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ARTICLE TYPE

New robust LMI synthesis conditions for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ gain-scheduled reduced-order DOF control of discrete-time LPV systems†

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Summary

This paper investigates the problems of stabilization and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ reduced order dynamic output-feedback (DOF) control of discrete-time linear systems. The synthesis conditions are formulated in terms of parameter-dependent linear matrix inequalities (LMIs) combined with scalar parameters, dealing with state-space models where the matrices depend polynomially on time-varying parameters and are affected by norm-bounded uncertainties. The motivation to handle these models comes from the context of networked control systems, particularly when a continuous-time plant is controlled by a digitally implemented controller. The main technical contribution is a distinct LMI based condition for the DOF problem, allowing an arbitrary structure (polynomial of arbitrary degree) for the measured output matrix. Additionally, an innovative heuristic is proposed to reduce the conservativeness of the stabilization problem. Numerical examples are provided to illustrate the potentialities of the approach to cope with several classes of discrete-time linear systems (time-invariant and time-varying) and the efficiency of the proposed design conditions when compared with other methods available in the literature.

KEYWORDS:

LPV discrete-time systems, mixed $\mathcal{H}_2/\mathcal{H}_\infty$control, gain-scheduled control, LMI relaxations, output-feedback control

1 | INTRODUCTION

The Lyapunov stability theory is a practical and relevant approach to verify the stability of dynamical systems.[1] Particularly in the case of linear systems, the stability (and performance) analysis and synthesis conditions for controllers and filters can be formulated in terms of linear matrix inequalities (LMIs),[2] that are attractive formulations from the optimization point of view due to convexity. In this context, although most of the control techniques usually employ state-feedback strategies,[3,4,5,6] when the measurement or estimation in real-time of all states are not possible, the use of this class of controllers becomes impracticable. This issue stimulated the growth of techniques based on output-feedback and observer-based control, which are more appropriated and have a workable implementation.[7,8,9,10] Moreover, the use of dynamic (full or reduced order) or static controllers depends on the system to be controlled. In some cases, e.g. large scale systems, full order controllers are intractable, making the reduced order (or even a static) controller the only available alternative.[11,12,8,13]

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[0]**Abbreviations:** DOF: dynamic output-feedback; LPV: linear parameter-varying; LMI: linear matrix inequality

Apart from ensuring stability, the vast majority of control projects also aims the optimization of some performance criteria, for example, the minimization of the decay rate, gain and phase margins or the search for the lowest attenuation level in terms of the $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ norms. Some techniques were proposed considering, separately, the problems of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control,[8,3,6,14] while other methods introduce the concept of optimizing both of these norms, which is called the problem of *mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control*.[15,16,17,13] To obtain better performance indexes, some approaches were developed adopting parameter-dependent optimization variables for the Lyapunov and slack variables matrices,[7,8,18] yielding less conservative results than the ones provided by methods based on the well known concept of *quadratic stability*, where a constant (parameter-independent) Lyapunov matrix is used to certificate the closed-loop stability.[19] When it is possible to read or estimate the time-varying parameters in real-time, gain-scheduled controllers can provide more stringent performances.[7,8]

This paper deals with discrete-time linear systems where the state-space matrices have entries that depend polynomially on time-varying parameters (that can be used for gain-scheduled) and are affected by norm-bounded time-varying uncertainties. The motivation to consider such models arises from the discretization of polytopic continuous-time linear parameter varying (LPV) systems (or uncertain time-invariant systems), which comes out with several practical applications, for example, in the context of networked control systems.[20,21] Using a Taylor series expansion of arbitrary fixed degree, as proposed by Braga et al,[22] the resulting discretized system can be represented with this particular structure. Although the discretization procedure is not investigated in this paper, this representation is adopted because it generalizes several models found in the literature. In this sense, note that not only discretized polytopic systems but also other dynamic models (such as polytopic, switched or precisely known systems) can be described in terms of polynomial matrices of fixed degree, with norm-bounded terms.[23,24,25,10,26,27,28]

The aim of this paper in terms of contributions is to provide parameter-dependent LMI conditions to treat the previously described systems considering the problem of stabilization, where a new heuristic procedure is introduced to find stabilizing gains, and the problem of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control. The proposed method presents a generalist nature regarding its applications. It can be particularized to deal with several classes of linear systems, such as time-varying and time-invariant polytopic systems, accordingly providing robust or gain-scheduled controllers, considering either dynamic or static output-feedback (DOF or SOF) and static state-feedback (SSF) approaches. Numerical examples illustrate the effectiveness of the proposed design methods and highlight the advantages when compared with other conditions from the literature.

Notation: The set of natural numbers is denoted by $\mathbb{N}$, the set of vectors (matrices) of order n ($n \times m$) with real entries is represented by $\mathbb{R}^n$ ($\mathbb{R}^{n\times m}$) and the set of symmetric positive definite matrices of order n with real entries is given by $\mathbb{S}^n_+$. For matrices or vectors, the symbol $'$ denotes the transpose, the expression $\mathrm{He}(X) := X + X'$ is used to shorten formulas, and the symbol $\star$ represents transposed blocks in a symmetric matrix. To state that a symmetric matrix P is positive (negative) definite, it is used $P > 0$ ($P < 0$). The space of discrete functions that are square-integrable is defined by ℓ_2.

In addition to the set of notations described above, the following lemma[29] is required in this paper to treat norm-bounded terms in the proposed synthesis conditions.

Lemma 1 (Zhou and Khargonekar[29]). Given a scalar $\eta > 0$ and matrices M and N of compatible dimensions, then

$$MN + N'M' \leq \eta MM' + \eta^{-1}N'N.$$

2 | PROBLEM STATEMENT

Consider the following linear discrete-time system affected by time-varying parameters

$$\begin{aligned} x(k+1) &= A_\Delta(\alpha(k))x(k) + B_\Delta(\alpha(k))u(k) + E_\Delta(\alpha(k))w(k) \\ z(k) &= C_z(\alpha(k))x(k) + D_z(\alpha(k))u(k) + E_z(\alpha(k))w(k) \\ y(k) &= C_y(\alpha(k))x(k) + E_y(\alpha(k))w(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input, $w(k) \in \mathbb{R}^r$ is the exogenous input, $z(k) \in \mathbb{R}^p$ is the controlled output, $y(k) \in \mathbb{R}^q$ is the measurement output and $\alpha(k) = [\alpha_1(k), \dots, \alpha_N(k)]'$ is a vector of bounded time-varying parameters, which lies in the unit simplex given by

$$\Lambda \equiv \left\{ \zeta \in \mathbb{R}^N : \sum_{i=1}^{N} \zeta_i = 1,\ \zeta_i \geq 0,\ i = 1, \dots, N \right\},$$

for all $k \geq 0$. Matrices $A_\Delta(\alpha(k))$, $B_\Delta(\alpha(k))$ and $E_\Delta(\alpha(k))$ are given by

$$\begin{aligned} A_\Delta(\alpha(k)) &= A(\alpha(k)) + \Delta A(\alpha(k)) \\ B_\Delta(\alpha(k)) &= B(\alpha(k)) + \Delta B(\alpha(k)) \\ E_\Delta(\alpha(k)) &= E(\alpha(k)) + \Delta E(\alpha(k)) \end{aligned} \tag{2}$$

where the terms $\Delta A(\alpha(k))$, $\Delta B(\alpha(k))$ and $\Delta E(\alpha(k))$ represent unstructured uncertainties whose norms have as upper bounds known values δ_A, δ_B and δ_E, as described bellow

$$\delta_A = \sup_{\alpha(k)\in\Lambda} ||\Delta A(\alpha(k))||_2, \quad \delta_B = \sup_{\alpha(k)\in\Lambda} ||\Delta B(\alpha(k))||_2, \quad \delta_E = \sup_{\alpha(k)\in\Lambda} ||\Delta E(\alpha(k))||_2. \tag{3}$$

The state-space matrices ($A(\alpha(k))$, $B(\alpha(k))$, $E(\alpha(k))$, $C_z(\alpha(k))$, $D_z(\alpha(k))$, $E_z(\alpha(k))$, $C_y(\alpha(k))$ and $E_y(\alpha(k))$) of system (1) are polynomial matrices of a fixed degree on $\alpha(k)$ with known monomial coefficients. For example, if all matrices are affine (polynomial dependence of degree one) and $\alpha(k) \in \Lambda$, then the presented LPV system fits into the so called linear time-varying polytopic representation, such that each one of the matrices can be written as the convex combination of N known vertices given as follows

$$M(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k) M_i, \ \alpha(k) \in \Lambda. \tag{4}$$

The purpose of this paper is the design of a stabilizing scheduled DOF controller with order $n_c \leq n_x$ given by

$$C := \begin{cases} x_c(k+1) = A_c(\alpha(k))x_c(k) + B_c(\alpha(k))y(k) \\ \quad u(k) = C_c(\alpha(k))x_c(k) + D_c(\alpha(k))y(k) \end{cases} \tag{5}$$

The computation of a DOF controller of order n_c can be reformulated, for example, using the strategy used by Måartensson[30] and El Ghaoui et al,[31] which restructures the problem of designing the controller (5) as the search for a static output-feedback gain, denoted by

$$\Theta(\alpha(k)) := \begin{bmatrix} A_c(\alpha(k)) & B_c(\alpha(k)) \\ C_c(\alpha(k)) & D_c(\alpha(k)) \end{bmatrix} \in \mathbb{R}^{(m+n_c)\times(q+n_c)}, \tag{6}$$

for the augmented system

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}_\Delta(\alpha(k))\tilde{x}(k) + \tilde{B}_\Delta(\alpha(k))\tilde{u}(k) + \tilde{E}_\Delta(\alpha(k))w(k) \\ \tilde{z}(k) &= \tilde{C}_z(\alpha(k))\tilde{x}(k) + \tilde{D}_z(\alpha(k))\tilde{u}(k) + \tilde{E}_z(\alpha(k))w(k) \\ \tilde{y}(k) &= \tilde{C}_y(\alpha(k))\tilde{x}(k) + \tilde{E}_y(\alpha(k))w(k) \end{aligned} \tag{7}$$

with $\tilde{x}(k) = \begin{bmatrix} x(k)' & x_c(k)' \end{bmatrix}'$, $\tilde{u}(k) = \begin{bmatrix} x_c(k+1)' & u(k)' \end{bmatrix}'$, $\tilde{y}(k) = \begin{bmatrix} x_c(k)' & y(k)' \end{bmatrix}'$, $\tilde{z}(k) = z(k)$ and

$$\begin{bmatrix} \tilde{A}_\Delta(\alpha(k)) & \tilde{E}_\Delta(\alpha(k)) & \tilde{B}_\Delta(\alpha(k)) \\ \tilde{C}_z(\alpha(k)) & \tilde{E}_z(\alpha(k)) & \tilde{D}_z(\alpha(k)) \\ \tilde{C}_y(\alpha(k)) & \tilde{E}_y(\alpha(k)) & 0 \end{bmatrix} := \left[\begin{array}{cc|c|cc} A_\Delta(\alpha(k)) & 0 & E_\Delta(\alpha(k)) & 0 & B_\Delta(\alpha(k)) \\ 0 & 0 & 0 & \mathrm{I} & 0 \\ \hline C_z(\alpha(k)) & 0 & E_z(\alpha(k)) & 0 & D_z(\alpha(k)) \\ \hline 0 & \mathrm{I} & 0 & 0 & 0 \\ C_y(\alpha(k)) & 0 & E_y(\alpha(k)) & 0 & 0 \end{array}\right].$$

Therefore, the closed-loop system is given by

$$\mathcal{H} := \begin{cases} \tilde{x}(k+1) = \tilde{A}_{cl}(\alpha(k))\tilde{x}(k) + \tilde{B}_{cl}(\alpha(k))w(k) \\ \quad \tilde{z}(k) = \tilde{C}_{cl}(\alpha(k))\tilde{x}(k) + \tilde{D}_{cl}(\alpha(k))w(k) \end{cases} \tag{8}$$

whose matrices are denoted by

$$\begin{bmatrix} \tilde{A}_{cl}(\alpha(k)) & \tilde{B}_{cl}(\alpha(k)) \\ \tilde{C}_{cl}(\alpha(k)) & \tilde{D}_{cl}(\alpha(k)) \end{bmatrix} = \begin{bmatrix} \tilde{A}_\Delta(\alpha(k)) & \tilde{E}_\Delta(\alpha(k)) \\ \tilde{C}_z(\alpha(k)) & \tilde{E}_z(\alpha(k)) \end{bmatrix} + \begin{bmatrix} \tilde{B}_\Delta(\alpha(k)) \\ \tilde{D}_z(\alpha(k)) \end{bmatrix} \Theta(\alpha(k)) \begin{bmatrix} \tilde{C}_y(\alpha(k))' \\ \tilde{E}_y(\alpha(k))' \end{bmatrix}'. \tag{9}$$

2.1 | Stability Analysis

System 1 with null inputs (i.e., $u(k) = w(k) = 0$) is said to be *asymptotically stable* if, given an initial condition $x(0)$, the trajectories converge to the origin as the time tends to infinity, that is

$$\lim_{k\to\infty} x(k) \to 0, \quad \forall x(0).$$

The stability analysis can be performed through the Lyapunov stability theory,[32] which is directly associated to the properties of the dynamic matrix $A_\Delta(\alpha(k))$. Sufficient conditions to verify the asymptotic stability are given as follows.

