and its representations on different grids. The simplicity and mathematical properties suggest that it can be a robust definition that minimizes the effects of mesh anisotropies on simulation results. The performance of the proposed subgrid characteristic length is successfully tested for decaying isotropic turbulence and a turbulent channel flow using artificially refined grids. Finally, a simple extension of the method for unstructured meshes is proposed and tested for a turbulent flow around a square cylinder. Comparisons with existing subgrid characteristic length scales show that the proposed definition is much more robust with respect to mesh anisotropies and has a great potential to be used in complex geometries where highly skewed (unstructured) meshes are present. Published by AIP Publishing. https://doi.org/10.1063/1.5012546

I. INTRODUCTION

The Navier-Stokes (NS) equations are an excellent mathematical model for turbulent flows. However, direct numerical simulations are not feasible yet for most practical turbulent flows because the nonlinear convective term produces far too many scales of motion. Hence, in the foreseeable future, numerical simulations of turbulent flows will have to resort to models of the small scales. The most popular example thereof is Large-Eddy Simulation (LES). Briefly, the LES equations arise from applying a spatial filter, with filter length \( \Delta \), to the NS equations, resulting in

\[
\partial_t \overline{u} + (\overline{u} \cdot \nabla) \overline{u} = \nu \Delta \overline{u} - \nabla \overline{\tau} - \nabla \cdot \tau, \quad \nabla \cdot \overline{u} = 0. \tag{1}
\]

Here, \( \overline{u} \) is the filtered velocity and \( \tau = \overline{u} \otimes \overline{u} - \overline{u} \otimes \overline{u} \) is the subgrid stress (SGS) tensor that represents the effect of the unresolved scales. It is assumed that the filter \( u \rightarrow \overline{u} \) commutes with differentiation. Since the SGS tensor, \( \tau \), depends not only on the filtered velocity, \( \overline{u} \), but also on the full velocity field, \( u \), we encounter a closure problem. We thus have to approximate \( \tau \) by a tensor depending only on the filtered velocity, i.e., \( \tau \approx \overline{\tau} \).

Because of its inherent simplicity and robustness, the eddy-viscosity assumption is by far the most used closure model,

\[
\tau(\overline{u}) \approx -2\nu_e \mathbf{S}(\overline{u}), \tag{2}
\]

where \( \nu_e \) denotes the eddy-viscosity and \( \mathbf{S}(\overline{u}) = 1/2(\nabla \overline{u} + \nabla \overline{u}^T) \) is the rate-of-strain tensor. Notice that \( \tau(\overline{u}) \) is considered traceless without loss of generality because the trace can be included as part of the filtered pressure, \( \overline{p} \). Then, most of the existing eddy-viscosity models can be expressed as follows:

\[
\nu_e = (C_m \Delta)^2 D_m(\overline{u}), \tag{3}
\]

where \( C_m \) is the model constant, \( \Delta \) is the subgrid characteristic length, and \( D_m(\overline{u}) \) is the differential operator with units of frequency associated with the model. Here, no summation over \( m \) is implied.

In the last few decades research has primarily focused on either the calculation of the model constant, \( C_m \), or the development of more appropriate model operators, \( D_m(\overline{u}) \). An example of the former is the approach proposed by Lilly\(^1\) to determine the model constant of the Smagorinsky model.\(^2\)
For an isotropic mesh, i.e., $\Delta = \Delta x = \Delta y = \Delta z$, and under the assumption that the cutoff wave number $k_c = \pi / \Delta$ lies within the inertial range of a universal Kolmogorov spectrum, $E(k) \approx C K^{\alpha/3} k^{-5/3}$, a model’s constant, $C_m$, can be found by assuming that its dissipation is equal to the turbulent kinetic energy dissipation, $\varepsilon$. In this way, Lilly\textsuperscript{1} obtained the Smagorinsky constant $C_S = (2/3 C_k)^{\alpha/3} / \pi$ (taking a value of $C_K \approx 1.58$ for the Kolmogorov constant\textsuperscript{3} leads to $C_S \approx 0.17$).

The classical Smagorinsky model\textsuperscript{2} has the disadvantage that its differential operator, $D_m(\mathbf{u}) = |\nabla(\mathbf{u})|$, does not vanish near solid walls. Early attempts to overcome this inherent problem of the Smagorinsky model made use of wall functions.\textsuperscript{4,5} Later, Germano et al.\textsuperscript{6} proposed the dynamic procedure, in which the constant $C_m$ is computed with the help of the Jacobi identity (in least-squares sense), as was originally proposed by Lilly.\textsuperscript{7} However, this approach leads to highly variable coefficient fields with a significant fraction of negative values for $\nu_v$. This can cause numerical instability in simulations. Thus, averaging with respect to the homogeneous direction(s) and ad hoc clipping of $\nu_v$ are, in general, necessary. Therefore, the original dynamic procedure cannot be applied to geometrically complex flows without homogeneous directions. Several attempts to overcome these intrinsic limitations can be found in the literature: namely, the dynamic localization model and the Lagrangian dynamic model were, respectively, proposed by Ghosal et al.\textsuperscript{8} and Meneveau et al.\textsuperscript{9} In the same vein, Park et al.\textsuperscript{10} introduced two global dynamic approaches: a dynamic global model based on the Germano identity\textsuperscript{11} and a dynamic global model with two test filters based on the “global equilibrium” between the viscous dissipation and the SGS dissipation. Later, You and Moin\textsuperscript{12} presented a dynamic global approach using only one test filter. Tejada-Martínez and Jansen\textsuperscript{13,14} proposed an approach where the filter width ratio, the sole model parameter in a dynamic Smagorinsky model, is computed dynamically too. To do so, they assumed scale invariance and made use of a secondary test filter.

To construct models that vanish near solid walls, one can alternatively change the differential operator, $D_m(\mathbf{u})$. Examples thereof are the WALE model,\textsuperscript{15} Vreman’s model,\textsuperscript{16} the QR model,\textsuperscript{17} and the $\sigma$-model.\textsuperscript{18} This list can be completed with a novel eddy-viscosity model proposed by Ryu and Iaccarino\textsuperscript{19} and two eddy-viscosity models recently proposed by the authors of this paper: namely, the S3PQR model\textsuperscript{20} and the vortex-stretching-based eddy-viscosity model.\textsuperscript{21}

Surprisingly, in the LES community, little attention has been paid to the computation of the subgrid characteristic length, $\Delta$, which is also a key element of any eddy-viscosity model [see Eq. (3)]. Due to its simplicity and applicability to unstructured meshes, nowadays the most widely used approach to compute the subgrid characteristic length is the one proposed by Deardorff,\textsuperscript{22} i.e., the cube root of the cell volume. For a Cartesian grid, it reads

$$\Delta_{\text{vol}} = (\Delta x \Delta y \Delta z)^{1/3}.$$  \hspace{1cm} (4)

Extensions of this approach for anisotropic grids were proposed by Schumann,\textsuperscript{23} Lilly,\textsuperscript{24} and Scotti et al.\textsuperscript{25} It was found that for small anisotropies, Deardorff’s length scale is reasonably accurate, whereas corrections are required for highly anisotropic meshes, such as the pancake-like meshes that are often used to resolve near-wall turbulence or shear layers. For instance, the following correction was proposed by Scotti et al.\textsuperscript{25}

$$\Delta_{\text{Sco}} = f(a_1, a_2)\Delta_{\text{vol}},$$  \hspace{1cm} (5)

where $f(a_1, a_2) = \sqrt[4/3]{27(\ln a_1)^2 - \ln a_1 \ln a_2 + (\ln a_2)^2}$ and $a_1 = \Delta x / \Delta z$ and $a_2 = \Delta y / \Delta z$, assuming that $\Delta x \leq \Delta z$ and $\Delta y \leq \Delta z$. Nevertheless, it is “still assumed, however, that the small scale limit of the simulation is in the Kolmogorov inertial sub-range in all directions. This is a serious limitation as the most common reason for applying anisotropic resolution is an expectation of anisotropic and/or inhomogeneous turbulence, typically in regions close to a boundary.”\textsuperscript{26}

The definition of $\Delta$ given in Eq. (5) was tested by Scotti et al.\textsuperscript{26} for forced isotropic turbulence using highly anisotropic grids. They compared the correction factor $f(a_1, a_2)$ with the correction $f_{\text{dyn}}(a_1, a_2)$ obtained by applying the above-mentioned dynamic approach to $C_m \Delta_{\text{vol}} f_{\text{dyn}}(a_1, a_2)$ (the dynamic approach is usually applied to find the model constant, $C_m$). They reached the conclusion that the dynamic model reproduces the correct trend for pancake-like grids ($a_2 = 1$, $\Delta x \ll \Delta y \ll \Delta z$) but fails for pencil-like ones ($a_1 = a_2$, $\Delta x = \Delta y \ll \Delta z$). Another limitation of the approach proposed by Scotti et al.\textsuperscript{25} [see Eq. (5)] is that it is applicable only to structured Cartesian grids. To circumvent this, Colosqui and Oberai\textsuperscript{27} proposed an extension applicable to unstructured meshes. They assume that the second-order structure function satisfies Kolmogorov’s hypotheses.

