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COORDINATION OF PASSIVE SYSTEMS UNDER QUANTIZED MEASUREMENTS*

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Abstract. In this paper we investigate a passivity approach to collective coordination and synchronization problems in the presence of quantized measurements and show that coordination tasks can be achieved in a practical sense for a large class of passive systems.

Key words. coordination, quantization, passivity, nonsmooth analysis

AMS subject classifications. 34D05, 34D06, 34H05, 49J52, 49J53, 93C10, 93C15, 93D05, 93D20

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1. Introduction.

In the very active area of consensus, synchronization, and coordinated control there has been an increasing interest in the use of quantized measurements and control ([22, 26, 18, 5, 24, 6] and references therein). As a matter of fact, since these problems deal with systems or agents which are distributed over a network, it is very likely that the agents must exchange information over a digital communication channel, and quantization is one of the basic limitations induced by finite bandwidth channels.

The use of quantized measurements induces a partition of the space of measurements: whenever the measurement function crosses the boundary between two adjacent sets of the partition, a new value is broadcast through the channel. As a consequence, when the networked system under consideration evolves in continuous time, as is often the case with, e.g., problems of coordinated motion, the use of quantized measurements results in a completely asynchronous exchange of information among the agents of the network. Despite the asynchronous information exchange and the use of a discrete set of information values, meaningful examples of synchronization or coordination can be obtained [12, 14, 7].

In view of the several contributions to quantized coordination problems available for discrete-time systems [22, 26, 18, 5, 24, 6], one might think to derive the sampled-data model of the system and then apply the discrete-time results. However, a sampled-data approach to the design of coordinated motion algorithms presents a few drawbacks: it might require synchronous sampling at all the nodes of the network and consequent accurate synchronization of all the node clocks; it might also require fast sampling rates, which may not be feasible in a networked system with a large number of nodes and connections. Finally, the sampled-data model may not fully preserve some of the features of the original model. For the reasons above,

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in this paper we focus on coordination problems under quantized measurements for continuous-time systems.

A few works on this class of problems have recently appeared. The work [12] deals with consensus algorithms using binary control algorithms. In [14] the attention is turned to quantized measurements, and the consensus problem under quantized *relative* measurements is tackled. A similar problem is studied in [7] in the context of discontinuous control systems. The paper also introduces hysteretic quantizers to prevent the occurrence of chattering due to the presence of sliding modes. More recently, the work [25] has studied the quantized consensus algorithm for double integrators. A remarkable advancement in the study of consensus algorithms over *time-varying* communication graphs and using quantized measurements has been provided by [17].

Despite the unquestionable interest of the results in papers such as [14, 7, 25, 12, 17], they present an important limitation: they focus on agents with simple dynamics such as single [14, 7, 12, 17] or double integrators [25]. The goal of this paper is to investigate a more general approach to coordinated motion and synchronization which takes into account simultaneously complex dynamics for the agents of the network and quantized measurements.

In this paper we focus on the approach to coordinated motion proposed in [1]. In that paper, the author has shown how a number of coordination tasks could be achieved for a class of passive nonlinear systems and has been using this approach for related problems in subsequent work [4, 3]. Others have been exploiting passivity ([8, 20, 37, 27] to name a few) in connection with coordination problems. Our interest for the approach in [1] stems from the fact that it lends itself to dealing with the presence of quantized measurements very naturally. As a matter of fact, quantized measurements can be taken into account by introducing in the feedback law static discontinuous maps (the so-called quantizers). Although the discontinuous nature of the quantizers prevents the conditions for coordination in [1] from being fulfilled, one can argue that an approximate or "practical" [7] coordination task is achievable under suitably modified conditions. This is the idea which is pursued in this paper.

A second aim of this paper is to study practical state synchronization under quantized output feedback. Passivity [8, 34, 33], or the weaker notion of semi-passivity [31, 30, 35], has also played an important role in synchronization problems. Here we mainly focus on the models considered in [8, 33].

The main contribution of this paper is to show that some of the results of [1] and [33] hold in a practical sense in the presence of quantized measurements. Because the latter introduces discontinuities in the system, a rigorous analysis is carried out relying on notions and tools from nonsmooth control theory and differential inclusions. As far as the coordination problem is concerned, although the passivity approach of [1] allows us to consider a large variety of coordination control problems, in this paper we mainly focus on agreement problems in which agents aim at converging to the same position.