Lemma 2 (Willems[32]: Asymptotic Stability)**.** System (1) with null inputs (i.e., $u(k) = w(k) = 0$) is asymptotically stable if there exists a parameter-dependent matrix $P(\alpha(k)) \in \mathbb{S}_+^{n_x}$, such that

$$A_\Delta(\alpha(k))' P(\alpha(k+1)) A_\Delta(\alpha(k)) - P(\alpha(k)) < 0, \quad \forall \alpha(k) \in \Lambda \tag{10}$$

or, equivalently (by Schur complement)

$$\begin{bmatrix} P(\alpha(k)) & A_\Delta(\alpha(k))' P(\alpha(k+1)) \\ P(\alpha(k+1)) A_\Delta(\alpha(k)) & P(\alpha(k+1)) \end{bmatrix} > 0. \tag{11}$$

If the parameters are time-invariant, i.e., $\alpha(k+1) = \alpha(k) = \alpha$, the conditions of Lemma 2 are necessary and sufficient to guarantee the Schur stability of the system, or equivalently, to ensure that the absolute value of all the eigenvalues of $A_\Delta(\alpha)$ are smaller than one.

It is possible to generalize Lemma 2 to, besides providing a stability certificate, also determine a bound to the convergence decay rate of the state trajectories to the origin.[33,34] First of all, it is known that, if the Lyapunov function, $V(\alpha(k))$, is such that

$$V(\alpha(k+1)) < \rho^2 V(\alpha(k)),$$

for $0 < \rho \leq 1$, then ρ sets up a bound for the decay rate of the states, that is,

$$||x(k)||_2 \leq \rho^k ||x(0)||_2, \quad \forall k \geq 1.$$

The conditions presented in the following lemma, when verified, guarantee the asymptotic stability with decay rate limited by ρ.

Lemma 3 (Rugh,[33] Elia and Mitter:[34] Decay rate)**.** System (1) with null inputs (i.e., $u(k) = w(k) = 0$) is asymptotically stable and has a decay rate bounded by ρ if there exists a parameter-dependent matrix $P(\alpha(k)) \in \mathbb{S}_+^{n_x}$, such that $0 < \rho \leq 1$ and

$$\hat{A}(\alpha(k))' P(\alpha(k+1)) \hat{A}(\alpha(k))' - \rho^2 P(\alpha(k)) < 0, \quad \forall \alpha(k) \in \Lambda$$

or, equivalently (by Schur complement)

$$\begin{bmatrix} \rho^2 P(\alpha(k)) & \hat{A}(\alpha(k))' P(\alpha(k+1)) \\ P(\alpha(k+1)) \hat{A}(\alpha(k)) & P(\alpha(k+1)) \end{bmatrix} > 0.$$

Regarding time-invariant parameters, the decay rate bounded by ρ can be interpreted as the radius of the circle centered at the origin, that contains all the poles of $A_\Delta(\alpha)$. This region, illustrated in Figure 1 , is delimited by

$$|\lambda_i(A_\Delta(\alpha)/\rho)| < 1,$$

where $\lambda_i(\cdot)$, $i = 1, \ldots, n$, are the eigenvalues of the matrix $A_\Delta(\alpha)/\rho$ for a fixed value of α.[35]

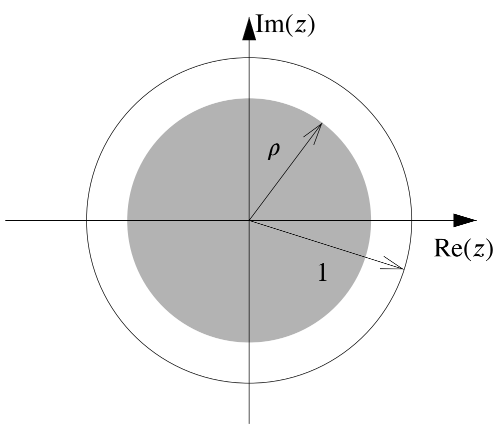


FIGURE 1 Circle of radius ρ centered at the origin in the complex plane.

The conditions of Lemma 3, applied to linear time-invariant (LTI) systems are necessary and sufficient to guarantee that the all the eigenvalues of the system lie inside the region of interest, for all $\alpha \in \Lambda$.

2.2 | Performance Indexes

As performance criteria for the design of stabilizing DOF controllers in the form of (5), it is considered the problem of minimizing upper bounds (guaranteed costs) for the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of the closed-loop system (8). The $\mathcal{H}_\infty$ norm is used to represent a robustness criterion regarding the disturbance rejection, while the $\mathcal{H}_2$ norm is applied in order to specify an optimization criterion associated to the energy of the impulse response of the system.

If the system (8) is asymptotically stable, then the performance criterion based on the $\mathcal{H}_2$ in an infinite horizon [14] is defined by

$$||\mathcal{H}||_2^2 = \limsup_{T\to\infty} \mathcal{E}\left\{\frac{1}{T}\sum_{k=0}^{T} \tilde{z}(k)'\tilde{z}(k)\right\},$$

where T is a positive integer that represents the time horizon and $\mathcal{E}\{\cdot\}$ is the mathematical expectation, considering that $w(k)$ is a standard white noise (Gaussian zero-mean in which the covariance matrix is equal to the identity matrix).

Concerning the $\mathcal{H}_\infty$ norm of system (8), an upper bound μ_∞ for this norm can be computed taking the definition presented as follows (see, for instance, the work of de Caigny et al [7]) that guarantees that, for any input $w(k) \in \ell_2$, the output of the system $\tilde{z}(k) \in \ell_2$ satisfies

$$||\tilde{z}(k)||_2 < \mu_\infty ||w(k)||_2,\ \ \mu_\infty > 0,\ \forall \alpha(k) \in \Lambda,\ k \geq 0.$$

3 | STABILIZATION OF LPV SYSTEMS WITH NORM-BOUNDED TERMS

This section presents sufficient LMI conditions for the synthesis of reduced order DOF stabilizing controllers for system (1).

One of the contributions of this paper, discussed with more details in the end of Section 5, is that, the proposed design conditions do not impose any structural constraint in matrix $\tilde{C}_y(\alpha(k))$, which is a very common practice in the methods from the literature when dealing with static output-feedback control. The main artifice that enabled this improvement is the introduction of matrix $Q(\alpha(k)) \in \mathbb{R}^{(q+n_c)\times(n_x+n_c)}$ to the problem, enabling a linearization procedure of the inequalities associated to the output-feedback problem (otherwise it would be necessary to work with bilinear matrices inequalities (BMIs)). As the dimensions of matrix Q are equal to the dimensions of the measured output matrix $\tilde{C}_y(\alpha(k))$ from the original system, an intuitive choice is $Q(\alpha(k)) = \tilde{C}_y(\alpha(k))$. However, alternative choices for matrix $Q(\alpha(k))$, which are used in the design conditions of this paper, are presented in the following remark.

Remark 1. Matrices $Q_i(\alpha(k)) \in \mathbb{R}^{(q+n_c)\times(n_x+n_c)}$, $i = 1, \ldots, p$ (with p being the quantity of matrices that the synthesis condition requires), are stipulated by the user to make possible the synthesis conditions in terms of LMIs. Two options for $Q_i(\alpha(k))$ are proposed:

- The first one, and more intuitive, is
$$Q_i(\alpha(k)) = \tilde{C}_y(\alpha(k)),\ i = 1, \ldots, p \tag{12}$$
- The second one is given by
$$Q_i(\alpha(k)) = \begin{bmatrix} 0_{(q+n_c)\times\sigma_Q} & \mathrm{I}_{(q+n_c)} & 0_{(q+n_c)\times(n_x-\sigma_Q-q)} \end{bmatrix}, \tag{13}$$
where a new input parameter, $0 \leq \sigma_Q \leq n_x - q$, is introduced with the purpose of defining the position of the identity matrix.

Based on this initial information, Theorem 1 is proposed to deal with the case of DOF control of polynomial LPV systems with uncertain norm-bounded terms.

Theorem 1. There is a DOF gain $\Theta(\alpha(k))$ such that system (8), for a noise input $w(k) = 0$, is asymptotically stable if there exist matrices[1] $P(\alpha(k)) \in \mathbb{S}_+^{(n_x+n_c)}$, $F(\bar{\alpha}(k))$ and $G(\bar{\alpha}(k)) \in \mathbb{R}^{(n_x+n_c)\times(n_x+n_c)}$, $L(\alpha(k)) \in \mathbb{R}^{(m+n_c)\times(q+n_c)}$ and $S(\alpha(k)) \in \mathbb{R}^{(q+n_c)\times(q+n_c)}$, given matrix $Q(\alpha(k))$, scalar variables η_A and η_B and given scalar parameters $\gamma \neq 0$ and ξ, such that

$$\mathcal{Q} + \mathcal{C}\mathcal{B} + \mathcal{B}'\mathcal{C}' < 0, \qquad \forall \alpha(k) \in \Lambda, \tag{14}$$

[1] The parameter vector $\bar{\alpha}(k)$ represents $\bar{\alpha}(k) = \big(\alpha(k),\ \alpha(k+1)\big)$.

holds, considering that $\mathcal{Q}$ is given by

$$\mathcal{Q} = \begin{bmatrix} \Gamma_{11} & \star & \star & \star & \star \\ \Gamma_{21} & P(\alpha(k)) - G(\bar{\alpha}(k)) - G(\bar{\alpha}(k))' & \star & \star & \star \\ (\tilde{B}(\alpha(k))L(\alpha(k)))' & 0 & 0 & \star & \star \\ \xi F(\bar{\alpha}(k)) & G(\bar{\alpha}(k)) & 0 & -\eta_A \mathrm{I} & \star \\ \xi L(\alpha(k))Q(\alpha(k)) & L(\alpha(k))Q(\alpha(k)) & L(\alpha(k)) & 0 & -\eta_B \mathrm{I} \end{bmatrix}$$

with

$$\begin{aligned} \Gamma_{11} &= \xi \mathrm{He}\left(\tilde{A}(\alpha(k))F(\bar{\alpha}(k)) + \tilde{B}(\alpha(k))L(\alpha(k))Q(\alpha(k))\right) - P(\alpha(k+1)) + \eta_A \delta_A^2 \mathrm{I} + \eta_B \delta_B^2 \mathrm{I} \\ \Gamma_{21} &= -\xi F(\bar{\alpha}(k)) + (\tilde{A}(\alpha(k))G(\bar{\alpha}(k)) + \tilde{B}(\alpha(k))L(\alpha(k))Q(\alpha(k)))' \end{aligned}$$

and matrices $\mathcal{C}$ and $\mathcal{B}$ are given, respectively, by

$$\mathcal{C} = \begin{bmatrix} 0 \\ 0 \\ \gamma \mathrm{I} \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{B}' = \begin{bmatrix} \xi(S(\alpha(k))Q(\alpha(k)) - \tilde{C}_y(\alpha(k))F(\bar{\alpha}(k)))' \\ (S(\alpha(k))Q(\alpha(k)) - \tilde{C}_y(\alpha(k))G(\bar{\alpha}(k)))' \\ S(\alpha(k))' \\ 0 \\ 0 \end{bmatrix}.$$

In the affirmative case, the stabilizing DOF gain-scheduled controller is given by $\Theta(\alpha(k)) = L(\alpha(k))S(\alpha(k))^{-1}$.

Proof. For ease of notation, the dependence on the time-varying parameters is omitted hereafter. Furthermore, P^+ is used to represent $P(\alpha(k+1))$ and $\bar{G}$ and $\bar{F}$ are used instead of $G(\bar{\alpha}(k))$ and $F(\bar{\alpha}(k))$, respectively. Note that the feasibility of (14) guarantees that $\gamma(S + S') < 0$ (in the entry (3,3) of the left-hand side matrix of (14)), implying that S^{-1} and, consequently, the controller Θ exist whenever (14) holds.

The initial step of the proof is to recover the inequalities that treat the original matrices of the system $(\tilde{A}_\Delta, \tilde{B}_\Delta)$, which embrace the polynomial terms and the norm-bounded uncertainties. For that, it is necessary to manipulate the conditions in order to recover the terms ΔA and ΔB from their bounds δ_A and δ_B, employing the expressions presented in (3). First, note that the inequality in (14) can be rewritten as

$$\begin{bmatrix} \xi \mathrm{He}\left(\tilde{A}\bar{F} + \tilde{B}LQ\right) + \eta_A \delta_A^2 \mathrm{I} + \eta_B \delta_B^2 \mathrm{I} + \Psi_{11} & \star & \star & \star & \star \\ (\tilde{A}\bar{G} + \tilde{B}LQ)' + \Psi_{21} & \Psi_{22} & \star & \star & \star \\ (\tilde{B}_\Delta L)' + \Psi_{31} & \Psi_{32} & \Psi_{33} & \star & \star \\ \xi \bar{F} & \bar{G} & 0 & -\eta_A \mathrm{I} & \star \\ \xi LQ & LQ & L & 0 & -\eta_B \mathrm{I} \end{bmatrix} < 0 \tag{15}$$

with

$$\Psi_{11} = -P^+, \quad \Psi_{21} = -\xi\bar{F}, \quad \Psi_{31} = \xi\gamma(SQ - \tilde{C}_y\bar{F}), \quad \Psi_{22} = P - \bar{G} - \bar{G}', \quad \Psi_{32} = \gamma(SQ - \tilde{C}_y\bar{G}), \quad \Psi_{33} = \gamma(S + S'). \tag{16}$$

Thus, applying the Schur complement on inequality (15), one has

$$R_B + \eta M_B M_B' + \eta^{-1} N_B' N_B < 0, \tag{17}$$

with

$$M_B' = \begin{bmatrix} \delta_B \mathrm{I} & 0 & 0 & 0 \end{bmatrix}, \quad N_B = \begin{bmatrix} \xi LQ & LQ & L & 0 \end{bmatrix}, \quad \eta = \eta_B,$$

$$R_B = \begin{bmatrix} \xi \mathrm{He}(\tilde{A}\bar{F} + \tilde{B}LQ) + \Psi_{11} + \eta_A \delta_A^2 \mathrm{I} & \star & \star & \star \\ (\tilde{A}\bar{G} + \tilde{B}LQ)' + \Psi_{21} & \Psi_{22} & \star & \star \\ (\tilde{B}_\Delta L)' + \Psi_{31} & \Psi_{32} & \Psi_{33} & \star \\ \xi\bar{F} & \bar{G} & 0 & -\eta_A \end{bmatrix},$$

and matrices Ψ_{ij} given in (16).