Alternative definitions of the subgrid characteristic length scale, $\Delta$, include the maximum of the cell sizes,

$$\Delta_{\text{max}} = \max(\Delta x, \Delta y, \Delta z),$$ \hspace{1cm} (6)

the $L^2$-norm of the tensor $\Delta = \text{diag}(\Delta x, \Delta y, \Delta z)$ divided by $\sqrt{3}$,

$$\Delta_{L2} = \sqrt{\langle \Delta x^2 + \Delta y^2 + \Delta z^2 \rangle / 3},$$ \hspace{1cm} (7)

and the square root of the harmonic mean of the squares of the grid sizes

$$\Delta_{\text{Lapl}} = \sqrt{3/(\Delta x^2 + 1/\Delta y^2 + 1/\Delta z^2)},$$ \hspace{1cm} (8)

which is directly related to the largest eigenvalue of the discrete approximation of the (negative) Laplacian, $-\nabla^2$. The first definition, $\Delta_{\text{max}}$, was originally proposed in the first presentation of the Detached-Eddy Simulation (DES) method by Spalart et al.\textsuperscript{28} as a safer and robust definition of $\Delta$.

More recent definitions of the subgrid characteristic length scale are also found in the context of DES. Namely, Chauvet et al.\textsuperscript{29} introduced the concept of sensitizing $\Delta$ to the local velocity field. In particular, they made $\Delta$ dependent on the orientation of the vorticity vector, $\omega = (\omega_x, \omega_y, \omega_z) = \nabla \times \mathbf{u}$,

$$\Delta_{\omega} = \sqrt{\langle \omega_x^2 \Delta y \Delta z + \omega_y^2 \Delta x \Delta z + \omega_z^2 \Delta x \Delta y \rangle / |\omega|^2},$$ \hspace{1cm} (9)

with some minor corrections to prevent indeterminate forms of type $0/0$ (see the original paper\textsuperscript{29} for details). The formulation was subsequently generalized for unstructured meshes.
by Deck. The definition of $\Delta_\omega$ detects the alignment of the vorticity vector, $\omega$, with an axis, e.g., if $\omega = (0, 0, \omega_z)$, then $\Delta_\omega$ reduces to $\sqrt{l_x l_y}$. This approach was motivated by the fact that $\Delta_{\text{max}}$ results in an excessive generation of SGS dissipation in the initial region of shear layers typically resolved on highly anisotropic grids. In a DES, this results in an artificial delay of Kelvin-Helmholtz instabilities because the model switches from Reynolds-averaged Navier-Stokes (RANS) to LES mode further downstream. However, similar to Deardorff’s definition of $\Delta$ given in Eq. (4), the definition given in Eq. (9) may still involve the smallest of the grid spacings. This may lead to very low values of eddy-viscosity.

To circumvent this problem, recently Mockett et al. proposed the following flow-dependent subgrid characteristic length scale:

$$\tilde{\Delta}_\omega = \frac{1}{\sqrt{3}} \max_{n,m=1,...,8} |l_n - l_m|,$$  \hspace{1cm} (10)

where $l = \omega/|\omega| \times r_n$ and $r_n$ $(n = 1, \ldots, 8$ for a hexahedral cell) are the locations of the cell vertices. The quantity $\tilde{\Delta}_\omega$ represents the diameter of the set of cross-product points, $l_n$, divided by $\sqrt{3}$. In the above-described case with $\omega = (0, 0, \omega_z)$, it reduces to $\tilde{\Delta}_\omega = \sqrt{(l_x^2 + l_y^2)/3}$. Therefore, it is $O(\Delta_{\text{max}})$ (for the typical situation where $\Delta_x > \Delta_y$ and $\Delta_z > \Delta_y$) or $\Delta_\omega = \sqrt{l_x l_y}$. Thus “unlike $\Delta_\omega$, definition (10) never leads to a strong effect of the smallest grid-spacing on the subgrid-scale $\Delta$ even though it achieves the desired decrease compared to the $\Delta_{\text{max}}$ definition in the quasi-2D flow regions treated on strongly anisotropic grids.”

More recently, Shur et al. proposed to modify the definition of $\Delta_\omega$ given in Eq. (10) by introducing a nondimensional function $0 \leq F_{KH}(\text{VTM}) \leq 1$, resulting in the Shear Layer Adapted (SLA) subgrid-scale

$$\tilde{\Delta}_{\text{SLA}} = \tilde{\Delta}_\omega F_{KH}(\text{VTM}),$$  \hspace{1cm} (11)

where the Vortex Tilting Measure (VTM) is given by

$$\text{VTM} = \frac{|(\hat{S} \cdot \omega) \times \omega|}{\omega^2 \sqrt{\hat{S}}},$$  \hspace{1cm} (12)

where $\hat{S}$ is the traceless part of the rate-of-strain tensor, $S = 1/2(\nabla \vec{u} + \nabla \vec{u}^T)$, i.e., $\hat{S} = S - 1/3(\text{tr}(S))1$. Note that for incompressible flows $\text{tr}(\hat{S}) = \nabla \cdot \vec{u} = 0$, therefore, $\hat{S} = S$.

Finally $Q_A$ denotes the second invariant of a second-order tensor $\hat{A}$, $Q_A = 1/2[\text{tr}^2(A) - \text{tr}(A^2)]$. The vortex tilting measure is bounded, $0 \leq \text{VTM} \leq 1$, and it takes zero value when the vorticity is aligned with an eigenvector of $\hat{S}$ with eigenvalue $\lambda_i$, i.e., $\hat{S} \omega = \lambda_i \omega$. Therefore, the VTM can be viewed as a measure of how much the rate-of-strain tensor tilts the vorticity vector towards another direction. Finally, the function $F_{KH}$ is aimed at unlocking the Kelvin-Helmholtz instability in the initial part of shear layers. Different functions $F_{KH}$ were proposed by Shur et al. with the basic requirements that $0 \leq F_{KH}(\text{VTM}) \leq 1$, $F_{KH}(0) = 0$, and $F_{KH}(1) = 1$.

Despite the above-mentioned length scales, so far no consensus has been reached on how to define the subgrid characteristic length scale, particularly when considering anisotropic or unstructured grids. In this work, we therefore propose a new flow-dependent subgrid characteristic length scale that is based on the subgrid stress tensor, $\tau$, and its representations on different grids. This simple and robust definition of $\Delta$ reduces the effect of mesh anisotropies on the performance of SGS models.

The structure of this paper is as follows. In Sec. II, all the above-mentioned definitions of the subgrid characteristic length are compared and classified on the basis of a list of desirable properties. These properties are based on physical, numerical, and/or practical arguments. Then, within this framework, a new subgrid characteristic length, based on the Taylor-series expansion of the SGS tensor in the computational space, is proposed in Sec. III. Moreover, in Sec. IV, a simple extension of this length scale for unstructured grids is proposed. In Sec. V, the newly proposed length scale is tested in wall-resolved large-eddy simulations on highly anisotropic structured grids (test cases: decaying isotropic and a plane-channel flow) and unstructured grids (test case: turbulent flow around a square cylinder), confirming that it is a robust definition that reduces the effects of mesh anisotropies on the performance of LES models. Finally, relevant results are summarized and conclusions are given in Sec. VI.

II. PROPERTIES OF THE SUBGRID CHARACTERISTIC LENGTH

Starting from the classical Smagorinsky model, many eddy-viscosity models [see Eq. (2)] have been proposed (see the work of Trias et al. for a recent review). The definition of $\nu_e$ given in Eq. (3) provides a general template for most of them. Therefore, a subgrid characteristic length, $\Delta$, which is commonly associated with the local grid size, is required. Hence, for isotropic grids, $\Delta$ is equal to the mesh size, i.e., $\Delta = \Delta_x = \Delta_y = \Delta_z$. However, for anisotropic or unstructured grids, a consensus on defining the subgrid characteristic length has not been reached yet. Despite the fact that in some situations it may provide very inaccurate results, three and a half decades later, the approach proposed by Deardorff, i.e., the cube root of the cell volume [see Eq. (4)], is by far the most widely used.