A few other papers have appeared which deal with coordination problems for passive systems in the presence of quantization. The work [20] deals with a position coordination problem for Lagrangian systems when delays and limited data rates are affecting the system. The paper [16] deals with master-slave synchronization of passifiable Lurie systems when the master and the slave communicate over a limited data rate channel. The main difference of our paper compared with [20, 16] is that in the former each system in the network transmits quantized information in a completely asynchronous fashion and no common sampling time is required. From a mathematical point of view, our approach yields a discontinuous system as opposed to a sampled-data one.

The organization of the paper is as follows. The passivity approach to coordination problems is recalled in section 2. In section 3 the coordination control problem in the presence of uniform quantizers is formulated and the main results are presented along with some examples. The synchronization problem for passive systems under quantized output feedback is studied in section 4. In section 5 a few guidelines for future research are discussed.

2. Preliminaries. Consider N systems connected over an undirected static graph $G = (V, E)$, where V is a set of N nodes and $E \subseteq V \times V$ is a set of M edges connecting the nodes. The standing assumption throughout the paper is that the graph G is *connected*. Each system i, with $i = 1, 2, \ldots, N$, is associated to the node i of the graph and the edges connect the nodes or systems which communicate.

Each system i is described by

$$\Sigma_i: \begin{array}{l} \dot{\xi}_i = f_i(\xi_i) + g_i(\xi_i)u_i, \\ w_i = h_i(\xi_i) + v_i, \end{array} \tag{2.1}$$

where the state $\xi_i \in \mathbb{R}^{n_i}$, the input $u_i \in \mathbb{R}^p$, the output $w_i \in \mathbb{R}^p$, the exogenous signal $v_i \in \mathbb{R}^p$, and the maps f_i, g_i, h_i are assumed to be locally Lipschitz satisfying $f_i(\mathbf{0}) = \mathbf{0}$, $g_i(\mathbf{0})$ full column-rank, $h_i(\mathbf{0}) = \mathbf{0}$. For the system Σ_i, we assume the following.

Assumption 2.1. There exists a continuously differentiable storage function $S_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ which is positive definite and radially unbounded such that

$$\nabla S_i(\xi_i)(f_i(\xi_i) + g_i(\xi_i)u_i) \leq -W_i(\xi_i) + h_i(\xi_i)^T u_i, \tag{2.2}$$

where W_i is a continous positive function which is zero at the origin.

Such a system Σ_i is called a strictly passive system (with $v_i = 0$). If W_i is a nonnegative function, then Σ_i is called a passive system.

Label one end of each edge in E by a positive sign and the other one by a negative sign. Now, consider the kth edge in E, with $k \in \{1, 2, \ldots, M\}$, and let i, j be the two nodes connected by the edge. For the coordination problem, which is detailed in subsection 2.1, the relative measurements of the integral form $\int_0^t w_i(\tau)\mathrm{d}\tau$ and $\int_0^t w_j(\tau)\mathrm{d}\tau$ are used. On the other hand, for the synchronization problem, which is briefly reviewed in subsection 2.2, we need the relative measurements of the signals w_i and w_j. Thus, depending upon specific problems, let z_k describe the difference between the signals w_i and w_j (or the difference between the signals $x_i(t) := \int_0^t w_i(\tau)\mathrm{d}\tau + x_i(0)$ and $x_j(t) := \int_0^t w_j(\tau)\mathrm{d}\tau + x_j(0)$ with constant vectors $x_i(0), x_j(0) \in \mathbb{R}^p$) and be defined as follows:

$$z_k = \begin{cases} w_i - w_j \ \ (\text{or } \ x_i - x_j\ ) & \text{if } i \text{ is the positive end of the edge } k, \\ w_j - w_i \ \ (\text{or } \ x_j - x_i\ ) & \text{if } i \text{ is the negative end of the edge } k\,. \end{cases}$$

Recall also that the incidence matrix D associated with the graph G is the $N \times M$ matrix such that

$$d_{ik} = \begin{cases} +1 & \text{if node } i \text{ is the positive end of edge } k, \\ -1 & \text{if node } i \text{ is the negative end of edge } k, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of D, the variables z can be concisely represented as

$$z = (D^T \otimes I_p)w \quad \left(\text{or } \ z = (D^T \otimes I_p)x\right), \tag{2.3}$$

where $w = [w_1^T \dots w_N^T]^T$ and $x = [x_1^T \dots x_N^T]^T$, respectively, and the symbol $\otimes$ denotes the Kronecker product of matrices.