Using Lemma 1 and considering the relation given in (3) ($\Delta\tilde{B}\Delta\tilde{B}' \leq \delta_B^2\mathrm{I}$), one has that (17) implies

$$R_B + \begin{bmatrix} \Delta\tilde{B} \\ 0 \\ 0 \\ 0 \end{bmatrix} N_B + N_B' \begin{bmatrix} \Delta\tilde{B}' & 0 & 0 & 0 \end{bmatrix} < 0. \tag{18}$$

Applying the Schur complement, one obtains

$$R_A + \eta M_A M_A' + \eta^{-1} N_A' N_A < 0, \tag{19}$$

with

$$M_A' = \begin{bmatrix} \delta_A\mathrm{I} & 0 & 0 \end{bmatrix}, \quad N_A = \begin{bmatrix} \xi\bar{F} & \bar{G} & 0 \end{bmatrix}, \quad \eta = \eta_A,$$
$$R_A = \begin{bmatrix} \xi\mathrm{He}(\tilde{A}\bar{F} + \tilde{B}_\Delta LQ) + \Psi_{11} & \star & \star \\ (\tilde{A}\bar{G} + \tilde{B}_\Delta LQ)' + \Psi_{21} & \Psi_{22} & \star \\ (\tilde{B}_\Delta L)' + \Psi_{31} & \Psi_{32} & \Psi_{33} \end{bmatrix},$$

and matrices Ψ_{ij} given in (16). Using Lemma 1 and knowing that $\Delta\tilde{A}\Delta\tilde{A}' \leq \delta_A^2\mathrm{I}$, one has

$$R_A + \begin{bmatrix} \Delta\tilde{A} \\ 0 \\ 0 \end{bmatrix} N + N' \begin{bmatrix} \Delta\tilde{A}' & 0 & 0 \end{bmatrix} < 0. \tag{20}$$

Finally, it is possible to rewrite (20) as

$$\hat{\mathcal{Q}} + \hat{\mathcal{C}}\hat{\mathcal{B}} + \hat{\mathcal{B}}'\hat{\mathcal{C}}' < 0 \tag{21}$$

with

$$\hat{\mathcal{Q}} = \begin{bmatrix} \xi\mathrm{He}(\tilde{A}_\Delta\bar{F} + \tilde{B}_\Delta LQ) - P^+ & \star & \star \\ -\xi\bar{F} + (\tilde{A}_\Delta\bar{G} + \tilde{B}_\Delta LQ)' & P - \bar{G} - \bar{G}' & \star \\ (\tilde{B}_\Delta L)' & 0 & 0 \end{bmatrix}, \quad \hat{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ \gamma\mathrm{I} \end{bmatrix}, \quad \hat{\mathcal{B}}' = \begin{bmatrix} \xi(SQ - \tilde{C}_y\bar{F})' \\ (SQ - \tilde{C}_y\bar{G})' \\ S' \end{bmatrix}.$$

Knowing that the stabilizing gain $\Theta = LS^{-1}$ and that the closed-loop dynamic matrix $\tilde{A}_{cl}$ can be rewritten as follows

$$\begin{aligned} \tilde{A}_{cl} &= \tilde{A}_\Delta + \tilde{B}_\Delta\Theta\tilde{C}_y & (22) \\ &= \tilde{A}_\Delta + \tilde{B}_\Delta LS^{-1}\tilde{C}_y + \tilde{B}_\Delta LQ - \tilde{B}_\Delta LQ & (23) \\ &= \tilde{A}_\Delta + \tilde{B}_\Delta LQ - \tilde{B}_\Delta L(Q - S^{-1}\tilde{C}_y), & (24) \end{aligned}$$

multiplying (21) on the right by

$$X = \begin{bmatrix} \mathrm{I} & 0 & \xi(S^{-1}\tilde{C}_y\bar{F} - Q)' \\ 0 & \mathrm{I} & (S^{-1}\tilde{C}_y\bar{G} - Q)' \end{bmatrix}', \tag{25}$$

and on the left by its transpose, one has

$$\begin{bmatrix} \xi\mathrm{He}(\tilde{A}_{cl}\bar{F}) - P^+ & \star \\ -\xi\bar{F} + (\tilde{A}_{cl}\bar{G})' & P - \bar{G} - \bar{G}' \end{bmatrix} < 0. \tag{26}$$

Pre- and post-multiplying (26), respectively, by T and T' with $T = \begin{bmatrix} \mathrm{I} & \tilde{A}_{cl} \end{bmatrix}$, one obtains

$$\tilde{A}_{cl}P\tilde{A}_{cl}' - P^+ < 0,$$

which guarantees that system (8) is asymptotically stable using a duality argument.[36] □

Another class of systems that Theorem 1 is able to handle as particular case is the polytopic time-varying systems (affine dependence on the parameters) without the presence of norm-bound terms, that is, $\Delta A(\alpha(k)) = \Delta B(\alpha(k)) = 0$ (extensively investigated in the literature). The next corollary presents an adaptation of Theorem 1 to deal with this particular scenario, with the inclusion of a decay rate bounded by ρ.

Corollary 1 (Polytopic LPV systems)**.** There is a DOF gain $\Theta(\alpha(k))$ such that system (8), for a noise input $w(k) = 0$, with $\Delta A(\alpha(k)) = \Delta B(\alpha(k)) = 0$, is asymptotically stable and has a decay rate bounded by ρ, if there exist matrices $P(\alpha(k)) \in \mathbb{S}_+^{(n_x+n_c)}$, $F(\bar{\alpha}(k))$ and $G(\bar{\alpha}(k)) \in \mathbb{R}^{(n_x+n_c)\times(n_x+n_c)}$, $L(\alpha(k)) \in \mathbb{R}^{(m+n_c)\times(q+n_c)}$, $S(\alpha(k)) \in \mathbb{R}^{(q+n_c)\times(q+n_c)}$ and $Q(\alpha(k))$, and given scalar parameters $\gamma \neq 0$, ξ and ρ, such that $0 < \rho \leq 1$ and

$$\check{\mathcal{Q}} + \check{\mathcal{C}}\check{\mathcal{B}} + \check{\mathcal{B}}'\check{\mathcal{C}}' < 0 \tag{27}$$

holds, considering that $\check{Q}$ is given by

$$\check{Q} = \begin{bmatrix} \Gamma_{11} & \star & \star \\ \Gamma_{21} & P(\alpha(k)) - G(\bar{\alpha}(k)) - G(\bar{\alpha}(k))' & \star \\ (\tilde{B}_\Delta(\alpha(k))L(\alpha(k)))' & 0 & 0 \end{bmatrix},$$

with

$$\begin{aligned} \Gamma_{11} &= \xi \mathrm{He}(\tilde{A}_\Delta(\alpha(k))F(\bar{\alpha}(k)) + \tilde{B}_\Delta(\alpha(k))L(\alpha(k))Q) - \rho^2 P(\alpha(k+1)) \\ \Gamma_{21} &= -\xi F(\bar{\alpha}(k)) + (\tilde{A}_\Delta(\alpha(k))G(\bar{\alpha}(k)) + \tilde{B}_\Delta(\alpha(k))L(\alpha(k))Q(\alpha(k)))' \end{aligned}$$

and matrices $\check{C}$ and $\check{B}$ are given, respectively, by

$$\check{C} = \begin{bmatrix} 0 \\ 0 \\ \gamma \mathrm{I} \end{bmatrix}, \quad \check{B}' = \begin{bmatrix} \xi(S(\alpha(k))Q(\alpha(k)) - \tilde{C}_y(\alpha(k))F(\bar{\alpha}(k)))' \\ (S(\alpha(k))Q(\alpha(k)) - \tilde{C}_y(\alpha(k))G(\bar{\alpha}(k)))' \\ S(\alpha(k))' \end{bmatrix}.$$

In affirmative case, the stabilizing DOF gain-scheduled controller is given by $\Theta(\alpha(k)) = L(\alpha(k))S(\alpha(k))^{-1}$.

Proof. As done in the proof of Theorem 1, for ease of notation, the dependence on the time-varying parameters is omitted hereafter. Knowing that $\tilde{A}_{cl}$ can be written as in (22), multiplying (27) on the right by X as given in (25) and on the left by its transpose, one has

$$\begin{bmatrix} \xi \mathrm{He}(\tilde{A}_{cl}\bar{F}) - \rho^2 P^+ & \star \\ -\xi \bar{F} + (\tilde{A}_{cl}\bar{G})' & P - \bar{G} - \bar{G}' \end{bmatrix} < 0. \tag{28}$$

Pre- and post-multiplying (28), respectively, by T and T' with $T = \begin{bmatrix} \mathrm{I} & \tilde{A}_{cl} \end{bmatrix}$, one obtains

$$\tilde{A}_{cl} P \tilde{A}_{cl}' - \rho^2 P^+ < 0,$$

that guarantees that system (8) with $\Delta A(\alpha(k)) = \Delta B(\alpha(k)) = 0$ is asymptotically stable and has a decay rate bounded by ρ using a duality argument.[37] □

Note that the parameter-dependent inequalities from Theorem 1 and Corollary 1 are linear with respect to the optimization variables only if the scalars γ, ξ and ρ are given (otherwise the conditions are BMIs). More details on this subject are presented in Section 6.

4 | MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ CONTROL OF LPV SYSTEMS WITH NORM-BOUNDED TERMS

Before presenting the main result of this section, it is necessary to introduce some parameter-dependent inequalities regarding the synthesis of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers.

First, consider the next inequalities associated to the $\mathcal{H}_2$ control design problem

$$\mu_2^2 \geq \mathrm{Tr}\,\{W(\alpha(k))\}, \tag{29}$$

$$\mathcal{Q}_G + \mathcal{C}_G \mathcal{S}_G + \mathcal{S}_G{}' \mathcal{C}_G{}' < 0, \tag{30}$$

$$\mathcal{Q}_T + \mathcal{C}_T \mathcal{S}_T + \mathcal{S}_T{}' \mathcal{C}_T{}' < 0, \tag{31}$$

where

$$\mathcal{Q}_G = \begin{bmatrix} \Gamma_{11} & \star & \star & \star & \star & \star & \star \\ \Gamma_{21} & \Gamma_{22} & \star & \star & \star & \star & \star \\ \Gamma_{31} & 0 & -\mathrm{I} & \star & \star & \star & \star \\ (\tilde{B}(\alpha(k))L(\alpha(k)))' & 0 & 0 & 0 & \star & \star & \star \\ \xi F(\bar{\alpha}(k)) & G(\bar{\alpha}(k)) & 0 & 0 & -\eta_A \mathrm{I} & \star & \star \\ \xi L(\alpha(k))Q_1(\alpha(k)) & L(\alpha(k))Q_1(\alpha(k)) & L(\alpha(k))\tilde{E}_y(\alpha(k)) & L(\alpha(k)) & 0 & -\eta_B \mathrm{I} & \star \\ 0 & 0 & \mathrm{I} & 0 & 0 & 0 & -\eta_E \mathrm{I} \end{bmatrix}$$

with

$$\begin{aligned}
\Gamma_{11} =&\ \xi\mathrm{He}(\tilde{A}(\alpha(k))F(\bar{\alpha}(k)) + \tilde{B}(\alpha(k))L(\alpha(k))Q_1(\alpha(k))) - P(\alpha(k+1)) + \eta_A\delta_A^2\mathrm{I} + \eta_B\delta_B^2\mathrm{I} + \eta_E\delta_E^2\mathrm{I}\\
\Gamma_{21} =&\ -\xi F(\bar{\alpha}(k)) + (\tilde{A}(\alpha(k))G(\bar{\alpha}(k)) + \tilde{B}(\alpha(k))L(\alpha(k))Q_1(\alpha(k)))',\\
\Gamma_{31} =&\ (\tilde{E}(\alpha(k)) + \tilde{B}(\alpha(k))L(\alpha(k))\tilde{E}_y(\alpha(k)))',\\
\Gamma_{22} =&\ P(\alpha(k)) - G(\bar{\alpha}(k)) - G(\bar{\alpha}(k))',
\end{aligned}$$

matrices $\mathcal{C}_G$ and $\mathcal{S}_G$ given by

$$\mathcal{C}_G = \begin{bmatrix} 0\\ 0\\ 0\\ \gamma\mathrm{I}\\ 0\\ 0\\ 0 \end{bmatrix}, \quad \mathcal{S}_G' = \begin{bmatrix} \xi(S(\alpha(k))Q_1(\alpha(k)) - \tilde{C}_y(\alpha(k))F(\bar{\alpha}(k)))'\\ (S(\alpha(k))Q_1(\alpha(k)) - \tilde{C}_y(\alpha(k))G(\bar{\alpha}(k)))'\\ (S(\alpha(k))\tilde{E}_y(\alpha(k)) - \tilde{E}_y(\alpha(k)))'\\ S(\alpha(k))'\\ 0\\ 0\\ 0 \end{bmatrix},$$

and

$$\mathcal{Q}_T = \begin{bmatrix} P(\alpha(k)) - H(\bar{\alpha}(k)) - H(\bar{\alpha}(k))' & \star & \star & \star\\ \tilde{C}_z(\alpha(k))H(\bar{\alpha}(k)) + \tilde{D}_z(\alpha(k))L(\alpha(k))Q_2(\alpha(k)) & -W(\alpha(k)) & \star & \star\\ 0 & (\tilde{E}_z(\alpha(k)) + \tilde{D}_z(\alpha(k))L(\alpha(k))\tilde{E}_y(\alpha(k)))' & -\mathrm{I} & \star\\ 0 & (\tilde{D}_z(\alpha(k))L(\alpha(k)))' & 0 & 0 \end{bmatrix}$$

and matrices $\mathcal{C}_T$ and $\mathcal{S}_T$ given by

$$\mathcal{C}_T = \begin{bmatrix} 0\\ 0\\ 0\\ -\mathrm{I} \end{bmatrix}, \quad \mathcal{S}_T' = \begin{bmatrix} (S(\alpha(k))Q_2(\alpha(k)) - \tilde{C}_y(\alpha(k))H(\bar{\alpha}(k)))'\\ 0\\ (S(\alpha(k))\tilde{E}_y(\alpha(k)) - \tilde{E}_y(\alpha(k)))'\\ S(\alpha(k))' \end{bmatrix}.$$