Alternative methods to compute the subgrid characteristic length scale, $\Delta$, have been reviewed in Sec. I. They are classified in Table I according to a list of desirable properties for a (proper) definition of $\Delta$. These properties are based on physical, numerical, and/or practical arguments. Namely, the first property, denoted as $P1$, entails both positiveness and locality. Although from a physical point of view negative values of $\nu_e$ may be justified with the backscatter phenomenon, from a numerical point of view, the condition $\nu_e \geq 0$ is, in general, considered appropriate because it guarantees stability. Further, the LES equations should be Galilean invariant. In order to preserve this physical principle, the flow-dependent definitions of $\Delta$ (see property $P3$) are based on invariants derived from the gradient of the resolved velocity field, $\mathbf{G} \equiv \nabla \vec{u}$. In doing so, the condition of locality is also achieved. From a practical point of view, locality is a desirable feature especially if the model is aimed to be applied in complex flows. The second property ($P2$) requires that $\Delta$ is properly bounded, i.e., given a
TABLE I. Properties of different definitions of the subgrid characteristic length, \( \Delta \). Namely, \textbf{P1}: positive (\( \Delta \geq 0 \)), local, and frame invariant; \textbf{P2}: bounded, i.e., given a structured Cartesian mesh where \( \Delta x \leq \Delta y \leq \Delta z \), \( \Delta x \leq \Delta \leq \Delta z \); \textbf{P3}: sensitive to the local flow field; \textbf{P4}: applicable to unstructured meshes; \textbf{P5}: computational cost.

<table>
<thead>
<tr>
<th>Formula</th>
<th>( \Delta_{\text{vol}} )</th>
<th>( \Delta_{\text{SCO}} )</th>
<th>( \Delta_{\text{max}} )</th>
<th>( \Delta_{L2} )</th>
<th>( \Delta_{Lapl} )</th>
<th>( \Delta_{\omega} )</th>
<th>( \Delta_{\text{eq}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textbf{P1}</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>\textbf{P2}</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>\textbf{P3}</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>\textbf{P4}</td>
<td>Yes</td>
<td>No</td>
<td>No ( ^a )</td>
<td>No</td>
<td>No ( ^a )</td>
<td>No ( ^b )</td>
<td>Yes</td>
</tr>
<tr>
<td>\textbf{P5}</td>
<td>Low</td>
<td>Medium</td>
<td>Low</td>
<td>Low</td>
<td>Low</td>
<td>Medium</td>
<td>High</td>
</tr>
</tbody>
</table>

\( ^a \) Possible with some adaptations.

\( ^b \) Deck \( ^b \) proposed a generalization for unstructured meshes.

structured Cartesian mesh where \( \Delta x \leq \Delta y \leq \Delta z \), we need \( \Delta x \leq \Delta \leq \Delta z \). These first two properties \textbf{P1} and \textbf{P2} are achieved by all the length scales shown in Table I. The third property (\textbf{P3}) classifies the methods to compute \( \Delta \) in two families: those that solely depend on geometrical properties of the mesh and those that are also dependent on the local flow topology, i.e., the velocity gradient, \( \mathbf{G} \).

Assuming that the grid is Cartesian, we can express the subgrid characteristic length scales that are fully mesh-based in terms of the properties of the following second-order diagonal tensor:

\[
\Delta = \text{diag}(\Delta x, \Delta y, \Delta z). \tag{13}
\]

We take \( \Delta x \leq \Delta y \leq \Delta z \) without loss of generality. Indeed the aforementioned mesh-based length definitions can be written as

\[
\Delta_{\text{vol}} = R_{\Delta}^{1/3}, \quad \Delta_{\text{SCO}} = f(a_1, a_2)R_{\Delta}^{1/3}, \quad \Delta_{\text{max}} = \lambda_1^{\Delta},
\]

\[
\Delta_{L2} = \sqrt{\frac{\text{tr}(\Delta^2)}{3}} = \sqrt{\frac{\mathcal{P}^2_{\Delta} - 2Q_{\Delta}}{3}}, \tag{14}
\]

where the correction function \( f(a_1, a_2) \) was defined in Eq. (5), \( a_1 = \Delta x/\Delta z \) and \( a_2 = \Delta y/\Delta z \). Moreover, \( P_{\Delta} = \text{tr}(\Lambda), \quad Q_{\Delta} = 1/2[\text{tr}^2(\Delta) - \text{tr}(\Delta^2)], \) and \( R_{\Delta} = \text{det}(\Delta) = 1/6[\text{tr}^3(\Delta) - 3\text{tr}(\Delta)\text{tr}(\Delta^2) + 2\text{tr}(\Delta^3)] \) represent the first, second, and third invariants of the second-order tensor \( \Delta \), respectively. The three eigenvalues, \( \lambda_{1 \Delta} \geq \lambda_{2 \Delta} \geq \lambda_{3 \Delta} \), of \( \Delta \) are solutions of the characteristic equation \( \text{det}(\Delta - \lambda I) = \lambda^3 - P_{\Delta} \lambda^2 + Q_{\Delta} \lambda - R_{\Delta} = 0 \).

The aforementioned characteristic length scale definitions that depend on both the mesh and the flow topology can also be expressed in terms of invariants, namely, the invariants of \( \Delta \), the velocity gradient \( \mathbf{G} \), and \( \mathbf{G} \)'s symmetric and anti-symmetric parts. To this end, note that the vorticity vector \( \mathbf{\omega} = (\omega_x, \omega_y, \omega_z) = \nabla \times \mathbf{u} \) can be expressed in terms of the rate-of-rotation tensor \( \Omega = 1/2(\mathbf{G} - \mathbf{G}^T) \) as \( \omega_k = -\epsilon_{ijk} \Omega_{ij} \), where \( \epsilon_{ijk} \) is the Levi-Civita symbol. Here, summation over repeated indices is implied. Finally, the symmetric part of \( \mathbf{G} \) is the rate-of-strain tensor, \( \mathbf{S} = 1/2(\mathbf{G} + \mathbf{G}^T) \), which are the flow-dependent quantities required to compute the definitions of \( \Delta \) of Chauvet et al. \( ^{29} \) [see the definition of \( \Delta_{\omega} \) given in Eq. (9)] and Mockett et al. \( ^{31} \) [see the definition of \( \Delta_{\omega} \) given in Eq. (10)] and the modification \( \Delta_{SLA} \) proposed by Shur et al. \( ^{32} \) [see Eq. (11)].

The last two desirable properties of subgrid characteristic length scales are of practical interest. Namely, property \textbf{P4} refers to the applicability of the method for unstructured meshes. Among the definitions of \( \Delta \) that do not depend on the local flow, only the approach of Deardorff \( ^{22} \) can be straightforwardly used for unstructured grids. Recent flow-dependent definitions of \( \Delta \) are potentially applicable for unstructured grids, although some of them have a relatively high computational cost. In this regard, to complete the list of properties, it is also desirable that the definition of \( \Delta \) is well conditioned and has a low (or moderate) computational cost (property \textbf{P5}). In this respect, flow-dependent definitions of \( \Delta \) may be problematic, having a significantly higher computational cost. Moreover, they require special attention for indeterminate forms of type 0/0.

The new definition of the subgrid characteristic length, \( \Delta_{\text{eq}} \), which is presented in Sec. III matches all the above-mentioned properties with an inherent simplicity and a moderate computational cost.

III. BUILDING A NEW SUBGRID CHARACTERISTIC LENGTH

Several approaches to compute the subgrid characteristic length, \( \Delta \), can be found in the literature (see Sec. I). Their properties have been analyzed and compared in Sec. II (see Table I). As remarked before, despite these existing length scales, no consensus has been reached on how to define the subgrid characteristic length scale, particularly for (highly) anisotropic or unstructured grids. In this section, we therefore propose a new flow-dependent subgrid characteristic length scale that is based on the subgrid stress tensor, \( \mathbf{\tau} \), and its representations on different grids.

The subgrid characteristic length, \( \Delta \), appears in a natural way when we consider the lowest-order approximation of the subgrid stress tensor, \( \mathbf{\tau} = \mathbf{u} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{u} \), i.e., the unclosed term in the filtered Navier-Stokes equations, Eq. (1). The approximation of the subgrid stress is obtained by approximating the residual velocity \( \mathbf{u}' = \mathbf{u} - \mathbf{u} \).

To start, we restrict ourselves to one spatial direction and consider a box filter. The residue of the box filter can be related to the error of the midpoint rule for numerical integration, denoted by \( \epsilon \) here. We have \( \mathbf{\tau} = \int_{-\Delta x/2}^{+\Delta x/2} \mathbf{u}(x) \mathbf{dx} = \Delta x \mathbf{u}(x) + \epsilon \) with \( \epsilon = \frac{\Delta x^2}{2} \partial_x \partial_x \mathbf{u}(c) \), where \( c \) lies somewhere in between \( x - \Delta x/2 \) and \( x + \Delta x/2 \). An expression for the residue of the one-dimensional box filter is then obtained by dividing.
this error by $-\Delta x$ and adding $u$. Thus to lowest order, we get $u'(x) = -\frac{\Delta^2}{2x} \partial_x \partial_x u(x) + O(\Delta^4)$.