In this paper we are interested in control laws which use quantized measurements. For each $k = 1, 2, \dots, M$, instead of z_k, the vector

$$\mathsf{q}(z_k) := (\mathsf{q}(z_{k1}) \dots \mathsf{q}(z_{kp}))^T \tag{2.4}$$

is available, where q is the quantizer map which is defined as follows. Given a positive real number Δ, we let $\mathsf{q} : \mathbb{R} \to \mathbb{Z}\Delta$ be the function

$$\mathsf{q}(r) = \Delta \left\lfloor \frac{r}{\Delta} + \frac{1}{2} \right\rfloor \tag{2.5}$$

with $\frac{1}{\Delta}$ the precision of the quantizer. As $\Delta \to 0$, $\mathsf{q}(r) \to r$. Observe that each entry of z_k is quantized independently of the others and the quantized information is then used in the control law.

Remark 2.2. The results of the paper continue to hold if each quantizer has its own resolution (that is, the information z_{kj} is quantized by a quantizer with resolution Δ_{kj}). However, to reduce the notational burden, we only deal with the case in which the quantizers all have the same resolution Δ.

In the following subsections, we review the results on the passivity approach to the coordination problems of [1] and to the synchronization problems of [33] without the quantized measurements.

2.1. Passivity approach to the coordination problem. In the coordination problems of [1], the signal w_i of each system Σ_i corresponds to the velocity of the system, and thus, x_i, $i = 1, \dots, N$, represents the positions which must be coordinated (recall that $x_i(t) := \int_0^t w_i(\tau)\mathrm{d}\tau + x_i(0)$). The coordination problem under consideration requires all the systems of the formation to move with a prescribed velocity v, i.e., $v_1 = v_2 = \cdots = v_N = v$. Define

$$y_i = \dot{x}_i - v, \tag{2.6}$$

the velocity tracking error. It can be checked from (2.1) and the definition of $\dot{x}_i$ that $y_i = h(\xi_i)$. The standing assumption is that, possibly after a preliminary feedback which uses information available locally, each system Σ_i is strictly passive, i.e., (2.2) holds with W_i positive definite. In other words, it is strictly passive from the control input u_i to the velocity error y_i.

For conciseness, (2.1), (2.6) are rewritten as

$$\begin{aligned} \dot{x} &= \underbrace{\begin{pmatrix} h_1(\xi_1) \\ \vdots \\ h_N(\xi_N) \end{pmatrix}}_{h(\xi)} + \underbrace{\begin{pmatrix} v \\ \vdots \\ v \end{pmatrix}}_{\mathbf{1}_N \otimes v} \\ \dot{\xi} &= \underbrace{\begin{pmatrix} f_1(\xi_1) \\ \vdots \\ f_N(\xi_N) \end{pmatrix}}_{f(\xi)} + \underbrace{\begin{pmatrix} g_1(\xi_1) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & g_N(\xi_N) \end{pmatrix}}_{g(\xi)} u, \end{aligned} \tag{2.7}$$

where $x = (x_1^T \dots x_N^T)^T$, $\xi = [\xi_1^T \dots \xi_N^T]^T$, $u = [u_1^T \dots u_N^T]^T$, $\mathbf{1}_N$ is the N-dimensional vector whose entries are all equal to 1 and $\mathbf{0}$ denotes a vector of appropriate dimension of all zeros.

The formation control problem consists of designing each control law u_i, with $i = 1, 2, \dots, N$, in such a way that it uses only the information available to the agent i and guarantees the following two specifications:

(i) $\lim_{t\to\infty} |\dot{x}_i(t) - v(t)| = 0$ for each $i = 1, 2, \dots, N$ with $v(t)$ a bounded and piecewise continuous reference velocity for the formation;

(ii) $z_k(t) \to \mathcal{A}_k$ as $t \to \infty$ for each $k = 1, 2, \dots, M$, where $\mathcal{A}_k \subset \mathbb{R}^p$ are the prescribed sets of convergence[1] and $z = (D^T \otimes I_p)x$ as defined in (2.3).