Regarding $\mathcal{H}_\infty$ control, consider the following inequality

$$\mathcal{Q} + \mathcal{C}\mathcal{S} + \mathcal{S}'\mathcal{C}' < 0, \tag{32}$$

where $\mathcal{Q}$ is given by

$$\mathcal{Q} = \begin{bmatrix} \Gamma_{11} & \star & \star & \star & \star & \star & \star & \star\\ \Gamma_{21} & \Gamma_{22} & \star & \star & \star & \star & \star & \star\\ \Gamma_{31} & \Gamma_{32} & -\mu_\infty^2\mathrm{I} & \star & \star & \star & \star & \star\\ \Gamma_{41} & 0 & \Gamma_{43} & -\mathrm{I} & \star & \star & \star & \star\\ (\tilde{B}(\alpha(k))L(\alpha(k)))' & 0 & (\tilde{D}_z(\alpha(k))L(\alpha(k)))' & 0 & 0 & \star & \star & \star\\ \xi U(\bar{\alpha}(k)) & V(\bar{\alpha}(k)) & 0 & 0 & 0 & -\eta_A\mathrm{I} & \star & \star\\ \xi L(\alpha(k))Q_3(\alpha(k)) & L(\alpha(k))Q_3(\alpha(k)) & 0 & L(\alpha(k))\tilde{E}_y(\alpha(k)) & L(\alpha(k)) & 0 & -\eta_B\mathrm{I} & \star\\ 0 & 0 & 0 & \mathrm{I} & 0 & 0 & 0 & -\eta_E\mathrm{I} \end{bmatrix}$$

with

$$\begin{aligned}
\Gamma_{11} =&\ \xi\mathrm{He}\left(\tilde{A}(\alpha(k))U(\bar{\alpha}(k)) + \tilde{B}(\alpha(k))L(\alpha(k))Q_3(\alpha(k))\right) - \hat{P}(\alpha(k+1)) + \eta_A\delta_A^2\mathrm{I} + \eta_B\delta_B^2\mathrm{I} + \eta_E\delta_E^2\mathrm{I},\\
\Gamma_{21} =&\ -\xi U(\bar{\alpha}(k)) + (\tilde{A}(\alpha(k))V(\bar{\alpha}(k)) + \tilde{B}(\alpha(k))L(\alpha(k))Q_3(\alpha(k)))',\\
\Gamma_{31} =&\ \xi(\tilde{C}_z(\alpha(k))U(\bar{\alpha}(k)) + \tilde{D}_z(\alpha(k))L(\alpha(k))Q_3(\alpha(k))),\\
\Gamma_{41} =&\ (\tilde{E}(\alpha(k)) + \tilde{B}(\alpha(k))L(\alpha(k))\tilde{E}_y(\alpha(k)))',\\
\Gamma_{22} =&\ \hat{P}(\alpha(k)) - V(\bar{\alpha}(k) - V(\bar{\alpha}(k))',\\
\Gamma_{32} =&\ \tilde{C}_z(\alpha(k))V(\bar{\alpha}(k)) + \tilde{D}_z(\alpha(k))L(\alpha(k))Q_3(\alpha(k)),\\
\Gamma_{43} =&\ (\tilde{E}_z(\alpha(k)) + \tilde{D}_z(\alpha(k))L(\alpha(k))\tilde{E}_y(\alpha(k)))',
\end{aligned}$$

and $\mathcal{C}$ and $\mathcal{S}$ given, respectively, by

$$\mathcal{C} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \gamma \mathrm{I} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{S}' = \begin{bmatrix} \xi(S(\alpha(k))Q_3(\alpha(k)) - \tilde{C}_y(\alpha(k))U(\bar{\alpha}(k)))' \\ (S(\alpha(k))Q_3(\alpha(k)) - \tilde{C}_y(\alpha(k))V(\bar{\alpha}(k)))' \\ 0 \\ (S(\alpha(k))\tilde{E}_y(\alpha(k)) - \tilde{E}_y(\alpha(k)))' \\ S(\alpha(k))' \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Combining inequalities (29)-(32), the following theorem presents a synthesis condition to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ gain-scheduled reduced-order DOF control problem for the discrete-time linear system (1).

Theorem 2 (Mixed $\mathcal{H}_2/\mathcal{H}_\infty$)**.** If there exist matrices $P(\alpha(k))$ and $\hat{P}(\alpha(k)) \in \mathbb{S}_+^{(n_x+n_c)}$, $W(\alpha(k)) \in \mathbb{S}_+^{n_z}$, $F(\bar{\alpha}(k))$, $G(\bar{\alpha}(k))$, $H(\bar{\alpha}(k))$, $U(\bar{\alpha}(k))$ and $V(\bar{\alpha}(k)) \in \mathbb{R}^{(n_x+n_c)\times(n_x+n_c)}$, $L(\alpha(k)) \in \mathbb{R}^{(m+n_c)\times(q+n_c)}$ and $S(\alpha(k)) \in \mathbb{R}^{(q+n_c)\times(q+n_c)}$, given matrices $Q_i(\alpha(k))$, $i = 1, \dots, 3$, scalar variables μ_2, μ_∞, η_A, η_B, and η_E, and given scalar parameters $\gamma \neq 0$ and ξ, such that inequalities (29), (30), (31) and (32), are satisfied for all $\alpha(k) \in \Lambda$, then $\Theta(\alpha(k)) = L(\alpha(k))S(\alpha(k))^{-1}$ is a gain-scheduled stabilizing DOF controller for system (8) and the following statements are true.

(i) For a given $\mu_2 > 0$, by minimizing μ_∞ subject to (29)-(32), one has that scalars μ_2 and μ_∞ are upper bounds for the norms $\mathcal{H}_2$ and $\mathcal{H}_\infty$, respectively.

(ii) For a given $\mu_\infty > 0$, by minimizing μ_2 subject to (29)-(32), one has that scalars μ_2 and μ_∞ are upper bounds for the norms $\mathcal{H}_2$ and $\mathcal{H}_\infty$, respectively.

(iii) For a given $\kappa \in [0, 1]$, and considering the problem of minimizing

$$\nu = \kappa\mu_\infty + (1-\kappa)\mu_2, \tag{33}$$

subject to (29)-(32), one has that scalars μ_2 and μ_∞ are upper bounds for the norms $\mathcal{H}_2$ and $\mathcal{H}_\infty$, respectively.

Proof. This proof is divided in two main parts, regarding $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. As done in the proof of Theorem 1 for ease of notation, the dependence on the time-varying parameters is omitted hereafter. As well in Theorem 1, note that the feasibility of (30) (or (32)) guarantees that $\gamma(S+S') < 0$, implying that S^{-1} and, consequently, the controller Θ exist whenever (30) (or (32)) holds.

The first step to be taken is to recover the inequalities that treat the original matrices of the system ($\tilde{A}_\Delta$, $\tilde{B}_\Delta$, $\tilde{E}_\Delta$), which embrace the polynomial terms and the norm-bounded uncertainties. This can be performed as shown in Theorem 1, that is, manipulating the conditions in order to recover the terms ΔA, ΔB, and ΔE from their bounds δ_A, δ_B, and δ_E, employing the expressions presented in (3). Considering the $\mathcal{H}_2$ norm, inequalities (29) and (31) do not require those manipulations, since only (30) (related to the controllability gramian) presents norm-bounded uncertainties. By applying the Schur complement on inequality (30), one has

$$R_E + \eta M_E M_E' + \eta^{-1} N_E' N_E < 0, \tag{34}$$

with

$$R_E = \begin{bmatrix} \xi\mathrm{He}(\tilde{A}F(\bar{\alpha}(k)) + \tilde{B}LQ_1) - P^+ + \eta_A\delta_A^2\mathrm{I} + \eta_B\delta_B^2\mathrm{I} & \star & \star & \star & \star & \star \\ -\xi\bar{F} + (\tilde{A}\bar{G} + \tilde{B}LQ_1)' & P - \bar{G} - \bar{G}' & \star & \star & \star & \star \\ (\tilde{E}_\Delta + \tilde{B}L\tilde{E}_y)' & 0 & -\mathrm{I} & \star & \star & \star \\ (\tilde{B}L)' + \xi\gamma(SQ_1 - \tilde{C}_y\bar{F}) & \xi\gamma(SQ_1 - \tilde{C}_y\bar{F}) & \gamma(S+S') & 0 & \star & \star \\ \xi\bar{F} & \bar{G} & 0 & 0 & -\eta\mathrm{I}_A & \star \\ \xi LQ_1 & LQ_1 & L\tilde{E}_y & L & 0 & -\eta_B\mathrm{I} \end{bmatrix},$$

$$M_E' = \begin{bmatrix} \delta_E\mathrm{I} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_E = \begin{bmatrix} 0 & 0 & \mathrm{I} & 0 & 0 & 0 \end{bmatrix}, \quad \eta = \eta_E.$$

Thus, knowing that (3) holds ($\Delta\tilde{E}\Delta\tilde{E}' \leq \delta_E^2\mathrm{I}$), Lemma 1 is applied in (34) to obtain

$$R_E + \begin{bmatrix} \Delta\tilde{E}' & 0 & 0 & 0 & 0 & 0 \end{bmatrix}' N_E + N_E' \begin{bmatrix} \Delta\tilde{E}' & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0. \tag{35}$$

Repeating this procedure, sequentially and in an analogous way as described above, in order to recover matrices $\tilde{B}_\Delta$ and $\tilde{A}_\Delta$, one has that (30) can be rewritten as

$$\hat{Q}_G + \hat{C}_G \hat{S}_G + \hat{S}_G' \hat{C}_G' < 0 \tag{36}$$

for

$$\hat{Q}_G = \begin{bmatrix} \xi \mathrm{He}(\tilde{A}_\Delta \bar{F} + \tilde{B}_\Delta L Q_1) - P^+ & \star & \star & \star \\ -\xi \bar{F} + (\tilde{A}_\Delta \bar{G} + \tilde{B}_\Delta L Q_1)' & P - \bar{G} - \bar{G}' & \star & \star \\ (\tilde{E}_\Delta + \tilde{B}_\Delta L \tilde{E}_y)' & 0 & -I & \star \\ (\tilde{B}_\Delta L)' & 0 & 0 & 0 \end{bmatrix}, \quad \hat{C}_G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \gamma \mathrm{I} \end{bmatrix}, \quad \hat{S}_G' = \begin{bmatrix} \xi(SQ_1 - \tilde{C}_y \bar{F})' \\ (SQ_1 - \tilde{C}_y \bar{G})' \\ (S\tilde{E}_y - \tilde{E}_y)' \\ S' \end{bmatrix}.$$

Pre- and post-multiplying (36) respectively by

$$\begin{bmatrix} \mathrm{I} & \tilde{A}_{cl} & 0 & \xi(S^{-1}\tilde{C}_y \bar{F} - Q_1)' + \tilde{A}_{cl}(S^{-1}\tilde{C}_y \bar{G} - Q_1)' \\ 0 & 0 & \mathrm{I} & (S^{-1}\tilde{E}_y - \tilde{E}_y)' \end{bmatrix}$$

and its transpose, with $\tilde{A}_{cl}$ and $\tilde{B}_{cl}$ as given in (9), one has

$$\begin{bmatrix} \tilde{A}_{cl} P \tilde{A}_{cl}' - P^+ & \tilde{B}_{cl} \\ \tilde{B}_{cl}' & -\mathrm{I} \end{bmatrix} < 0,$$

which recovers the controllability gramian presented by de Caigny et al.[7]

On the other hand, pre- and post-multiplying (31) respectively by

$$\begin{bmatrix} \tilde{C}_{cl} & \mathrm{I} & 0 & \xi \tilde{C}_{cl}(S^{-1}\tilde{C}_y \bar{H} - Q_2)' \\ 0 & 0 & \mathrm{I} & (S^{-1}\tilde{E}_y - \tilde{E}_y)' \end{bmatrix}$$

and its transpose, with $\tilde{C}_{cl}$ and $\tilde{D}_{cl}$ as given in (9), one has

$$\begin{bmatrix} \tilde{C}_{cl} P \tilde{C}_{cl}' - W & \tilde{D}_{cl} \\ \tilde{D}_{cl}' & -\mathrm{I} \end{bmatrix} < 0,$$

which recovers the cost inequality condition presented by de Caigny et al.[7] Therefore, it has been proved that the set of inequalities (29), (30) and (31) guarantees asymptotic stability of system (8) and provides an upper bound for its $\mathcal{H}_2$ norm given by μ_2.