On a three-dimensional, isotropic grid, i.e., $\Delta = \Delta x = \Delta y = \Delta z$, the above approximation of the residue becomes $u' = -\frac{\Delta^2}{2\gamma} \nabla \cdot \nabla u + O(\Delta^4)$. With the help of this approximation, it can be shown that the subgrid stress tensor is given by

$$\tau(\mathbf{u}) = \frac{\Delta^2}{12} \mathbf{G}^T \mathbf{G} + O(\Delta^4).$$

(15)

The leading-order term in Eq. (15) is the gradient model proposed by Clark et al., where $\Delta$ denotes the filter length. Equation (15) has been derived for the box filter. However, it can be shown that the same result is obtained for any convolution filter with a symmetric kernel.

We stress that in the above derivation, the grid is assumed to be isotropic, that is, $\Delta = \Delta x = \Delta y = \Delta z$. For an anisotropic grid, we can postulate that the lowest-order approximation of the subgrid stress also provides us with $\tau(\mathbf{u}) \approx \frac{\Delta^2}{12} \mathbf{G}^T \mathbf{G}$, that is, the approximation (1) depends quadratically on the velocity gradient, (2) is given by a symmetric tensor, (3) is invariant under a rotation of the coordinate system, and (4) is proportional to $\Delta^2$. Here, however, we do not yet know how to define the filter length, $\Delta$, because the grid is anisotropic. For the gradient model, however, we can define the filter length by mapping the anisotropic mesh onto an isotropic mesh. Therefore we consider the coordinate transformations $\tilde{x} = x/\Delta x$, $\tilde{y} = y/\Delta y$, and $\tilde{z} = z/\Delta z$. Expanding the subgrid stress as before, but now in the new, isotropic, coordinate system $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, and applying the chain rule for differentiation we obtain

$$\tau(\mathbf{u}) = \frac{1}{12} \mathbf{G}_A \mathbf{G}^T + O(\Delta^4).$$

(16)

Here, the velocity gradient on the anisotropic grid is defined as

$$\mathbf{G}_A \equiv \mathbf{G} \Delta,$$

(17)

where $\Delta$ is the second-order tensor containing the mesh information given by Eq. (13). Equation (16) does not require an explicit definition of the filter length $\Delta$. In fact the filter length is hidden in $\mathbf{G}_A$ and is not represented by a scalar but by the tensor $\Delta$. Since both Eqs. (15) and (16) represent the lowest-order approximation of the subgrid stress, we can equate them and thus define the filter length $\Delta$ in Eq. (15) for anisotropic meshes. Here it may be remarked that we equate tensors; hence, the equality is to be understood in least-square sense. This leads to the following flow-dependent definition of $\Delta$:

$$\Delta_{lsq} = \sqrt{\frac{G_{22} G_{33}^T \cdot \mathbf{G}^T \mathbf{G}}{G_{22}^T \cdot \mathbf{G}^T \mathbf{G}}}.$$  

(18)

We first remark that this length scale reduces to $\Delta$ on an isotropic mesh. Second, since $\Delta_{lsq}$ is formally based on the lowest-order approximation of the subgrid stress, we see it as a generic way to define the filter length. It can thus be applied in any turbulence model, not just in eddy-viscosity models, Eq. (3). With respect to the properties discussed in Sec. II, the characteristic length scale given by Eq. (18) depends on the velocity gradient $\mathbf{G}$. Therefore, it is locally defined and is frame invariant (P1). Moreover, $\Delta_{lsq}$ is obviously sensitive to flow orientation (property P3).

Furthermore, it may be noted that the numerator in Eq. (18) can be viewed as the Frobenius norm of the tensor $\mathbf{G}^T \mathbf{G} \Delta$, i.e., $G_{22} G_{33}^T \cdot \mathbf{G}^T \mathbf{G} = \text{tr}(G_{22} G_{33}^T (G^T G^T)) = \text{tr}(G_{22} \mathbf{G}^T \mathbf{G}) = \text{tr}(G_{22}^T G \mathbf{G})$. Moreover, $G_{22}^T \cdot \mathbf{G}^T \mathbf{G} = \text{tr}(G_{22}^T G \mathbf{G}) = \text{tr}(G^T G^T G) = G^T G$, so we can also express $\Delta_{lsq}$ as

$$\Delta_{lsq} = \sqrt{\frac{\Delta G^T G}{G^T G}}.$$  

(19)

From this definition, it is obvious that $\Delta_{lsq}$ is positive and well bounded (properties P1 and P2). Its applicability for unstructured meshes (property P4) relies on the proper adaptation of the tensor $\Delta$ (see Sec. IV). Regarding property P5, the computational cost of $\Delta_{lsq}$ is relatively small when compared to the other flow-dependent (property P3) length scales discussed in this paper, and special attention is only required for indeterminate forms of type 0/0.

The inherent simplicity and mathematical properties of the proposed length scale, as well as its basis in representations of the subgrid stress tensor on different grids, suggest that it can be a robust definition that minimizes the effects of mesh anisotropies on the performance of LES models.

Note that the definition of $\Delta_{lsq}$ provided in Eq. (18) was already presented and partially evaluated during the Stanford CTR Summer Program 2016 and the CEAA’16 conference.

To get a better understanding of $\Delta_{lsq}$, we consider several special cases. First of all, as remarked before, this length scale reduces to $\Delta$ on an isotropic mesh. Second, for purely rotating flows, i.e., $S = 0$ and $\mathbf{G} = \Omega$, $\Delta_{lsq}$ reduces to

$$\Delta_{lsq} = \sqrt{\frac{\omega_x^2 (\Delta y^2 + \Delta z^2) + \omega_y^2 (\Delta x^2 + \Delta z^2) + \omega_z^2 (\Delta x^2 + \Delta y^2)}{2|\omega|^2}},$$

(20)

which resembles the definition of $\Delta_{lsq}$ proposed by Chauvet et al. given in Eq. (9). Actually, similar to the definition of $\Delta$ proposed by Mockett et al. given in Eq. (10), $\Delta_{lsq}$ is $O(\max(\Delta x, \Delta y))$ instead of $\Delta_{lsq} = \sqrt{\Delta x \Delta y}$. Therefore, it also avoids a strong effect of the smallest grid-spacing. Finally, results obtained for a simple 2D mesh and flow,

$$\Delta = \begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \omega_0 \end{bmatrix},$$

(21)

are displayed in Fig. 1. Notice that the size of the control volume remains equal to unity; therefore, $\Delta_{vel} = 1$, regardless of the value of $\beta$. On the other hand, values of $\omega$ in Fig. 1 range from a pure shear flow ($\omega = 0$) to a simple shear flow ($\omega = 1/2$) and to a pure rotating flow ($\omega = 1$). For the two limiting situations, $\Delta_{lsq} = \sqrt{(\beta^2 + \beta^{-2})/2}$, whereas for $\omega = 1/2$, it reads $\Delta_{lsq} = \beta^{-1}$. Recalling that in the particular case $\Delta = \beta$ and $\Delta y = \beta^{-1}$, $\Delta_{lsq} = \sqrt{(\Delta x^2 + \Delta y^2)/2}$ for $\omega = 0$ (pure shear) and $\omega = 1$ (pure rotation), whereas $\Delta_{lsq} = \Delta y$ for the simple shear flow with $\omega = 1/2$. The latter corresponds quite well with the typical quasi-2D grid-aligned flow in the initial region of a shear layer. As it could be expected, the computed $\Delta_{lsq}$ is equal to the grid size in the direction orthogonal to the shear layer. The pure rotating flow ($\omega = 1$) is just a particular case of Eq. (20) with $\omega_x = \omega_y = 0$ and $\omega_z = 1$.
In order to study in more detail the effect of mesh anisotropies for different definitions of $\Delta$, let us consider a Cartesian mesh with $\Delta x = \Delta y = 1$ and $\Delta z = \alpha$. In this case, the geometry-dependent definitions of $\Delta$ result in

$$
\Delta_{\text{vol}} = \alpha^{1/3}, \quad \Delta_{\text{Sco}} = f(\min(\alpha, \alpha^{-1}), \min(1, \alpha^{-1})), \quad \Delta_{\max} = \max(1, \alpha), \quad \Delta_{L2} = \sqrt{\frac{2 + \alpha^2}{3}}, \quad \Delta_{\text{apl}} = \sqrt{\frac{3\alpha^2}{2\alpha^2 + 1}}.
$$

These functions are displayed in Fig. 2 using a log-log scale. Values of $\alpha > 1$ correspond to pencil-like meshes ($\Delta x = \Delta y \ll \Delta z$), whereas values of $\alpha < 1$ correspond to pancake-like meshes ($\Delta x \ll \Delta y = \Delta z$). Averaged results of $\Delta_{\text{sq}}$ are also displayed; they have been obtained from a large sample of random traceless velocity gradient tensors, $G$. Notice that this simple random procedure was able to produce fairly good predictions to determine the model constant, $C_m$, for different SGS models. Among all the geometry-dependent definitions, for the mesh considered here, the closest to $\Delta_{\text{sq}}$ by far is the definition of $\Delta_{L2}$ given in Eq. (7). Actually, for very simple flow configurations such as pure shear or pure rotation, $\Delta_{\text{sq}}$ reduces to $\Delta_{L2}$. The second closest is the correction proposed by Scotti et al. [25] [see Eq. (5)]. Finally, $\Delta_{\text{apl}}$ is the only definition that predicts values of $\Delta$ smaller than the classical Deardorff definition $\Delta_{\text{vol}}$.