In [1], where measurements without quantization are considered, the case $\mathcal{A}_k = \{\mathbf{0}\}$ is referred to as the *agreement problem*.

Let $P_k : \mathbb{R}^p \to \mathbb{R}$, for $k = 1, 2, \dots, N$, be nonnegative continuously differentiable (the latter assumption will be removed in the next section) and radially unbounded functions whose minimum is achieved at the points in $\mathcal{A}_k$. To be more precise, the functions P_k are assumed to satisfy

$$P_k(z_k) = 0 \text{ and } \nabla P_k(z_k) = \mathbf{0} \quad \text{if and only if} \quad z_k \in \mathcal{A}_k. \tag{2.8}$$

Define

$$\nabla P_k(z_k) = \psi_k(z_k). \tag{2.9}$$

The feedback laws proposed in [1] to solve the problem formulated above are

$$u_i = -\sum_{k=1}^{M} d_{ik}\psi_k(z_k)\ ,\ i = 1, 2, \dots, N. \tag{2.10}$$

Observe that, as required, each control law u_i uses only information which is available to the agent i. Indeed, $d_{ik} \neq 0$ if and only if the edge k connects i to one of its neighbors. In compact form, (2.10) can be rewritten as

$$u = -(D \otimes I_p)\psi(z), \tag{2.11}$$

where $\psi(z) = [\psi_1(z_1)^T \dots \psi_M(z_M)^T]^T$ and z is as in (2.3). Before ending the section, we recall that the system below with input $\dot{x}$ and output $-u$, namely (see Figure 2 in [1] for a pictorial representation of the system),

$$\begin{aligned} \dot{z} &= (D^T \otimes I_p)\dot{x}, \\ -u &= (D \otimes I_p)\psi(z), \end{aligned} \tag{2.12}$$

is passive from $\dot{x}$ to $-u$ with storage function $\sum_{k=1}^{M} P_k(z_k)$. We remark that the function $P_k(z_k)$ is chosen in such a way that the region where the variable z_k must converge for the system to achieve the prescribed coordination task coincides with the set of the global minima of $P_k(z_k)$. Hence, the coordination task guides the design of $P_k(z_k)$, which in turn allows us to determine the control functions (2.10) via (2.9). The functions $P_k(z_k)$ in the case of agreement problems via quantized control laws will be designed in section 3.

2.2. Passivity approach to the synchronization problem. In the synchronization problem of [33, Theorem 4], each system Σ_i in (2.1) (with $v_i = 0$) is assumed

[1] We refer the interested reader to [1] for examples of sets $\mathcal{A}_k$ related to some coordination problems. The sets $\mathcal{A}_k$ which are of interest in this paper will be introduced in (3.1).

to be linear, identical, and passive. For such setting, each (passive) system Σ_i is of the form

$$\begin{aligned} \dot{\xi}_i &= A\xi_i + Bu_i, \\ w_i &= C\xi_i, \qquad\qquad i = 1, 2, \ldots, N, \end{aligned} \tag{2.13}$$

where $\xi_i \in \mathbb{R}^n$, $u_i, w_i \in \mathbb{R}^p$, and the passivity of Σ_i implies that the following assumption holds.

Assumption 2.3. There exists an $(n \times n)$ matrix $P = P^T > 0$ such that $A^T P + PA \leq 0,\ B^T P = C$.

The synchronization problems can then be stated as designing each control law u_i, $i = 1, 2, \ldots, N$, using only the information available to the agent i such that, for every i, $\xi_i - \xi_0 \to \mathcal{A}$, where ξ_0 is the trajectory of the autonomous system $\dot{\xi}_0 = A\xi_0$ which is initialized by the average of the initial states, i.e., $\xi_0(0) = \frac{1}{N}\sum_i \xi_i(0)$, and $\mathcal{A} \subset \mathbb{R}^p$ is the prescribed set of convergence. In the case without the quantized measurements, which is treated in [33], $\mathcal{A} = \{\mathbf{0}\}$. The coordination problem that is reviewed in subsection 2.1 is related to the case when $\dot{\xi}_0 = \mathbf{0}$ [1]. For another viewpoint, we can consider that (2.13) corresponds to the case in the subsection 2.1, where the mapping $u \mapsto y$ is an identity operator, $v = \mathbf{0}$, and one takes into account dynamics on the subsystem x which are more complex than those of a single integrator.