In the second part of this proof, regarding the $\mathcal{H}_\infty$ norm, inequality (32) is manipulated to recover the original matrices of the system ($\tilde{A}_\Delta$, $\tilde{B}_\Delta$, $\tilde{E}_\Delta$), using the procedure enunciated before (Schur complement and Lemma 1), considering the following choices

$$M_E' = \begin{bmatrix} \Delta\tilde{E}' & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_E = \begin{bmatrix} 0 & 0 & 0 & \mathrm{I} & 0 & 0 & 0 \end{bmatrix}, \quad \eta = \eta_E,$$

$$M_B' = \begin{bmatrix} \Delta\tilde{B} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_B = \begin{bmatrix} \xi L\tilde{C}_y & L\tilde{C}_y & 0 & L\tilde{E}_y & L & 0 \end{bmatrix}, \quad \eta = \eta_B,$$

$$M_A' = \begin{bmatrix} \Delta\tilde{A}' & 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_A = \begin{bmatrix} \xi\bar{U} & \bar{V} & 0 & 0 & 0 \end{bmatrix}, \quad \eta = \eta_A,$$

and appropriated matrices R_E, R_B and R_A, obtained in an analogous way as described in Theorem 1. After performing these steps, one has that if (32) is verified, then the following inequality holds

$$\hat{\mathcal{Q}} + \hat{\mathcal{C}}\hat{\mathcal{B}} + \hat{\mathcal{B}}'\hat{\mathcal{C}}' < 0 \tag{37}$$

with

$$\hat{\mathcal{Q}} = \begin{bmatrix} \xi\mathrm{He}(\tilde{A}_\Delta \bar{U} + \tilde{B}_\Delta L Q_3) - \hat{P}^+ & \star & \star & \star & \star \\ -\xi\bar{U} + (\tilde{A}_\Delta \bar{V} + \tilde{B}_\Delta L Q_3)' & \hat{P} - \bar{V} - \bar{V}' & \star & \star & \star \\ \xi(\tilde{C}_z \bar{U} + \tilde{D}_z L Q_3) & \tilde{C}_z \bar{V} + \tilde{D}_z L Q_3 & -\mu_\infty^2 \mathrm{I} & \star & \star \\ (\tilde{E}_\Delta + \tilde{B}_\Delta L \tilde{E}_y)' & 0 & (\tilde{E}_z + \tilde{D}_z L \tilde{E}_y)' & -\mathrm{I} & \star \\ (\tilde{B}_\Delta L)' & 0 & (\tilde{D}_z L)' & 0 & 0 \end{bmatrix}, \hat{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \gamma S \end{bmatrix}, \hat{\mathcal{B}}' = \begin{bmatrix} \xi(Q_3 - S^{-1}\tilde{C}_y \bar{U})' \\ (Q_3 - S^{-1}\tilde{C}_y \bar{V})' \\ 0 \\ (\tilde{E}_y - S^{-1}\tilde{E}_y)' \\ \mathrm{I} \end{bmatrix}.$$

Pre- and post-multiplying (37), respectively, by

$$\begin{bmatrix} \mathrm{I} & \tilde{A}_{cl} & 0 & 0 & \xi(S^{-1}\tilde{C}_y \bar{U} - Q_3)' + \tilde{A}_{cl}(S^{-1}\tilde{C}_y \bar{V} - Q_3)' \\ 0 & \tilde{C}_{cl} & \mathrm{I} & 0 & \tilde{C}_{cl}(S^{-1}\tilde{C}_y \bar{V} - Q_3)' \\ 0 & 0 & 0 & \mathrm{I} & (S^{-1}\tilde{E}_y - \tilde{E}_y)' \end{bmatrix},$$

and its transpose, with $\tilde{A}_{cl}$, $\tilde{B}_{cl}$, $\tilde{C}_{cl}$, and $\tilde{D}_{cl}$ as given in (9), one has

$$\begin{bmatrix} \tilde{A}_{cl}\hat{P}\tilde{A}'_{cl} - \hat{P}^{+} & \star & \star \\ \tilde{C}_{cl}P\tilde{A}'_{cl} & -\mu_\infty^2 \mathrm{I} + \tilde{C}_{cl}\hat{P}\tilde{C}'_{cl} & \star \\ \tilde{B}'_{cl} & \tilde{D}'_{cl} & -\mathrm{I} \end{bmatrix} < 0$$

that can be recognized as the Bounded Real Lemma [38] applied to system (8), which guarantees the asymptotic stability and that μ_∞ is an upper bound for the $\mathcal{H}_\infty$ norm of system (8).

Therefore, by using any of the three statements of Theorem 2, regarding the optimization of $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ norms, the closed-loop system is asymptotically stable and guaranteed costs for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are given respectively by μ_2 and μ_∞. □

5 | EXTENSIONS AND MAIN ADVANTAGES OF THE PROPOSED TECHNIQUE

Theorems 1 and 2 can be straightforwardly extended to deal with other classes of dynamical systems besides linear systems with polynomial dependence on time-varying parameters. Remark 2 enumerates those other classes, and the different control problems that can be addressed. Additionally, Remark 2 below provides instructions to adapt the proposed conditions to solve these problems.

Remark 2. Extensions of the method. Theorems 1 and 2 can be adapted to deal with the following classes of systems and approaches of control:

1. **State-feedback controllers:** Replace $\tilde{C}_y(\alpha(k))$ and $Q_i(\alpha(k))$, $i = 1, \dots, p$ by I and $\tilde{E}_y(\alpha(k))$ by 0, respectively.
2. **Robust controllers:** Take $S(\alpha(k)) = S$ and $L(\alpha(k)) = L$, $\forall \alpha(k) \in \Lambda$.
3. **LTI systems:** Consider system (8) with time-invariant parameters ($\alpha(k) = \alpha, \forall k \in \mathbb{N}$), and make $P(\alpha(k+1)) = P(\alpha(k)) = P(\alpha)$ (and $\hat{P}(\alpha(k+1)) = \hat{P}(\alpha(k)) = \hat{P}(\alpha)$).
4. **Polytopic systems:** Consider all state-space matrices of system (8) in the polytopic form described by (4).
5. **Systems without norm-bounded uncertainties:** Make $\Delta A(\alpha(k)) = \Delta B(\alpha(k)) = \Delta E(\alpha(k)) = 0$.
6. **Switched systems:** Consider the problem of designing a controller for a switched system given by (8) replacing $\alpha(k)$ by $\psi(k)$, where $\psi(k)$ represents a switched rule such as $\psi(k) : \mathbb{N} \rightarrow \Lambda$ that arbitrarily chooses the subsystem (operation mode) activated at each instant of time. The stabilizing gain, given by $\Theta_{\psi(k)} = L_{\psi(k)}S^{-1}_{\psi(k)}$, is obtained by the proposed conditions by replacing the dependence on $\alpha(k)$ by $\psi(k)$ in the decision variables.

Besides the various applications of the proposed method mentioned above, it is possible to evidence some other relevant particularities of the technique. The first one is related to the mixed problem, where the only variables common to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ constraints are $S(\alpha)$ and $L(\alpha)$, that is, the ones used to construct the control gain. Actually, the Lyapunov matrices and slack variables are different and this unique feature helps to reduce the conservativeness of the method.

Another interesting aspect of the proposed method is the possibility to treat any output matrix $C_y(\alpha(k))$ without imposing a special structure or restrictions in the optimization variables. Note that the first output-feedback methods based on LMIs [39,40] required that this matrix was constant, parameter-independent and constrained to the form $C_y(\alpha(k)) = [\mathrm{I} \;\; 0]$. More recent methods [41,42] relieved these constraints, but, in general, they still require similarity transformations and they are not capable of dealing with the polynomial dependency on the parameters (only affine dependency). Besides that, another notable characteristic of the design conditions is the fact that the slack variables are dependent on the time-varying parameters in two consecutive time instants, which is not very common in synthesis conditions that allow the design of robust gains (see a discussion about this topic in Section 6). Finally, the proposed method can provide control gains with particular structures (for example with decentralized structure) by constraining the structures of the decision variable matrices $L(\alpha(k))$ and $S(\alpha(k))$, for instance using the approach from Geromel et al. [43]

As disadvantage of the proposed method, in general, the good results are obtained at the price of a larger computational effort due to: the search for the scalars γ, ξ and ρ that requires to solve the inequalities a certain amount of times; the increase of the degrees of the decision variables; and the use of slack matrices that augment the number of scalar optimization variables required to solve the synthesis conditions.

6 | FINITE DIMENSIONAL TESTS

To perform numerical tests using the synthesis conditions proposed in this paper, first it is necessary to make some considerations. The verification of the positivity (or negativity) of the linear inequality conditions that depend on time-varying parameters characterizes an optimization problem of infinite dimension since the optimization variables are functions of the parameter $\alpha(k)$ (and also of the advanced instant $\alpha(k+1)$), whose forms (structures) are unknown *a priori*. The first step to work around this issue is to eliminate the time-dependency of the LMI conditions by assuming that $\alpha(k) \in \Lambda$ for all $k \geq 0$. It is also important to emphasize that, concerning the variation in time of parameters $\alpha(k)$, two main scenarios are investigated in the literature: the case where $\alpha(k+1)$ depends on $\alpha(k)$ (bounded rate of variation) and when they are independent (arbitrarily fast variation).[44] In this paper, the last case is adopted for the numerical experiments by considering $\alpha(k+1) = \beta(k) \in \Lambda$ independent of $\alpha(k) \in \Lambda$.

Even after these considerations, the proposed conditions are not in a programmable form yet, since they are given as parameter-dependent (robust) LMIs. One effective tool to deal with this problem is the employment of polynomial approximations[45,46] (only sufficient in the case of time-varying parameters, but still effective) using, for instance, the parser ROLMIP (*Robust LMI Parser*).[47] This parser, that works jointly with Yalmip,[48] allows to fix the optimization variables as polynomials (more precisely, homogeneous polynomials) of a chosen degree of dependence on the parameters. Thereby, the last task is to check the positivity (or negativity) of the resulting polynomial matrix inequalities (a known NP-hard problem). Among several relaxations available, ROLMIP employs the one based on Pólya's theorem.[49]

Regarding the choice of the polynomial degrees for the optimization variables, some remarks are important. The variables $L(\alpha(k))$ and $S(\alpha(k))$ define the structure of the controller, and if a robust controller (parameter-independent) is desired, then the degrees associated with them must be zero. If at least one of the degrees is not zero, then the synthesized controller is gain-scheduled and the vector of parameters $\alpha(k)$ must be available on-line (measured or estimated). These choices must be taken *a priori* by the designer, considering the nature of the controlled plant. The degrees related with the other variables only influence the conservativeness of the solutions. As discussed in Oliveira and Peres[50], concerning the minimum degree necessary to provide feasible solutions (stabilizing gains), if the robust LMIs have a solution, then for a sufficiently large degree $g^\star$, finite but unknown, the synthesis conditions will have a solution, providing a stabilizing controller and a guaranteed cost. For $g > g^\star$ stability continues to be assured and improved or at least equal guaranteed costs can be obtained (which means monotonic behavior in terms of performance indexes), clearly, at the price of a larger computational effort. To perform a fair comparison with other methods from the literature, the degree of the Lyapunov matrices and the slack variables is kept as equal to one in the examples.

The choice of the order of the controllers (n_c) also must be specified *a priori* by the designer. If this value is small, the number of variables in the synthesis problem decreases, reducing the computational complexity. Additionally, some practical applications usually constrain the use of controllers with order great than 2, for instance, admitting only controllers with the same order as a proportionalâĂŞintegralâĂŞderivative (PID) controller. By designing static controllers (order 0, as the ones provided in the examples at Section 7), it is possible to broaden the application of the proposed method even for real-world plants that only admits the use of proportional controllers. That is, Theorems 1 and 2 are guaranteed to be implementable in any conventional control device used in industry.

As mentioned, the proposed conditions require that the parameters γ and ξ be given *a priori*, otherwise the conditions are BMIs. In principle there is no rule to choose these parameters in order to obtain the best results. However, particular choices can be used to recover specific methods from the literature. One of them is the combination of Theorems 1 and 2 presented in Morais et al,[4] concerning a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback control condition for polytopic LTI systems, as stated in the following corollary.

Corollary 2. If the combination of Theorems 1 and 2 presented by Morais et al[4] has a solution, then Theorem 2 adapted to obtain mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback controllers for polytopic LTI systems without norm-bounded terms ($\Delta A = 0$, $\Delta B = 0$ and $\Delta E = 0$) provides the same solution by setting $\gamma \to -\infty$ and $\xi \in (-1, 1)$.