IV. JACOBIAN-BASED EXTENSION FOR UNSTRUCTURED MESHES

In Sec. III, a new method to compute the subgrid characteristic length has been proposed. Although it has been derived in the context of Cartesian meshes, the idea can be extended to unstructured meshes by noticing that it basically consists in projecting the leading term of the Taylor series expansion of $\tau$ [see Eq. (16)] onto the basic gradient model [see Eq. (15)].

For non-uniform Cartesian grids, we considered the coordinate transformations $\hat{x} = x/\Delta x$, $\hat{y} = y/\Delta y$, and $\hat{z} = z/\Delta z$. This led to a new, isotropic, coordinate system $\hat{x}, \hat{y}, \hat{z}$. Then, applying the chain rule for differentiation yielded the approximation of the subgrid stress tensor of Eq. (16). More generally, let $\xi(x_i)$ be a monotonic differentiable function that defines a mapping from the physical space in the $i$-direction, $x_i$, to the so-called computational space, $\xi_i$. Using the chain rule, we obtain

$$
\frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial \xi_i} \frac{d \xi_i}{dx_i} = \frac{1}{J_i} \frac{\partial \phi}{\partial \xi_i},
$$

where $J_i$ is the Jacobian of the transformation $x_i \rightarrow \xi_i$. Here, no summation over $i$ is implied. Recalling that $|G|_{ij} = \partial u_i/\partial x_j$, the leading term of $\tau$ can be written more compactly as follows:

$$
\tau = \frac{1}{12} G_{\xi} G_{\xi}^T + O(\hat{\Delta}^4),
$$

where the gradient in the mapped space $\xi$ is represented by $G_{\xi} = G J_i$.

and $J$ is the Jacobian of the transformation $x \rightarrow \xi$. Notice that this first term is generic for all practical filters [44] in the context of LES, i.e., filters with a Fourier transform starting with $\hat{G}(k) = 1 - k^2 \hat{\Delta}^2 / 2 + O(k^4)$. At the discrete level, for a Cartesian grid, the filter length in each direction is taken equal to the mesh size in the same direction, i.e., $\hat{\Delta} = \Delta x_i$. In this case, $J = \Delta$ and $G_{\xi} = G = G\Delta$; therefore, the general expression given in Eq. (24) reduces to Eq. (16) for non-uniform Cartesian meshes and to the well-known gradient model [33] given in Eq. (15) for uniform grid spacings.

At this point, it becomes clear that the extension of the new subgrid characteristic length $\Delta_{\text{sq}}$ [see Eq. (18) in Sec. III] for unstructured meshes relies on the computation of the Jacobian, $J$, on such grids. It is important to note that the gradient tensor, $G$, is actually being computed in any LES code. Below, the method to compute the Jacobian, $J$, is solely based on the discrete gradient operators; therefore, it can be easily applied to any existing code. Namely, using the matrix-vector notation, the discrete gradient operator is given by a block matrix

$$
G \phi_h = \begin{pmatrix} G_x \\ G_y \\ G_z \end{pmatrix} \phi_h,
$$

where $\phi_h = (\phi_1, \phi_2, \ldots, \phi_n)^T \in \mathbb{R}^n$, $n$ is the number of unknowns in our domain, and $G_x$, $G_y$, and $G_z$ represent the discrete gradient operator for each spatial direction.
As a preview of things, we first consider the discretization of the gradient operator, $\mathbf{G}$, in one spatial direction with periodic boundary conditions. Let us consider three values of a smooth function $\phi(x)$: $\phi_{i-1} = \phi(x_{i-1})$, $\phi_i = \phi(x_i)$, and $\phi_{i+1} = \phi(x_{i+1})$ with $x_{i-1} = x_i - \Delta x$ and $x_{i+1} = x_i + \Delta x$. By a simple combination of Taylor series expansions of $\phi(x)$ around $x = x_i$, the well-known second-order accurate approximation of the derivative is as follows:

$$\frac{\partial \phi(x_i)}{\partial x} \approx \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}. \quad (27)$$

Then with a uniformly meshed periodic direction, $\mathbf{G}_x$ results in a skew-symmetric circulant matrix of the form

$$\mathbf{G}_x = \frac{1}{2\Delta x} \text{circ}(0, 1, 0, \ldots, 0, -1). \quad (28)$$

Thus, eigenvalues of $\mathbf{G}_x$ lie on the imaginary axis, $\lambda_k^{G_x} \in \mathbb{I}$. Then, the eigenvalues can be easily bounded with the help of the Gershgorin circle theorem, i.e., $|\lambda_k^{G_x}| \leq 1/\Delta x$. Notice that the upper bound exactly corresponds to the Jacobian, $J_x = 1/\Delta x$, of the mapping from the physical to the computational space for Cartesian grids. This idea can be extended to any grid or numerical method if we consider that, at the discrete level, the Jacobian, $\mathbf{J}$, is a diagonal matrix

$$\mathbf{J} \equiv \begin{pmatrix} J_x & 0 \\ 0 & J_y \\ 0 & J_z \end{pmatrix}, \quad (29)$$

which, similar to the Cartesian case, guarantees that the spectral norm of the gradient in the so-called computational space $\mathbf{G} \equiv \mathbf{J} \mathbf{G} = (\mathbf{G}_x, \mathbf{G}_y, \mathbf{G}_z)^T$ is equal to or smaller than unity, i.e., $\|\mathbf{G}\|_2 \leq 1$. This condition can easily be realized by using the Gershgorin circle theorem. Namely,

$$|\lambda_i^{G} - \lambda_{ij}^{G}| \leq \sum_{j \neq i} |G_{ij}^{\mathcal{G}}| \quad \text{where} \quad G_{ij}^{\mathcal{G}} = \mathcal{J}^{\mathcal{G}}_i G_{ij}^\mathcal{G}, \quad (30)$$

and $G_{ij}^\mathcal{G} = [\mathbf{G}]_{i,j}$, $\mathcal{G}_i = [\mathbf{G}_x]_{i,i}$, and $\mathcal{J}_i = [\mathbf{J}]_{i,i}$ are the coefficients of the matrices $\mathbf{G}_x$, $\mathbf{G}_y$, $\mathbf{G}_z$, and $\mathbf{J}_i$, respectively. Since $\mathbf{G}$ (also $\mathbf{G}$) is usually a zero-diagonal matrix, i.e., $G_{ii} = 0$ (summation not implied), the condition $\|\mathbf{G}\|_2 \leq 1$ simplifies to

$$|\lambda_i^{G} - \lambda_{ij}^{G}| \leq \sum_{j \neq i} |G_{ij}^{\mathcal{G}}| \leq 1 \quad \forall i = 1, \ldots, n, \quad (31)$$

where $n$ is the number of unknowns in our domain. Finally, recalling that the Jacobian must be positive, $\mathcal{J}_i > 0$, and extending the previous analysis to the $y$ and $z$ directions, the following definitions for the Jacobian

$$\mathcal{J}_i^y \equiv \frac{1}{\sum_{j \neq i} |G_{ij}^{\mathcal{G}}|}, \quad \mathcal{J}_i^z \equiv \frac{1}{\sum_{j \neq i} |G_{ij}^{\mathcal{G}}|}, \quad \mathcal{J}_i^\mathcal{G} \equiv \frac{1}{\sum_{j \neq i} |G_{ij}^{\mathcal{G}}|}, \quad (32)$$

guarantee that inequalities (31) are always satisfied. Here, no summation over $i$ is implied. Therefore, the spectral norm of $\mathbf{G}_\mathcal{G}$ is equal to or smaller than unity, i.e., $\|\mathbf{G}\|_2 \leq 1$. In this way, the local Jacobian for the node $i$, $\mathcal{J}_i$, is given by

$$\mathcal{J}_i \equiv \begin{pmatrix} \mathcal{J}_i^y \\ \mathcal{J}_i^z \\ \mathcal{J}_i^\mathcal{G} \end{pmatrix}. \quad (33)$$

Notice that the definitions of the Jacobian given in Eq. (32) are solely based on the coefficients of the discrete gradient operator, $\mathbf{G}$. Therefore, there is no restriction regarding the type of grid and the numerical method. Moreover, it is worth noticing that for a Cartesian uniform mesh, this formula reduces to $\mathbf{J} = \text{diag}(\Delta x, \Delta y, \Delta z)$ similar to the definition of $\Delta$ given in Eq. (13).