In addition to output synchronization, it is well known that the *states* of interconnected passive systems synchronize under observability assumption [8]. The largest invariant set of the interconnected systems, when the measurements are not quantized and (C, A) is observable, is the set $\{\xi \in \mathbb{R}^{nN} : \xi_1 = \cdots = \xi_N\}$. In the case of quantized measurements, the invariant set is larger. Our main result in section 4 provides an estimate of the invariant set of the interconnected systems with quantized measurements. To this purpose, we rely on a result of exponential synchronization under static output feedback control laws and time-varying graphs which has been investigated in [33]. In the following statement, we recall Theorem 4 of [33] specialized to the case of time-invariant undirected graphs.

THEOREM 2.4. *Let Assumption* 2.3 *hold and suppose that the pair* (C, A) *is observable. Let the communication graph be undirected and connected, and denote* $z = (D^T \otimes I_p)w$ *as in* (2.3) *with* $w = [w_1^T \ldots w_N^T]^T$. *Then the solutions of*

$$\dot{\xi}_i = A\xi_i - B\sum_{k=1}^{M} d_{ik} z_k,\ i = 1, 2, \ldots, N, \tag{2.14}$$

satisfy $\lim_{t\to+\infty}\|\xi_i(t) - \frac{\mathbf{1}_N^T \otimes I_n}{N}\xi(t)\| = 0$, *where* $\xi = [\xi_1^T \xi_2^T \cdots \xi_N^T]^T$ *and the convergence is exponential. More precisely, the solutions converge exponentially to the solution of* $\dot{\xi}_0 = A\xi_0$ *initialized to the average of the initial conditions of the systems* (2.14), *i.e.,* $\xi_0(0) = \mathbf{1}_N^T \otimes I_n \xi(0)/N$.

Let $\tilde{\xi} = \xi - \frac{\mathbf{1}_N \mathbf{1}_N^T \otimes I_n}{N}\xi = (\Pi \otimes I_n)\xi$, with $\Pi = I_N - \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}$, be the disagreement vector. From (2.14), $\tilde{\xi}(t)$ obeys the equation

$$\dot{\tilde{\xi}} = \underbrace{[I_N \otimes A - (I_N \otimes B)(DD^T \otimes I_p)(I_N \otimes C)]}_{\tilde{A}}\tilde{\xi} \tag{2.15}$$

and the convergence result can be restated as $\lim_{t\to+\infty}||\tilde{\xi}(t)|| = 0$. The proof of the result rests on showing that the Lyapunov function $V(\tilde{\xi}) = \tilde{\xi}^T(I_N \otimes P)\tilde{\xi}$ along the solutions of (2.15) satisfies the inequality $\dot{V}(\tilde{\xi}) \leq -\lambda_2 ||(\Pi \otimes I_p)\tilde{\xi}||^2$, where λ_2 is the algebraic connectivity of the graph, i.e., the smallest nonzero eigenvalue of the Laplacian $L = DD^T$. Then the thesis descends from the observability assumption and Theorem 1.5.2 in [32].

3. Quantized coordination control.

3.1. A practical agreement problem. Despite the generality allowed by the passivity approach of [1], in this paper we focus on an agreement problem. By an agreement problem we mean a special case of coordination in which all the variables x_i connected by a path converge to each other. In the problem formulation in section 2, this amounts to $\mathcal{A}_k = \{\mathbf{0}\}$ for all $k = 1, 2, \ldots, M$. When using *statically*[2] quantized measurements (2.4), however, it is a well established fact [22, 14, 7] that a coordination algorithm leads to a practical agreement result, meaning that each variable z_k converges to a compact set containing the origin, rather than to the origin itself. Motivated by this observation, we set in this paper a weaker convergence goal, namely, for each $k = 1, 2, \ldots, M$, we ask the target set $\mathcal{A}_k$ to be of the form

$$\mathcal{A}_k = \bigtimes_{j=1}^{p} [-a, a], \tag{3.1}$$

where a is a positive constant and the symbol $\times$ denotes the Cartesian product. Then the design procedure of section 2 prescribes to choosing a nonnegative potential function $P_k(z_k)$ which is radially unbounded on its domain of definition and such that (2.8) holds. If such a function exists, then the control law is chosen via (2.9). To take into account the presence of quantized measurements, the nonlinearities ψ_k on the right-hand side of (2.9) should take the form

$$\psi_k(z_k) = \chi_k(\mathtt{q}(z_k)) \tag{3.2}$$

with χ_k to be defined later.