Proof. First, observe that, by fixing $L(\alpha) = Z$ and $S(\alpha) = F(\alpha) = G(\alpha) = H(\alpha) = U(\alpha) = V(\alpha) = X$, inequalities (29) and (31) are equivalent to (14) and (15) from Morais et al.[4] Furthermore, using the same change of variables, inequalities (36) and (37), adapted to solve the state-feedback control problem for polytopic LTI systems without norm-bounded terms, can be respectively rewritten as

$$\begin{bmatrix} \xi\mathrm{He}(\tilde{A}_\Delta(\alpha)X + \tilde{B}_\Delta(\alpha)L) - P(\alpha) & \star & \star \\ -\xi X + (\tilde{A}_\Delta(\alpha)X + \tilde{B}_\Delta(\alpha)L)' & P(\alpha) - X - X' & \star \\ (\tilde{E}_\Delta(\alpha))' & 0 & -I \end{bmatrix} < \mathcal{T}_2 \frac{1}{\gamma}(X + X')^{-1}\mathcal{T}_2', \tag{38}$$

$$\begin{bmatrix} \xi\mathrm{He}(\tilde{A}_\Delta(\alpha)X + \tilde{B}_\Delta(\alpha)L) - \hat{P}(\alpha) & \star & \star & \star \\ -\xi X + (\tilde{A}_\Delta(\alpha)X + \tilde{B}_\Delta(\alpha)L)' & \hat{P}(\alpha) - X - X' & \star & \star \\ \xi(\tilde{C}_z(\alpha)X + \tilde{D}_z(\alpha)L) & \tilde{C}_z(\alpha)X + \tilde{D}_z(\alpha)L & -\mu_\infty^2\mathrm{I} & \star \\ \tilde{E}_\Delta(\alpha)' & 0 & \tilde{E}_z(\alpha)' & -\mathrm{I} \end{bmatrix} < \mathcal{T}_\infty \frac{1}{\gamma}(X + X')^{-1}\mathcal{T}_\infty', \tag{39}$$

with $\mathcal{T}_2' = \begin{bmatrix}(\tilde{B}_\Delta L)' & 0 & 0\end{bmatrix}$ and $\mathcal{T}_\infty' = \begin{bmatrix}(\tilde{B}_\Delta L)' & 0 & (\tilde{D}_z(\alpha)L)' & 0\end{bmatrix}$. By setting $\gamma \to -\infty$ and $\xi \in (-1, 1)$, both left-hand sides of (38) and (39) are negative definite and therefore equivalent to Equations (21) and (17) presented in Morais et al,[4] considering $P(\alpha) = \hat{P}(\alpha) = W(\alpha)$ and $W(\alpha) = M(\alpha)$. □

Other technique that can be recovered by Theorem 2 is the method described in Section 5.3 of de Caigny et al,[7] which combines Theorems 8 and 9 to solve the problem of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ static output-feedback control for polytopic LPV systems, as shown in the next corollary.

Corollary 3. If the synthesis condition discussed in Section 5.3 (that combines Theorems 8 and 9) of de Caigny et al[7] has a solution, then Theorem 2 adapted to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$gain-scheduled static output-feedback control for polytopic LPV systems provides the same solution by setting $\gamma \to -\infty$ and $\xi = 0$.

Proof. First, observe that, by fixing $L(\alpha(k)) = Z(\alpha(k))$, $F(\bar{\alpha}(k)) = G(\bar{\alpha}(k)) = H(\bar{\alpha}(k)) = U(\bar{\alpha}(k)) = V(\bar{\alpha}(k)) = S(\alpha(k)) = G(\alpha(k))$, $\xi = 0$, and $\tilde{E}_y = 0$, $\tilde{C}_y = Q_i = [\mathrm{I}\ 0]$, inequality (29) is equivalent to Equation (34) from de Caigny et al.[7] Furthermore, inequalities (36) and (37) adapted to solve the static output-feedback control problem for polytopic LPV systems without norm-bounded terms can be respectively rewritten as

$$\begin{bmatrix} P(\alpha(k+1)) & \star & \star \\ (\tilde{A}_\Delta(\alpha(k))G(\alpha(k)) + \tilde{B}_\Delta(\alpha(k))Z(\alpha(k)))' & G(\alpha(k)) + G(\alpha(k))' + P(\alpha(k)) & \star \\ (\tilde{E}_\Delta(\alpha(k)))' & 0 & I \end{bmatrix} > \mathcal{T}_G \frac{1}{\gamma}(-G(\alpha(k)) - G(\alpha(k))')^{-1}\mathcal{T}_G', \tag{40}$$

$$\begin{bmatrix} \hat{P}(\alpha(k+1)) & \star & \star & \star \\ (\tilde{A}_\Delta(\alpha(k))G(\alpha(k)) + \tilde{B}_\Delta(\alpha(k))Z(\alpha(k)))' & G(\alpha(k)) + G(\alpha(k))' - \hat{P}(\alpha(k)) & \star & \star \\ 0 & \tilde{C}_z(\alpha(k))G(\alpha(k)) + \tilde{D}_z(\alpha(k))Z(\alpha(k)) & \mu_\infty^2\mathrm{I} & \star \\ \tilde{E}_\Delta(\alpha(k))' & 0 & \tilde{E}_z(\alpha(k))' & \mathrm{I} \end{bmatrix} > \mathcal{T}_\infty \frac{1}{\gamma}(-G(\alpha(k)) - G(\alpha(k))')^{-1}\mathcal{T}_\infty', \tag{41}$$

with $\mathcal{T}_G' = \begin{bmatrix}(\tilde{B}_\Delta(\alpha(k))Z(\alpha(k)))' & 0 & 0\end{bmatrix}$ and $\mathcal{T}_\infty' = \begin{bmatrix}(\tilde{B}_\Delta(\alpha(k))Z(\alpha(k)))' & 0 & (\tilde{D}_z(\alpha(k))Z(\alpha(k)))' & 0\end{bmatrix}$. By setting $\gamma \to -\infty$, it is possible to verify that both (38) and (39) left sides are positive definite.

Next, pre- and post-multiplying Equation (44) of de Caigny et al[7] respectively by

$$\begin{bmatrix} \eta^{1/2}I & 0 & 0 & 0 \\ 0 & \eta^{1/2}I & 0 & 0 \\ 0 & 0 & 0 & \eta^{1/2}I \\ 0 & 0 & \eta^{-1/2}I & 0 \end{bmatrix}$$

and its transpose, one recovers inequality (41), with $\eta P(\alpha(k+1)) = P(\alpha(k+1)), \eta P(\alpha(k)) = P(\alpha(k))$, $\eta G(\alpha(k)) = G(\alpha(k))$, $\eta Z(\alpha(k)) = Z(\alpha(k))$ and $\eta = \mu_\infty^2$. Additionally, inequality (31) can be rewritten as

$$\begin{bmatrix} G(\bar{\alpha}(k)) + G(\bar{\alpha}(k))' - P(\alpha(k)) & \star & \star \\ \tilde{C}_z(\alpha(k))G(\bar{\alpha}(k)) + \tilde{D}_z(\alpha(k))Z(\alpha(k)) & W(\alpha(k)) & \star \\ 0 & \tilde{E}_z(\alpha(k))' & \mathrm{I} \end{bmatrix} > \mathcal{T}_{T_1}(-G(\alpha(k)) - G(\alpha(k))')^{-1}\mathcal{T}_{T_1}' > 0, \tag{42}$$

with $\mathcal{T}_{T_1}' = \begin{bmatrix}0 & (\tilde{D}_z(\alpha(k))Z(\alpha(k)))' & 0\end{bmatrix}$. Then, (42) assures that

$$\begin{bmatrix} G(\bar{\alpha}(k)) + G(\bar{\alpha}(k))' - P(\alpha(k)) & \star \\ \tilde{C}_z(\alpha(k))G(\bar{\alpha}(k)) + \tilde{D}_z(\alpha(k))Z(\alpha(k)) & W(\alpha(k)) \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{E}_z(\alpha(k))' \end{bmatrix} (\mathrm{I})^{-1} \begin{bmatrix}0 & \tilde{E}_z(\alpha(k))'\end{bmatrix} > 0,$$

that is equivalent to Equation (51) of de Caigny et al.[7] by means of a simple congruence transformation. □

Motivated by the results of Corollaries 2 and 3, in the synthesis conditions presented in Sections 3 and 4 of this paper, the following set of values is used

$$\gamma = -10^5, \ \ \xi \in \{-0.9, -0.8, \dots, 0.8, 0.9\}. \tag{43}$$

If computational resources are available, a scalar search in γ or a finer grid on ξ could improve the results.

With respect to the introduction of a decay rate bounded by ρ in the synthesis conditions, a new heuristic procedure to find stabilizing controllers when the conditions from the literature fail is proposed. The heuristic consists of allowing the value of ρ to be greater than one in the synthesis conditions. Note that this choice, at first, does not make sense, since robust stability is only assured if $0 < \rho \leq 1$. However, note that, in general, synthesis conditions are only sufficient (for uncertain systems) and the main sources of conservativeness are the structure of the optimization variables and the linearizing technique to obtain LMIs. Therefore, the resulting value of ρ is merely an upper bound for the optimal value $\rho^\star$ and there may be a gap between ρ and $\rho^\star$. In other words, even if ρ is greater than 1, $\rho^\star$ can be lower than 1. On the other hand, it is known that robust stability analysis conditions can always determine if $\rho < 1$ with the increase of the degree of the polynomial variables.[50,51] Using this important feature, a stability analysis condition can be applied *a posteriori* to certificate the closed-loop stability (i.e., if $\rho < 1$). In resume, a convenient exploitation of the analysis conditions together with a slightly relaxed sufficient condition can lead to better results than a sole sufficient synthesis condition. This heuristic, along with other relevant details, are introduced in the next subsection.

6.1 | New heuristic design procedure

After designing a DOF gain by solving the optimization problem given in (27) with a bound ρ that does not assure stability, that is, making $\rho > 1$,two tests are applied, in sequence, to confirm the stability of the closed-loop system.

1st This test is used to verify the absolute value of the eigenvalues associated to the matrices $\tilde{A}_{cl_i}$, $i = 1, \dots, N$, which are obtained from $\tilde{A}_{cl}(\alpha)$ for the following choices: $\alpha_i = 1$, $\alpha_j = 0$, $\forall i, j = 1, \dots, N, j \neq i$. Note that, for a polytopic system, matrices $\tilde{A}_{cl_i}$ are called "vertices" of the matrix $\tilde{A}_{cl}(\alpha)$. After computing the matrices, the maximum absolute value of their eigenvalues is determined. This procedure, which is only a *necessary* condition for the stability, is used as a "cheap test" to be computationally verified, allowing to discard immediately the possibility of robust stability if some $\tilde{A}_{cl_i}$ is unstable.

2nd The first test is only a necessary condition to certificate the closed-loop stability and the polynomial (or polytopic) matrix $\tilde{A}_{cl}(\alpha)$ can still be unstable. Therefore, the second test aims to provide a robust stability certificate by solving a robust stability analysis LMI condition. However, this procedure demands the execution of one extra test, requiring a greater computational effort.

It is important to mention that the algorithm described in this section is used only to deal with LPV and LTI systems without the norm-bounded terms, using robust stability analysis conditions adapted for those systems. The stability analysis LMI conditions used in Algorithm 1 to provide the results of the numerical examples of this paper are given in Theorem 2 of de Oliveira et al[52] for LTI systems and their extension given by Daafouz and Bernussou for LPV systems.[3] Furthermore, the input parameter degP is used to set the degree of parameter dependence of the Lyapunov matrix in the analysis condition (greater degrees provide in general less conservative results, at the cost of greater computational effort).

In this sense, Algorithm 1 presents a procedure that verifies, *a posteriori*, the stability of the closed-loop system, considering a controller obtained from a heuristic search allowing that the upper bound for the decay rate ρ is greater than one.

Remark 3. A technique similar to the one proposed by Algorithm 1 can be applied in the search for mixed $\mathcal{H}_2/\mathcal{H}_\infty$controllers by considering the inclusion of parameter ρ^2 multiplying $P(\alpha(k+1))$ and $\hat{P}(\alpha(k+1))$ in Theorem 2. Although this adaptation can allow the synthesis of a larger set of stabilizing gains, note that, in this case, the values computed for μ_2 and μ_∞ are not upper bounds for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the original system (in fact they are meaningless). Thus, $\mathcal{H}_\infty$ and $\mathcal{H}_2$ analysis condition must be applied to the closed-loop system to *i)* assure the robust stability (the controller is actually a stabilizing one) and *ii)* determine the actual $\mathcal{H}_\infty$ and $\mathcal{H}_2$ guaranteed costs for the closed-loop system.

7 | NUMERICAL EXAMPLES

All the conditions proposed in this paper were programmed using the software MATLAB (R2014a) with the aid of the parsers ROLMIP[47] and Yalmip[48] and of the solver SeDuMi 1.3.[53] All the test were performed in an Ubuntu Linux computer, Intel Core i7-4770 (3.40 GHz), 8.0 GB RAM. All the gains presented along the section were truncated with four decimal digits.

Algorithm 1 Stability test of systems with $\rho > 1$

Require: $\tilde{A}(\alpha(k))$, $\tilde{B}(\alpha(k))$, $\tilde{C}_y(\alpha(k))$, $Q(\alpha(k))$, ξ, γ, ρ and degP
Ensure: Feasibility and $\Theta(\alpha(k))$

```
solve the optimization problem (27)
if optimization problem is feasible then
    calculate Ã_cl(α(k)) according with (9)
    if max(|λ_j(Ã_cl_i)| ≥ 1), for j = 1, …, n_x + n_c, and i = 1, …, N then
        return unfeasible
    else
        p = 0
        while p ≤ degP do
            solve the analysis condition with P(α(k)) of degree p
            if analysis condition is feasible then
                return feasible and Θ(α(k))
            else
                p = p + 1
            end if
        end while
    end if
else
    return unfeasible
end if
```

Regarding matrices $Q_i(\alpha(k))$, since different choices can provide better or worse results in the general case, tests were made combining (12) and (13) with different indexes i for all the examples presented in this paper. It was observed that the performances obtained were, in general, slightly better when choosing $Q_i(\alpha(k)) = \tilde{C}_y(\alpha(k))$, $i = 1, \dots, p$. Thus, almost all the numerical examples presented in this paper were performed using the most intuitive choice, that is, (12). Particularly, in Example 6.3 a comparison between the choice $Q_i(\alpha(k)) = \tilde{C}_y(\alpha(k))$, $i = 1, \dots, p$, and another one, combining both (12) and (13), is presented.