In this way, the subgrid characteristic length scale proposed in Sec. III is straightforwardly extended to unstructured meshes by simply replacing $\Delta$ in Eq. (18) by the local Jacobian, $\mathcal{J}_i$, defined in Eqs. (32) and (33).

V. NUMERICAL RESULTS

A. Decaying homogeneous isotropic turbulence

The numerical simulation of decaying isotropic turbulence was chosen as a first case to test the novel definition of the subgrid characteristic length scale, $\mathcal{G}_{lsq}$, proposed in Eq. (18). The configuration corresponds to the classical experiment of Comte-Bellot and Corrsin (CBC). Large-eddy simulation results have been obtained using the Smagorinsky model for a set of (artificially) stretched meshes. Namely, results for pancake-like meshes with $32 \times 32 \times N_z$ and $N_z = \{32, 64, 128, 256, 512, 1024, 2048\}$ are displayed in Fig. 3 (top). As expected, for increasing values of $N_z$, the results obtained using the classical definition of Deardorff, given in Eq. (4), diverge. This is because the value of $\Lambda_{vold}$ tends to zero for increasing $N_z$, and, therefore, the subgrid-scale model switches off. This is not the case for the definition of $\mathcal{G}_{lsq}$ proposed in this work. Interestingly, the results rapidly converge for increasing values of $N_z$. Therefore, the proposed definition of the subgrid characteristic length, $\mathcal{G}_{lsq}$, seems to minimize the effect of mesh anisotropies on the performance of subgrid-scale models.

Similar behavior is observed in Fig. 3 (bottom) for pancake-like meshes with $32 \times N_y \times N_z$ grid cells, where $N_z = \{32, 64, 128, 256, 512, 768\}$. In this case, the improper behavior of Deardorff’s definition is even more evident because $\nu_e$ in Eq. (3) scales as $O(\Delta^{4/3})$ instead of the $O(\Delta^{2/3})$ scaling for the pancake-like meshes. Therefore, the model switches off even more rapidly. Furthermore, it is worth mentioning that in this case, this numerical artifact is visible for a wide range of wavenumbers, whereas for pancake-like meshes, only the smallest resolved scales are affected in a significant manner. On the other hand, LES results obtained with $\mathcal{G}_{lsq}$ also tend to converge for increasing values of $N_z$. Nevertheless, compared with the results obtained with pancake-like meshes, significant changes are observed for the first three meshes, i.e., $N_z = \{32, 64, 128\}$. This delay in the convergence of the LES results may be attributed to the fact that more scales are converging for increasing values of $N_z$. Therefore, the newly proposed subgrid characteristic length scale strongly reduces the artificial effects caused by mesh anisotropies while providing a natural convergence.
FIG. 3. Three-dimensional kinetic energy spectra as a function of computational wavenumber, for decaying isotropic turbulence corresponding to the experiment of Comte-Bellot and Corrsin. LES results have been obtained using the Smagorinsky model for a set of anisotropic meshes with pancake-like (top) and pencil-like (bottom) control volumes. Results obtained with the novel definition of $\Delta_{lsq}$ proposed in Eq. (18) are compared with the classical definition proposed by Deardorff given in Eq. (4). For clarity, the latter results are shifted one decade down.

In order to analyze in more detail the effect of mesh anisotropies for different definitions of $\Delta$, two physical quantities of interest have been studied: namely, the resolved kinetic energy and the resolved enstrophy. Results displayed in Fig. 4 have been obtained with the same pancake-like meshes (i.e., $32 \times 32 \times N_z$ grid cells, where $N_z = \{32, 64, 128, 256, 512, 1024, 2048\}$). In this case, apart from the new definition, $\Delta_{lsq}$, and the Deardorff length scale, $\Delta_{vol}$, two additional definitions have also been tested: the definition proposed by Scotti et al., $\Delta_{Sco}$ given in Eq. (5) and the definition proposed by Mockett et al., $\tilde{\Delta}_{\omega}$ given in Eq. (10). It must be noted that for this comparison, we have chosen $\Delta_{Sco}$ because among all the definitions reviewed in Sec. I that solely depend on geometrical properties of the mesh (see property P3 in Table I) this is the length scale that provides the best results. Regarding the flow-dependent definitions, we have chosen $\Delta_{\omega}$ because this definition was actually proposed as an improvement of the definition by Chauvet et al. given in Eq. (9). The definition proposed by Shur et al., $\Delta_{SLA}$, given in Eq. (11) has not been considered here because it is just a modification of $\Delta_{\omega}$ specifically adapted to trigger the Kelvin-Helmholtz instability in the initial part of shear layers.

As explained above, energy spectra obtained using Deardorff’s length scale $\Delta_{vol}$ diverge for increasing values of $N_z$ due to the fact that $\Delta_{vol}$ tends to zero and, therefore, the subgrid-scale model switches off. This effect becomes even more evident for the resolved enstrophy (see Fig. 4, bottom) since this lack of SGS dissipation mainly affects the smallest resolved scales. This physically improper behavior is strongly mitigated by other definitions of $\Delta$. Namely, the definition proposed by Scotti et al., $\Delta_{Sco}$, displays the weakest dependence with respect to $N_z$. As explained in Sec. I, this definition of $\Delta$ was proposed as a correction of the Deardorff definition, $\Delta_{vol}$, for anisotropic meshes with the assumption of an isotropic turbulent regime. Therefore, it is not surprising that this definition behaves very robustly for a simulation of decaying homogeneous isotropic turbulence. On the other hand, we can observe that the novel definition, $\Delta_{lsq}$, and the definition proposed by Mockett et al., $\tilde{\Delta}_{\omega}$, display a very similar behavior. Results for both resolved kinetic energy and enstrophy rapidly converge for increasing values of $N_z$. Even more interestingly, taking the CBC results as an indication

FIG. 4. Resolved kinetic energy (top) and enstrophy (bottom) as a function of the number of grid points, $N_z$, for decaying isotropic turbulence corresponding to the experiment of Comte-Bellot and Corrsin. LES results have been obtained using the Smagorinsky model for a set of anisotropic meshes with pancake-like control volumes, i.e., $32 \times 32 \times N_z$. Results obtained with the novel definition of $\Delta_{lsq}$ proposed in Eq. (18) are compared with the definitions proposed by Deardorff, $\Delta_{vol}$ [Eq. (4)], Scotti et al., $\Delta_{Sco}$ [Eq. (5)], and Mockett et al., $\tilde{\Delta}_{\omega}$ [Eq. (10)], respectively.
of the trend the data should have, both definitions lead to significantly better solutions compared with the original 32^3 mesh and the solution obtained with the definition proposed by Scotti et al.\textsuperscript{25} \( \Delta_{Sco} \).

### B. Turbulent channel flow

To test the performance of the proposed definition of \( \Delta_{iso} \) with the presence of walls, simulations of a turbulent channel flow have also been considered. In this case, the code is based on a fourth-order symmetry-preserving finite-volume discretization\textsuperscript{30} of the incompressible Navier-Stokes equations on structured staggered grids. Regarding the spatial discretization of the eddy-viscosity models, the approach proposed by Trias et al.\textsuperscript{40} has been used in conjunction with the S3QR model recently proposed by Trias et al.\textsuperscript{29} Namely,

\[
v^3_{S3QR} = (C_{s3qr}\Delta)^2 Q_{GG}^{-1} R_{GG}^{5/6}, \tag{34}
\]

where \( C_{s3qr} = 0.762, Q_{GG} \) and \( R_{GG} \) are the second and third invariants of the symmetric second-order tensor \( GG \), and \( G \) is the gradient of the resolved velocity field, i.e., \( G \equiv \nabla u \). Similar to Vreman’s model,\textsuperscript{16} the S3QR model is also based on the invariants of the second-order tensor \( GG \). However, it was designed to have the proper cubic near-wall behavior. Apart from this, it fulfills a set of desirable properties, namely, positiveness, locality, Galilean invariance, and it automatically switches off for laminar, 2D, and axisymmetric flows. Furthermore, it is well conditioned, has a low computational cost, and has no intrinsic limitations for statistically inhomogeneous flows.