The presence of quantized measurements, i.e., of $\mathtt{q}(z_k)$, makes the right-hand side of (2.9) discontinuous and asks for a redefinition of the requirements (2.8). In this paper, we look for *locally Lipschitz* radially unbounded nonnegative functions P_k which satisfy

$$P_k(z_k) = 0 \text{ and } \mathbf{0} \in \partial P_k(z_k) \quad \text{if and only if} \quad z_k \in \mathcal{A}_k, \tag{3.3}$$

where $\partial P_k(z_k)$ is the Clarke generalized gradient (for this and other notions of nonsmooth analysis used throughout the paper see [2, 12, 13]), which is needed since $P_k(z_k)$ is now not continuously differentiable. Similarly to (2.8), we are asking $\mathcal{A}_k$ to be the set of all local and global minima for $P_k(z_k)$.

A candidate function $P_k(z_k)$ with the properties (3.3) and such that a function χ_k exists for which (2.9), (3.2) hold is the function

$$P_k(z_k) = \sum_{j=1}^{p} \int_0^{z_{kj}} \mathtt{q}(s)ds,$$

where z_{kj} is the jth component of the vector $z_k \in \mathbb{R}^p$. (See Figure 3.1 for a picture of $P_k(z_k)$.)

[2]The use of dynamic quantizers can lead to asymptotic results. See [5, 24] for a few results for discrete-time systems. Dealing with continuous-time systems and dynamic quantizers poses a few extra challenges, which are not addressed in this paper.

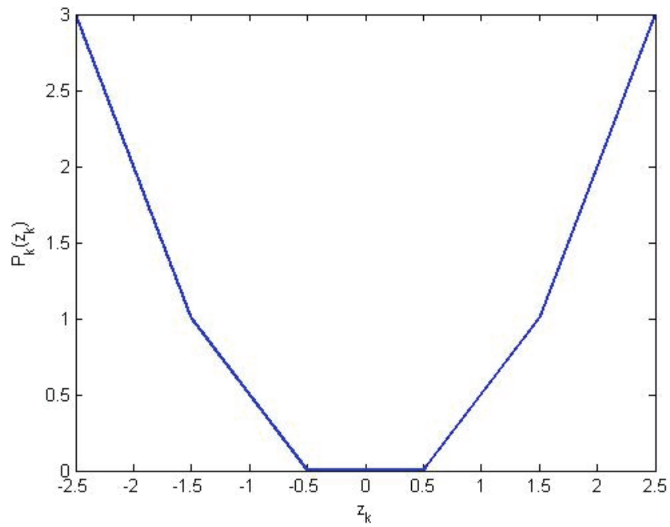

FIG. 3.1. *The graph of $P_k(z_k)$ with $z_k \in \mathbb{R}$ and $\Delta = 1$.*

Such a function is defined on all $\mathbb{R}^p$, is radially unbounded, and is locally Lipschitz. By Rademacher's theorem [11, Chapter 3] it is differentiable almost everywhere. In all the points of $\mathbb{R}^p$ where it is differentiable $\nabla P_k(z_k) = \mathsf{q}(z_k)$, i.e., (2.9), (3.2) holds with $\chi_k = \mathrm{Id}$ ($\mathrm{Id} : \mathbb{R}^p \to \mathbb{R}^p$ is the identity function). Bearing in mind the definitions (2.5) and (3.1), (3.3) holds if and only if $a = \frac{\Delta}{2}$. In what follows we examine the evolution of system (2.7) under the control law:

$$u_i = -\sum_{k=1}^{M} d_{ik}\mathsf{q}(z_k)\ ,\ i = 1, 2, \ldots, N. \tag{3.4}$$

3.2. Closed-loop system. Similarly to (2.11), we write the quantized control law in compact form as

$$u = -(D \otimes I_p)\mathsf{q}(z), \tag{3.5}$$

where $\mathsf{q}(z) = (\mathsf{q}(z_1)^T \ldots \mathsf{q}(z_M)^T)^T$. The closed-loop system then takes the following expression:

$$\begin{aligned} \dot{x} &= h(\xi) + \mathbf{1}_N \otimes v, \\ \dot{\xi} &= f(\xi) + g(\xi)(-(D \otimes I_p)\mathsf{q}(z)), \end{aligned} \tag{3.6}$$

where $z = (D^T \otimes I_p)x$ and the maps f, g, h are as in (2.7).