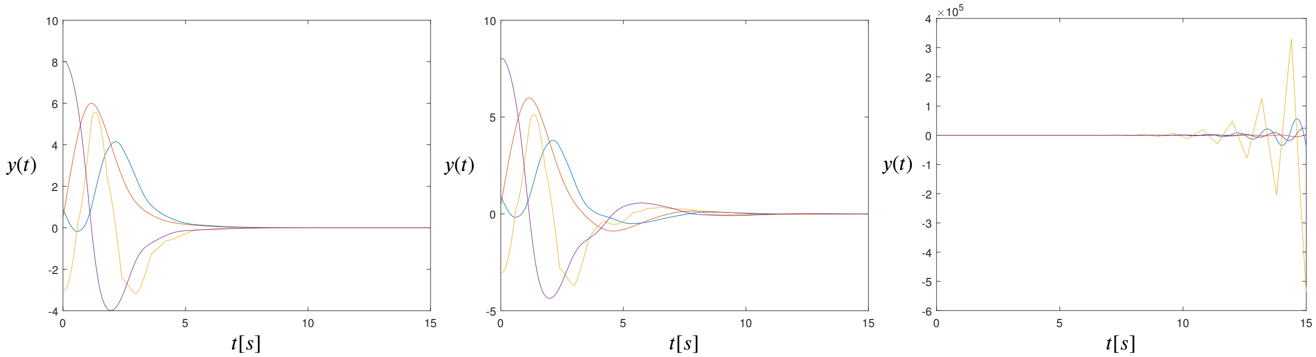
7.1 | Robust stabilization of LPV systems with norm-bounded terms

This example was produced to highlight the application of the proposed method in the stabilization of system in the form (1) arising from a discretization procedure of a continuous-time LPV system obtained by employing a Taylor series expansion truncated at a fixed degree. In this sense, consider the continuous-time polytopic system borrowed from Braga et al[22] and adapted for the time-varying case, with the following state-space matrices

$$A_c = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p(t)/2 & p(t)/2 & 0 & 0 \\ p(t)/3 & -p(t)/3 & 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{44}$$

where $p(t)$ is a parameter that can vary arbitrarily fast inside the interval 5.04 ± 1.008, yielding a polytopic LPV continuous-time system of two vertices. In order to obtain the discrete-time polynomial matrices $A_\Delta(\alpha)$ and $B_\Delta(\alpha)$, from (44), the discretization method proposed by Braga et al[22] is employed with a sampling of $T = 0.6$s.

Usually, to solve the problem of digital control of polytopic LTI or LPV continuous-time systems, some papers in the literature discretize only the vertices of the continuous-time model using a constant sampling-time and a first-order Taylor approximation (also known as Euler's approximation), thus producing another polytope (i.e., an affine parameter-dependent system) in the discrete-time domain. After that, a discrete-time condition for control design is applied, neglecting the residues of the discretization procedure (see a discussion on this issue in Braga et al[22]). However, since the residues are not taken into account, it is not theoretically possible to certify that the computed gains assure closed-loop stability for all the values that can be assumed by the time-varying parameters.



(a) $K_{T1(\ell=3,g=2)}$ (b) $K_{T1(\ell=4,g=1)}$ (c) K_{55}

FIGURE 2 Output trajectories of the continuous-time system (44) in closed-loop with the digital controllers obtained from (a) Theorem 1 with $g = 2$ and $\ell = 3$ considering the bounds for the residues ($\delta_A = 0.1728$ and $\delta_B = 0.0108$); (b) Theorem 1 with $g = 1$ and $\ell = 4$ considering the bounds for the residues ($\delta_A = 0.0817$ and $\delta_B = 0.0057$); and (c) Theorem 8 of de Caigny et al[55] with $\ell = 1$ neglecting the bounds for the residues ($\delta_A = 1.2847$ and $\delta_B = 0.1018$).

On the opposite way to what is done in most of the literature papers, the technique proposed in Theorem 1 of this paper ensures that, when considering the norm-bounded terms $\Delta A(\alpha)$ and $\Delta B(\alpha)$ of (3) that represent the residues of the truncated Taylor series, the synthesized controller stabilizes not only the discretized version of the LPV model, but also the hybrid set composed by the continuous-time plant and the discrete-time control law[2].

To illustrate the aforementioned comments, note that, by choosing a Taylor series expansion of degree $\ell = 1$, Theorem 1 is not able to provide a stabilizing solution, probably because of the high values of the bounds for the residues ($\delta_A = 1.2847$ and $\delta_B = 0.1018$). Considering the same discretized system and ignoring δ_A and δ_B, Theorem 8 of de Caigny et al[55] adapted for stabilization only, provides the following stabilizing gain

$$K_{55} = \begin{bmatrix} -3.5759 & 2.0704 & -7.7713 & 2.7337 \end{bmatrix}.$$

On the other hand, using a higher degree of Taylor series expansion, for example, $\ell = 3$, the bounds for the residues ($\delta_A = 0.1728$ and $\delta_B = 0.0108$) are smaller, allowing to find a stabilizing static output-feedback robust controller with Theorem 1, but only if the decision variables have degree $g \geq 2$ of polynomial dependence on the time-varying parameters. In this case, one obtains the following gain

$$K_{T1(\ell=3,g=2)} = \begin{bmatrix} -2.725 & -0.2351 & -4.3355 & -2.1367 \end{bmatrix}.$$

by Theorem 1 with $g = 2$, demanding $V = 196$ scalar variables and $L = 318$ LMI rows to solve the optimization problem. Increasing the degree of Taylor series expansion: $\ell = 4$, the upper bounds for the residues ($\delta_A = 0.0817$ and $\delta_B = 0.0057$) are further reduced and it is possible to design the following stabilizing robust gain

$$K_{T1(\ell=4,g=1)} = \begin{bmatrix} -2.588 & -1.1957 & -3.6591 & -1.8218 \end{bmatrix}.$$

with Theorem 1 using affine decision variables ($g = 1$), implying on less computational effort to solve the optimization problem ($V = 170$ scalar variables and $L = 212$ LMI rows).

In order to illustrate if controllers K_{55}, $K_{T1(\ell=3,g=2)}$, $K_{T1(\ell=4,g=1)}$ are actually stabilizing, time simulations were performed in Matlab considering the initial condition $x_0 = \begin{bmatrix} 1 & 0.5 & -3 & 8 \end{bmatrix}'$. Figure 2 presents a time-response of the output $y(t)$ of the closed-loop hybrid system (composed by the continuous-time plant plus discrete-time controller) for the particular case where the uncertain parameter $p(t)$ is given by $p(t) = 5.04 - 1.008\sin(4t)$. Converting the evolution of $p(t)$ into a parameter $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ belonging to a unit simplex, one can write $\alpha_1(t) = (\sin(4t) + 1)/2$ and $\alpha_2(t) = 1 - \alpha_1(t)$. As expected, the performed simulations show that using the controllers obtained with Theorem 1, the output trajectories converge to zero (stable closed-loop system). However, the controller obtained by Theorem 8 of de Caigny et al[55] cannot stabilize the closed-loop system because the bounds δ_A and δ_B have not been taken into account. Concerning the control of discretized systems obtained from polytopic

[2]Since the sampled-data control of continuous-time LPV systems is not the focus of this paper, this example is only presented to illustrate one of the several utilities of the proposed method. The demonstration of the closed-loop continuous-time LPV system stability using discrete-time controllers can be seen in the Appendix B of Braga et al.[54]

continuous-time systems, the result presented in Figure 2 justifies the use of a polynomial representation of the system (coming from the Taylor series expansion of degree ℓ) and also the consideration of the norm-bounded terms (that represent the residues of the truncated Taylor series). Furthermore, the stabilizing gain obtained for the discretized representation using a Taylor series expansion degree $\ell = 3$ and employing Theorem 1 with degree $g = 2$ in the decision variables shows that the augmentation of the degree of dependency on the time-varying parameters can be a useful tool to provide feasible solutions. Nevertheless, the flexibility of Theorem 1 also allows to provide a stabilizing controller by using less computational effort (for instance, decision variables with $g = 1$) by employing a more accurate discrete-time representation ($\ell = 4$) of the continuous-time LPV system.

7.2 | Statistical Comparisons of the Stabilizing Methods

The aim of this subsection is to perform a comparison of the state- and output-feedback control design conditions given in Section 3 with other methods from the literature considering LTI and LPV systems. To this end, numerical stabilization tests were implemented using the database of polytopic systems proposed by Morais et al.[4] This database is composed by systems that are unstable in open loop, robustly stabilized by some robust (parameter-independent) state-feedback gain but that are not quadratically stabilized. The tests were performed initially for systems with dimension $n_x \in \{2, 3\}$ states and $N \in \{2, \dots, 5\}$ vertices. For each combination of n_x and N, the basis contains 100 different sets of systems, which were used in the simulations.

Case 1: State-Feedback Stabilization of Polytopic LTI systems

The first part of this example addresses the SSF stabilization of LTI systems using Corollary 1 of Morais et al[4] (MB$^+$), Theorem 3 of de Oliveira et al[52] (dOBG), and the method proposed in Theorem 1. The scalar searches for the method MB$^+$ are performed in the parameter ξ considering nineteen values equally spaced in the interval $[-0.9, 0.9]$, and for the proposed method, the set given in (43), with $\xi = \{-0.2, -0.1, 0, 0.1, 0.2\}$. Regarding the pole allocation, two different cases are considered: (1) $\rho = 1$ (T1) and, (2) $\rho = \{1.05,\ 1.1\}$ (C1), using Algorithm 1 with degP=1.

The first columns of Table 1 show the result of the first part of this example. One can note that the results obtained from the relaxation of the pole allocation with radius greater than one provide better performance in all combinations of dimensions $(n_x|N)$ when compared to the other methods. A general analysis shows that, even performing less LMI tests than the compared technique (19 *versus* 10), the proposed method, in both cases analyzed, provided the best results. Additionally, as shown in Morais et al[4], the method proposed by de Oliveira et al[52] is more conservative than the other techniques.

Case 2: State-Feedback Stabilization of Polytopic LPV systems

In the second part of this example, the aim is to obtain state-feedback stabilizing gains for polytopic LPV systems. Tests were performed using Theorems 3 (robust control) and 4 (gain-scheduled control) proposed by Daafouz and Bernussou[5] (DB), and Theorem 1 adapted to deal with systems without norm-bounded uncertainties. The scalar searches of Theorem 1 were performed using (43), with $\xi = \{-0.2, -0.1, 0, 0.1, 0.2\}$. As it was done before, two situations were analyzed regarding the decay rate bounded by ρ, using Algorithm 1 with $degP = 1$. In the first one $\rho = 1$ was considered (T1) and, in the second one, $\rho = \{1.05,\ 1.1\}$ (C1).

It is important to notice that to perform the robust stability tests described in Algorithm 1, applied when $\rho > 1$, the first step is to obtain the dynamic matrix in closed-loop. With this purpose, it is necessary to explicitly determine the gain (for example, in terms of a polynomial structure), which involve the computation of the inverse of matrix $S(\alpha(k))$. Even though it is possible to eliminate the term $S(\alpha(k))^{-1}$ by, for example, applying congruence transformations (which would imply in $S(\alpha(k))$ multiplying other matrices), in this paper, a constant structure (zero degree) for matrix $S(\alpha(k))$ is imposed in order to simplify the computation of its inverse. However, although the synthesis conditions formulated in this way (degree zero in $S(\alpha(k))$ and degree one in $L(\alpha(k))$) can design a gain-scheduled controller, they are more conservative when compared with the case when both matrices that compose the gain are parameter-dependent (degree one in both $S(\alpha(k))$ and $L(\alpha(k))$). Table 1 shows the results obtained by the methods under investigation, which were applied for both cases of controller structures, whether they are robust (zero degree in $S(\alpha(k))$ and $L(\alpha(k))$) or gain-scheduled (degree zero in $S(\alpha(k))$ and degree one in $L(\alpha(k))$ for the tests with $\rho > 1$ and degree one in both matrices for the tests with $\rho = 1$). Observing the results, it is possible to notice that, just as it occurs for the LTI case, the obtained solutions from the relaxation of the decay rate ($\rho > 1$) show advantages in all combinations of dimensions $(n_x|N)$ when compared to the other methods, for both robust and gain-scheduled controllers. When compared to the method of Daafouz and Bernussou,[5] the adapted condition from Theorem 1 without the relaxation of the decay rate ($\rho = 1$) shows a slighter less conservative result in both robust and gain-scheduled cases.