Figure 5 shows the results obtained from numerical simulations of a turbulent channel flow at \( Re_x = 395 \) for a set of (artificially) refined grids. The results are compared with the DNS data of Moser et al.\textsuperscript{38} The dimensions of the channel are taken equal to those of the DNS, i.e., \( 2\pi \times 2 \times \pi \). The starting point corresponds to a 32^3 mesh, which suffices to obtain good agreement with the DNS data. Therefore, the computational grid is very coarse in comparison with the DNS which was performed on a 256 × 193 × 192 grid, i.e., the DNS used about 290 times more grid points than the first simulation. The grid points are uniformly distributed in the stream-wise and the span-wise directions, whereas the wall-normal points are distributed using hyperbolic sine functions. For the lower half of the channel, the distribution of points is given by

\[
y_j = \sinh(\gamma_j/N_y)/\sinh(\gamma/2) \quad j = 0, 1, \ldots, N_y/2, \tag{35}\]

where \( N_y \) denotes the number of grid points in the wall-normal direction. The stretching parameter, \( \gamma \), is taken equal to 7. Then, the grid points in the upper half of the channel are computed by means of symmetry. With this distribution and \( N_y = 32 \), the first off-wall grid point is located at \( y^{+} \approx 2.6 \), i.e., inside the viscous sublayer \( (y^{+} < 5) \), whereas \( \Delta x^{+} \approx 77.5 \) and \( \Delta z^{+} \approx 38.8 \). Hence, the grid is highly anisotropic in the near-wall region, e.g., \( \Delta x^{+}/\Delta y^{+} \approx 14.7 \) for the first off-wall control volume.

Apart from the first simulation, two additional meshes (with \( N_x = 128 \) and \( N_z = 512 \)) have been used to investigate the effect of \( \Delta \). We chose to refine in the span-wise direction because simulation results should not be too much affected compared with the other two directions. Again, as can be seen from Fig. 5, the results obtained with the new definition of \( \Delta \) are much more robust to mesh anisotropies. It is remarkable that almost no changes are observed in the mean velocity profile when the newly proposed length scale is employed, whereas significant changes are observed for Deardorff’s classical definition. Similar behavior is observed for the resolved turbulent kinetic energy, especially in the bulk region where results obtained with the new length scale are almost independent of the value of \( N_z \).

To study the effect of mesh anisotropies for other definitions of \( \Delta \), results of the mean streamwise velocity and turbulent kinetic energy at channel mid-height are displayed in Fig. 6. Similar to the previous test case, results obtained using the definitions proposed by Scotti et al.\textsuperscript{25} \( \Delta_{Sco} \), given in Eq. (5) and by Mockett et al.\textsuperscript{31} \( \Delta_{moc} \), given in Eq. (10) are also shown for comparison. In this case, similar to the simulation of decaying homogeneous isotropic turbulence, the results obtained using Deardorff’s definition, \( \Delta_{vol} \), are strongly influenced by the mesh anisotropy. Again, other definitions tend to mitigate this. Despite the fact that it is based on the assumption of isotropic turbulence, the robustness of the definition

\[
\text{FIG. 5. Results for a turbulent channel flow at } Re_x = 395 \text{ obtained with a set of anisotropic meshes using the S3PQ model. Solid lines correspond to the direct numerical simulation of Moser et al.}^{38} \text{ Results obtained with the novel definition of } \Delta_{iso} \text{ proposed in Eq. (18) are compared with the classical definition proposed by Deardorff given in Eq. (4). For clarity, the former results are shifted up. Top: mean streamwise velocity, } \langle u \rangle. \text{ Bottom: turbulent kinetic energy, } k.\text{ }
\]
FIG. 6. Results for a turbulent channel flow at $Re = 395$ obtained with a set of anisotropic meshes using the S3PQ model. Solid lines correspond to the direct numerical simulation of Moser et al. Results obtained with the novel definition of $\Delta_{lsq}$ proposed in Eq. (18) are compared with the definitions proposed by Deardorff, $\Delta_{vol}$ [Eq. (4)], Scotti et al., $\Delta_{Sco}$ [Eq. (5)], and Mockett et al., $\tilde{\Delta}_{\omega}$ [Eq. (10)], respectively. Top: mean streamwise velocity at channel mid-height, $\langle u \rangle |_{y^+ = 1}$. Bottom: turbulent kinetic energy at channel mid-height, $k |_{y^+ = 1}$.

C. Flow around a square cylinder

Finally, to test the performance of the proposed length scale with unstructured meshes, the turbulent flow around a square cylinder has been considered. In this case, the Reynolds number, $Re = U D / \nu = 22 \times 10^3$, is based on the inflow velocity, $U$, and the cylinder width, $D$. This is a challenging test case for LES. Apart from the well-known von Kármán vortex shedding in the wake region, this regime is characterized by the clear presence of Kelvin-Helmholtz vortical structures produced by the flow separation at the leading edge of the cylinder. The size of these vortices grows quickly, triggering turbulence before they reach the downstream corner of the cylinder. Actually, they break up into finer structures before being engulfed into the much larger von Kármán vortices. An additional motivation to choose this configuration is the fact that it has been studied before in many numerical and experimental studies. The reader is referred to our DNS study and references therein for further details about the flow dynamics.

In the current work, we have carried out LESs on unstructured meshes using the NOISEtte code for the simulation of compressible turbulent flows in problems of aerodynamics and aeroacoustics. It is based on the family of high-accuracy finite-volume EBR (Edge-Based Reconstruction) schemes for unstructured meshes. The EBR schemes provide at a low computing cost a higher accuracy than most Godunov-type second-order schemes on unstructured meshes. On translationally invariant (structured) meshes, the EBR schemes coincide with high-order (up to sixth-order) finite-difference schemes. Hybrid schemes with a combination of upwind and central-difference parts are used in LESs with automatic adaptation of the weights of the components in order to preserve numerical stability at a minimal numerical dissipation.

For the simulations, the 2D unstructured mesh displayed in Fig. 7 has been extruded in the span-wise direction. The definition $\tilde{\Delta}_{\omega}$ is not so robust when predicting the turbulent kinetic energy (see Fig. 6, bottom) where the results obtained with the new definition $\Delta_{lsq}$ are almost not affected when $N_z$ increases. In summary, the results obtained using the new length scale, $\Delta_{lsq}$, are at least as good as the best results obtained by other definitions with the advantage of having a much lower computational cost compared with $\tilde{\Delta}_{\omega}$ and being much easier to be used for unstructured grids.

FIG. 7. Left: 2D section of the unstructured mesh used to perform the set of LESs of the turbulent flow around a square cylinder at $Re = 22 \times 10^3$. This $(x, y)$-section contains 19,524 nodes. Right: zoom around the obstacle.
resulting meshes have $19\,524 \times N_z$ control volumes, where $N_z$ is the number of control volumes in the span-wise direction. In this study, we have considered three values for $N_z$, i.e., $N_z = \{50, 100, 1000\}$. The first two meshes are reasonable for an LES \cite{43} whereas the mesh with $N_z = 1000$ is clearly too fine (even for a DNS \cite{41}). The 2D base mesh (see Fig. 7, left) is basically composed of triangular elements, except for the region around the square cylinder where there are skewed quadrilateral elements (see Fig. 7, right). The dimensions of the computational domain are slightly smaller than in the DNS study: \cite{41} $27D \times 27D \times 3D$ in the stream-wise, cross-stream, and span-wise directions, respectively. The upstream face of

FIG. 8. Average eddy-viscosity, $\langle \nu_e \rangle$, divided by the kinematic viscosity, $\nu$. Results for a turbulent flow around a square cylinder at $Re = 22\,000$ carried out using the S3PQ model \cite{20} on a set of unstructured meshes obtained from the extrusion in the span-wise direction of the 2D mesh displayed in Fig. 7. Results obtained with the new definition, $\Delta_{lsq}$, proposed in Eq. (18) (left) are compared with the classical definition proposed by Deardorff given in Eq. (4) (right). This is done for two meshes differing in the number of grid points in the span-wise direction: $N_z = 100$ (top) and $N_z = 1000$ (bottom).

FIG. 9. The same as in Fig. 8. Zoom around the obstacle. Notice that the scale range has been properly modified.
the cylinder is located at 6.5D from the inflow and centered in the cross-stream direction. The origin of coordinates is placed at the center of the cylinder.

Similar to the turbulent channel flow, LES results have been obtained with the S3PQ model\textsuperscript{20} using two definitions of \( \Delta \): the new definition, \( \Delta_{lsq} \), proposed in Eq. (18) and the classical definition proposed by Deardorff, given in Eq. (4). Results are compared with the experimental data of Lyn et al.\textsuperscript{44} and our incompressible DNS results\textsuperscript{31} which are taken as a reference. This DNS was carried out with a constant velocity profile, \( u = (U, 0, 0) \), at the inflow, convective boundary conditions, \( \partial u/\partial t + U \partial u/\partial x = 0 \), at the outflow, Neumann boundary condition in the cross-stream direction, \( \partial u/\partial y = 0 \), periodic boundary conditions in the span-wise direction, and a no-slip condition at the surface of the cylinder. To make the comparison possible, present LESs are carried out with analogous boundary conditions and at a nearly incompressible Mach number (\( M = 0.1 \)). Results are presented in dimensionless form where the reference length and velocity are the cylinder width, \( D \), and the inflow velocity, \( U \), respectively.

Results for the average eddy-viscosity, \( \langle \nu_e \rangle \), divided by the kinematic viscosity, \( \nu \), are displayed in Figs. 8 and 9. Results for two meshes, i.e., \( N_z = 100 \) (top) and \( N_z = 1000 \) (bottom), are shown. As expected, values of \( \nu_e \), obtained with Deardorff’s length scale are strongly affected by such abnormal mesh anisotropies. In this regard, the ability of the new subgrid characteristic length, \( \Delta_{lsq} \), to adapt to these situations is remarkable. At first sight, the results displayed in Fig. 8 look almost identical for \( \Delta_{lsq} \), whereas very significant differences are observed for the Deardorff definition. A closer inspection (see Fig. 9) reveals how both definitions of \( \Delta \) respond to the abrupt mesh transition between the near obstacle region and the rest of the domain (see Fig. 7, right). The new subgrid characteristic length tends to mitigate the effects of this mesh transition compared with the results obtained with the Deardorff definition. This difference becomes more evident for the mesh with \( N_z = 1000 \).

Sharp discontinuities in \( \nu_e \) may have severe negative effects. Numerically, they can lead to more stringent time steps and potentially cause instabilities. The former increases the computational cost of the simulation, whereas the latter can be solved using a proper discretization of the viscous term.\textsuperscript{40} From a physical point of view, this abnormal behavior of \( \nu_e \) can negatively affect the quality of the results in an uncontrolled manner. In this regard, results of the average stream-wise velocity, \( \langle u \rangle \), in the near cylinder region are displayed in Fig. 10. In this case, both the influence of the definition of \( \Delta \) and the number of grid points in the span-wise direction, \( N_z \), are rather small. This is a consequence of the fact that the turbulence model itself has a relatively low impact in the near-wall region, especially near the upstream corner. Therefore, discrepancies with the DNS results in this region can simply be attributed to insufficient grid resolution. This region is actually characterized by the formation of small vortices in the shear layer due to the Kelvin-Helmholtz instability that are rapidly convected downstream. The size of these vortices grows quickly, triggering turbulence before they reach the downstream corner of the cylinder. Interestingly, much better agreement with the DNS results is achieved in this region. Apart from this, it is also interesting to observe that results obtained with the new subgrid characteristic length, \( \Delta_{lsq} \), become closer to the DNS results when \( N_z \) increases. The results obtained with the Deardorff definition display an opposite behavior. These trends remain the same in the wake region where differences between both approaches are more visible. This is clearly observed in the average stream-wise velocity profiles displayed in Fig. 11. The anomalous behavior of the Deardorff definition when refining in the span-wise direction becomes more evident further downstream. The most relevant results in this regard are the average stream-wise velocity profiles in the domain centerline displayed in Fig. 12. Results obtained with \( N_z = 1000 \) and the Deardorff definition of \( \Delta \) are completely different from those obtained with \( N_z = 50 \) and \( N_z = 100 \), showing that the definition of \( \Delta \) itself can have a very negative impact on the performance of an SGS model. Such an abnormal behavior is not observed with the new subgrid characteristic length, \( \Delta_{lsq} \).

These results confirm the findings of the first two test cases (i.e., decaying isotropic turbulence and a turbulent channel flow): compared with the Deardorff definition, \( \Delta_{lsq} \), the
proposed definition, $\Delta_{lsq}$, is much more robust with respect to mesh anisotropies. For those two cases, it was also seen that results using $\Delta_{lsq}$ are at least as good as the best results obtained with other definitions. Furthermore, it was also observed that these trends are even more evident for turbulent statistics. Here, the results for the stream-wise Reynolds stresses, $\langle u' u' \rangle$, in the near-wall region displayed in Fig. 13 seem to confirm this. The robustness of the new definition, $\Delta_{lsq}$, in the shear layer region where there is almost a perfect match for the three meshes is remarkable ($N_z = \{50, 100, 1000\}$), while differences are observed for the classical Deardorff definition, $\Delta_{vol}$. The new definition, $\Delta_{lsq}$, is even more robust if we consider that in this shear layer region, there is an abrupt mesh transition from structured hexahedral elements to unstructured triangular prism (see Fig. 7, right).

VI. CONCLUDING REMARKS

In this work, a novel definition of the subgrid characteristic length, $\Delta$, has been proposed with the aim to answer the following research question: Can we find a simple and robust definition of $\Delta$ that minimizes the effect of mesh anisotropies on the performance of SGS models? In this respect, due to its simplicity and mathematical properties, we consider the flow-dependent $\Delta_{lsq}$ given in Eq. (18) a very good candidate. Namely, it is locally defined, frame invariant, well bounded (see properties P1 and P2 in Sec. II), and well conditioned,
and it has a low computational cost (property \( P_5 \)). Moreover, a simple extension of this length scale for unstructured grids (property \( P_4 \)) has been proposed in Sec. IV: it basically consists in replacing \( \Delta \) in Eq. (18) by the local Jacobian, \( J_\delta \), defined in Eqs. (33) and (32). Finally, from the definition of \( \Delta_{lsq} \), it is obvious that it is dependent on the local flow topology given by the gradient of the resolved velocity, \( \nabla \mathbf{u} \) (property \( P_3 \)). In this respect, analytical analysis for simple flow configurations points out the adequacy of the proposed definition. Numerically, it has been successfully tested for simulations of decaying homogeneous isotropic turbulence and a turbulent channel flow at \( \tau_e = 395 \) using (artificially) refined grids. Comparisons with the classical length scale of Deardorff have shown that the proposed definition is much more robust with respect to mesh anisotropies. Due to these findings and its simplicity, we think the currently proposed length scale has a great potential to be used in subgrid-scale models in complex geometries where highly skewed (unstructured) meshes are present.

ACKNOWLEDGMENTS

This work has been financially supported by the Ministerio de Economía y Competitividad, Spain (No.ENE2014-60577-R). F.X.T. is supported by a Ramón y Cajal postdoctoral Contract (No. RYC-2012-11996) financed by the Ministerio de Economía y Competitividad, Spain. A.G. is supported by the Russian Science Foundation (Project No. 15-11-30039). M.H.S. is supported by the Free Competition in the Physical Sciences (Project No. 613.001.212), which is financed by the Netherlands Organization for Scientific Research (NWO). Part of this research was conducted at the CTR Summer Program 2016: F.X.T., M.H.S., and R.W.C.P.V. thank the Center for Turbulence Research (CTR) at Stanford University for hospitality and financial support. This work has been carried out using computing resources of the Barcelona Supercomputing Center and the Federal collective usage center ”Complex for Simulation and Data Processing for Mega-science Facilities” at NRC “Kurchatov Institute”, http://ckp.nrcki.ru/.

35M. H. Silvis, F. X. Trias, M. Abkar, H. J. Bae, A. Lozano-Durán, and R. W. C. Verstappen, “Exploring nonlinear subgrid-scale models and
new characteristic length scales for large-eddy simulation,” in Proceedings
of the Summer Program 2016, Center for Turbulence Research, Stanford
University, 2016.
36F. X. Trias, A. Gorobets, A. P. Duben, and A. Oliva, “Building a new subgrid
characteristic length for LES,” in Fourth International Workshop Com-
putational Experiment in Aeroacoustics, Svetlogorsk, Russia, September
2016.
37G. Comte-Bellot and S. Corrsin, “Simple Eulerian time correlation of
full- and narrow-band velocity signals in grid-generated, isotropic turbu-
38R. D. Moser, J. Kim, and N. N. Mansour, “Direct numerical simulation
of turbulent channel flow up to $Re_T = 590,” Phys. Fluids 11, 943–945
(1999).
40F. X. Trias, A. Gorobets, and A. Oliva, “A simple approach to discretize
41F. X. Trias, A. Gorobets, and A. Oliva, “Turbulent flow around a square
cylinder at Reynolds number 22000: A DNS study,” Comput. Fluids 123,
42I. Abalakin, P. Bakhvalov, and T. Kozubskaya, “Edge-based reconstruction
schemes for unstructured tetrahedral meshes,” Int. J. Numer. Methods Fluids
81(6), 331–356 (2016).
43M. Minguez, C. Brun, R. Pasquetti, and E. Serre, “Experimental and high-
order LES analysis of the flow in near-wall region of a square cylinder,” Int.
44D. Lyn, S. Einav, W. Rodi, and J. Park, “A laser Doppler velocimetry study
of ensemble-averaged characteristics of the turbulent near wake of a square