Case 3: Output-Feedback Stabilization of Polytopic LPV systems

This case considers the problem of designing static output-feedback controllers for LPV systems without norm-bounded uncertainties. The scope of the tests regarding the conditions from Theorem 1 are the same as mentioned in Case 2. The compared methods are the condition adapted of Equation (49) from de Caigny et al[7] (by eliminating the last column and row) for robust and gain-scheduled output-feedback control (dCC$^+$), and the technique provided by Theorem 1 of Dong and Yang[41] for robust output-feedback controllers (DY). By analyzing the results shown in the last columns of Table 1 , the first information is that, when compared to the state-feedback stabilization control for LPV and LTI systems, the output-feedback conditions present more conservative results, which is expected since the output-feedback problem is more involved. Additionally, as it happened for Cases 1 and 2, the percentage results obtained by Theorem 1 considering a relaxation of the decay rate ($\rho > 1$) are advantageous in all combinations of dimensions ($n_x|N$) and both controller structures (robust or gain scheduled), when compared to the conditions without this relaxation ($\rho = 1$) and the methods from de Caigny et al,[7] for robust and gain-scheduled controllers and from Dong and Yang[41], for robust controllers.

TABLE 1 Stabilization results for LTI and LPV systems.

		LTI systems - SSF				LPV systems - SSF						LPV systems - SOF						
		Robust				Robust			Gain-scheduled			Robust				Gain-scheduled		
n_x	N	dOB$^+$	MB$^+$	**T1**	**C1**	DB	**T1**	**C1**	DB	**T1**	**C1**	DY	dCC$^+$	**T1**	**C1**	dCC$^+$	**T1**	**C1**
2	2	80	85	85	91	8	7	29	42	44	59	1	3	3	17	17	20	41
	3	90	93	93	95	14	16	37	60	61	73	1	2	2	31	23	23	52
	4	91	91	91	96	18	19	44	55	57	72	1	4	4	27	16	18	49
	5	89	93	93	99	16	17	48	54	55	68	0	2	4	28	22	25	48
3	2	85	86	86	92	11	11	43	57	58	71	0	0	0	20	6	7	31
	3	97	97	97	100	14	15	50	50	55	78	0	1	0	20	7	11	39
	4	95	95	95	96	23	24	59	66	68	78	0	1	0	20	9	15	55
	5	92	96	96	99	26	26	55	61	63	83	0	4	5	23	12	12	48
Total (%)		89.9	92	92	96	16.1	17.0	45.6	55.6	57.6	72.8	0.3	2.1	2.3	23.3	14.1	16.4	45.4

7.3 | Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control of polynomial LPV systems

This example is concerned with the LPV continuous-time system that represents a linearized dynamic equation of a VTOL helicopter presented by Keel et al,[56] (where more details about the physical description of the dynamic equations can be found), with the following state-space matrices:

$$A_c = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & p(t) & -0.7070 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B_c = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix} \tag{45}$$

where $p(t)$ is a parameter that can vary arbitrarily fast inside the interval $p(t) = 0.3181\alpha_1(t) + 0.4181\alpha_2(t)$, yielding a polytopic LPV continuous-time system of two vertices. For purpose of mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ discrete-time control, the following matrices are also necessary

$$C_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; [D_z|E_z] = \begin{bmatrix} 1 & 0 & | & 0.1 \\ 0 & 1 & | & 0.2 \end{bmatrix}, \; \begin{bmatrix} E_y \\ \hline E_c \end{bmatrix} = \begin{bmatrix} 0.1 & 0.05 & 0 & 0 \\ \hline 0.05 & 0.01 & 0 & 0 \end{bmatrix}', \; C_y(\alpha(k)) = 0.5I\alpha_1(k) + I\alpha_2(k), \tag{46}$$

where the uncertainties in the output matrix can represent, for example, a failure in the measuring sensor. Additionally, to obtain the discrete-time model, consider $t = kT$ and convert matrices A_c and B_c of (45) and E_c of (46) in polynomial matrices $A_\Delta(\alpha(k))$, $B_\Delta(\alpha(k))$ and $E_\Delta(\alpha(k))$ through the discretization method proposed by Braga et al,[22] considering a Taylor series expansion of degree $\ell = 3$ for a sampling period of $T = 0.01$s.

The aim is to obtain static output-feedback controllers considering robust ($L(\alpha(k)) = L$ and $S(\alpha(k)) = S$) and gain-scheduling ($L(\alpha(k))$ and $S(\alpha(k))$ with degree 1 of parameter dependence) approaches, guaranteeing upper bounds for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. The flexibility of the method associated with the scalar parameter search is evaluated by employing the third statement of Theorem 2 with the set of scalars presented in (43) or fixing the value of $\xi = 0$ (no search is performed).

In the first part of this example, the value of parameter κ presented in the cost functional (33) is fixed in its extreme values: one or zero. With these choices, function ν considers the system performance regarding only the $\mathcal{H}_\infty$ or the $\mathcal{H}_2$ norm, respectively. The results obtained making $Q_i(\alpha(k)) = \tilde{C}_y(\alpha(k))$, $i = 1, \dots, 3$ are shown in Table 2 . Note that, comparing the values of ν obtained with $\kappa = 1$, that is, applying Theorem 2 by minimizing only the $\mathcal{H}_\infty$ guaranteed costs and letting free the $\mathcal{H}_2$ criterion, one can obtain upper bounds to the $\mathcal{H}_\infty$ norm similar to the ones that can be computed using Theorem 1 from Rosa et al.[8]

TABLE 2 Values of objective function ν associated with the synthesis of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output-feedback controllers computed by Theorem 2 for Example 3.

	Robust		**Gain-scheduled**	
κ	(43)	(43) with $\xi = 0$	(43)	(43) with $\xi = 0$
0 (pure $\mathcal{H}_2$)	0.5680	1.0165	0.4188	0.9291
1 (pure $\mathcal{H}_\infty$)	1.8163	8.1364	0.9560	6.6239

In the second part, parameter κ is evaluated in the interval $[0\ \ 1]$, in order to observe the behavior of the objective function ν inside this interval. Figure 3 shows these results considering $Q_i(\alpha(k)) = \tilde{C}_y(\alpha(k))$, $i = 1, \dots, 3$.

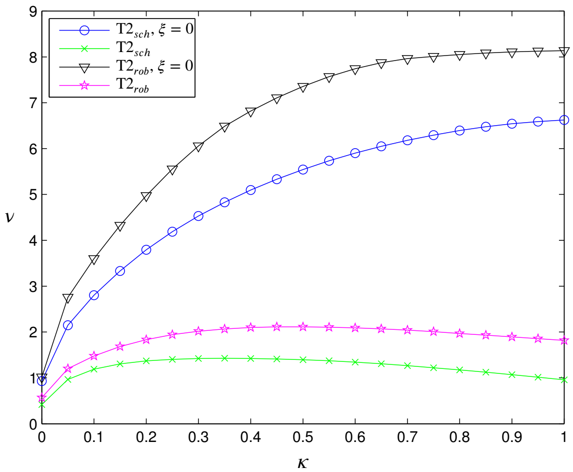


FIGURE 3 Values of objective function ν associated with the synthesis of robust and parameter-dependent mixed $\mathcal{H}_2/\mathcal{H}_\infty$ static output-feedback controllers computed by Theorem 2.

Note that, as expected, the use of gain-scheduled instead of robust controllers combined with the search in the scalar variables allows to achieve less conservative results for this example, for all values of κ. Another remark concerning these results is that Theorem 2 has better performance when prioritizing the minimization of $\mathcal{H}_2$ norm (small values of κ), for both robust and gain-scheduled controllers. It is interesting to observe that the curves associated with line searches in ξ show smaller variation for the values of ν when compared with the case $\xi = 0$.

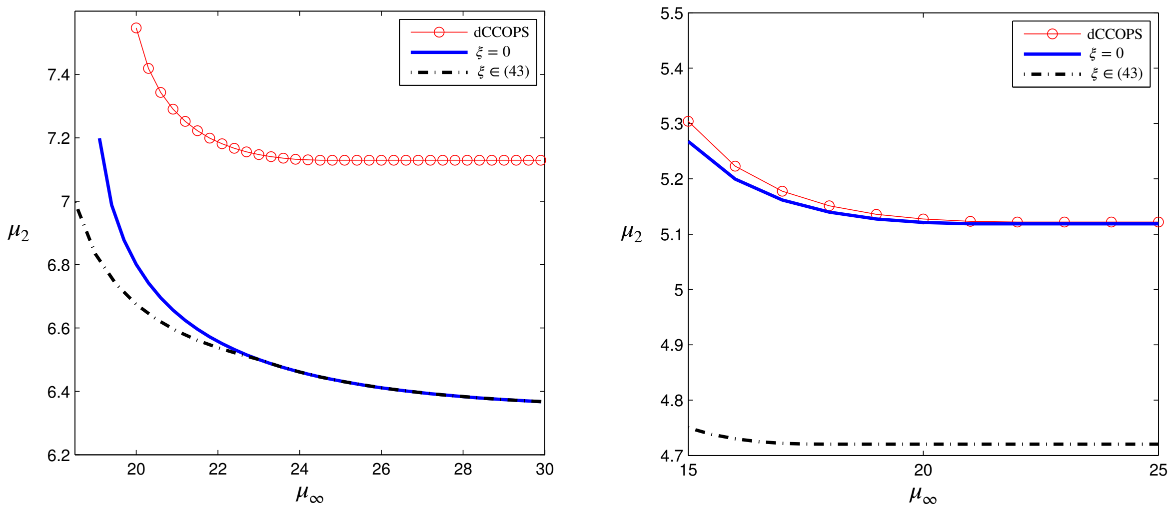
Furthermore, in order to observe the impact of the choice of matrices $Q_i(\alpha(k))$ in the synthesis conditions, $Q_1(\alpha(k))$ and $Q_3(\alpha(k))$ are made equal to $\tilde{C}_y(\alpha(k))$ and $Q_2(\alpha(k)) = [\mathrm{I}\ 0]$. Considering (43) with $\xi = 0$ and $\kappa = 0$, the values of the objective function v for robust and gain-scheduled controllers are 1.0526 and 0.9029, respectively. Thus, comparing these results to the ones presented in Table 2, it is possible to note that in the robust controller case, the performance associated to the choices $Q_i(\alpha(k)) = \tilde{C}_y(\alpha(k))$, $i = 1, \dots, 3$ is slightly better than the second combination of $Q_i(\alpha(k))$. However, for the gain-scheduled controller case, it is observed that the second choice provides the best results. Thus, it is shown that different choices of matrices Q_i can provide better or worse performances, depending on the system under investigation.

7.4 | Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control of polytopic LPV systems

Consider the polytopic time-varying system borrowed from de Caigny et al,[57] whose vertices of the matrices are given by

$$[A_1|A_2] = 0.43\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 2 & -1 & 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & -2 & -1 \end{array}\right], [E_1|E_2|B_i] = \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right], \quad \left[\begin{array}{c} C_{yi} \\ \hline C_{z_i} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 1 & 1 \end{array}\right], \quad \begin{array}{c} E_{z_i} = D_{z_i} = 0, \\ i = 1,\ 2. \end{array}$$

The aim of this example is to show a comparison between the performance of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output-feedback controllers, considering the second statement of Theorem 2 and the method described in Section 5.3 of de Caigny et al[7] given by the combination of Theorems 8 and 9 of the same work. The synthesis procedure, adopted for both techniques in this example, consists in minimizing μ_2, for a given $\mu_\infty > 0$, as shown in the second statement of Theorem 2. Two cases are considered regarding the search in the scalar variables for Theorem 2. In the first one no search on ξ is performed (only $\xi = 0$ is used) and the second considers the best result obtained by performing the search in the set given in (43). Figure 4 shows the results of the optimization variable μ_2 for an interval of given values of μ_∞, considering the synthesis of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust as well as parameter-dependent static output-feedback controllers with degree 1 of parameter dependence.



(a) Robust. (b) Parameter-dependent.

FIGURE 4 $\mathcal{H}_2$ guaranteed costs (μ_2) for given values of the $\mathcal{H}_\infty$ guaranteed costs (μ_∞) associated with the synthesis of robust and parameter-dependent static output-feedback controllers computed by Theorem 2 and de Caigny.[7]

Note that, as expected, less conservative results are obtained by designing gain-scheduled controllers. It can be noted that the proposed technique provides less conservative results when compared with the method of de Caigny et al,[7] even without performing scalar searches. Using the search for the scalars in the set (43), the lowest guaranteed bounds are obtained in both robust and gain-scheduled cases.

8 | CONCLUSION

This paper introduced new parameter-dependent LMI conditions for the synthesis of reduced-order dynamic output-feedback controllers for discrete-time systems with polynomially dependence on time-varying parameters plus norm-bounded time-varying uncertainty terms. The main motivation for handling this class of systems comes from the problem of control design for discretized uncertain continuous-time systems obtained via Taylor series expansion method. The main advantage of the method is its great versatility with respect to applications in other contexts. Making a few simple adjustments, the proposed technique can treat various other types of systems, such as polytopic LPV and LTI systems, also being capable to solve (full or reduced order) dynamic and static output or state-feedback control problems, providing gain-scheduled or robust controllers. Another novelty of this paper is the possibility of considering the output matrix $C_y(\alpha(k))$ polytopic or polynomial with generic degree, while other techniques in the literature require this matrix to be independent of parameters, to have a particular structure or the existence of particular similarity transformations. This ability can be useful, for instance, in the context of networked control systems where the output matrix is frequently subject to uncertainties. Furthermore, the new heuristic procedure for control synthesis is an additional contribution of the paper. The benefits of the proposed technique are demonstrated in terms of numerical experiments, whose results proved to be advantageous for the design of controllers, both in statistical evaluations (by employing a new heuristic procedure) and in terms of lower $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed costs when compared with other methods for some examples from the literature.

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