3.2.1. Control scenario and implementation. Before proceeding to the analysis of the system, it is important to motivate in more detail the control scenario we consider and how the overall control scheme is implemented.

For each pair of neighboring agents, one of the two is equipped with a sensor which continuously takes the relative measurement with respect to its neighbor, e.g., a sonar or a radar. Not all the agents are equipped with these sensors since they might have very dedicated tasks in the formation and space must be saved for other hardware needed to accomplish these tasks. On the other hand, since these agents need information to maintain their positions in the formation, they receive such information in quantized form from their neighbors via a digital communication channel. The implementation of the control law in (3.6) can be given by the *quantization-based distributed control protocol* as follows.

Initialization. At time $t_0 = 0$, all sensors measure $z_k(t_0)$, $k = 1, 2, \dots, p$. The processing units collocated with the sensors compute $\mathsf{q}(z_k(t_0))$ and the resulting value is broadcast to its neighbors. Each agent i computes the local control law u_i as in (3.4) and the control value is held until new information is available. Note that the closed-loop system evolves according to

$$\begin{aligned}\dot{x} &= h(\xi) + \mathbf{1}_N \otimes v,\\ \dot{\xi} &= f(\xi) + g(\xi)(-(D \otimes I_p)\mathsf{q}(z(t_0)))\end{aligned}$$

for all $t > t_0$ until new information is available.

Quantization-based transmission and control update. Let $\ell = 1$ and let t_ℓ be the smallest time at which $t_\ell > t_{\ell-1}$ and the processing unit of a sensor in the kth edge detects that $\mathsf{q}(z_k(t_\ell)) \neq \mathsf{q}(z_k(t_{\ell-1}))$. In this case, the quantized information $\mathsf{q}(z_k(t_\ell))$ is transmitted to its neighbor and the local control law of the ith and jth agents, where (i, j) is the pair of nodes linked by the kth edge, is updated by

$$\begin{aligned}u_i(t_\ell) &= u_i(t_{\ell-1}) - d_{ik}\big(\mathsf{q}(z_k(t_\ell)) - \mathsf{q}(z_k(t_{\ell-1}))\big),\\ u_j(t_\ell) &= u_j(t_{\ell-1}) - d_{jk}\big(\mathsf{q}(z_k(t_\ell)) - \mathsf{q}(z_k(t_{\ell-1}))\big).\end{aligned}$$

There is no other information exchange and hence the rest of the agents maintain their local control values. The local control law is now fixed until new information is transmitted again. For all $t > t_\ell$ and until this new transmission occurs, the evolution of the closed-loop system is given by

$$\begin{aligned}\dot{x} &= h(\xi) + \mathbf{1}_N \otimes v,\\ \dot{\xi} &= f(\xi) + g(\xi)(-(D \otimes I_p)\mathsf{q}(z(t_\ell))).\end{aligned}$$

The quantization-based event-triggered control update process is iterated with the index value ℓ incremented by one.

A few remarks are in order:

(i) The construction outlined above results in a sequence of unevenly spaced sampling times t_ℓ, $\ell \in \mathbb{N}$, at which sensors located at the systems broadcast quantized information to neighboring systems. This information is used by local controllers to update the control value. The control laws turn out to be piecewise constant functions of time whose value is updated whenever new information is received.

(ii) Notice that even the agent which measures z_k implements a control law in which $\mathsf{q}(z_k)$ is used instead of z_k itself. This is mainly motivated by our need to preserve a "symmetric" structure in the closed-loop system. In fact, given agents i, j and their relative distance z_k, in the case of unquantized information, agent i would use z_k in the control law, and agent j, $-z_k$. Similarly, in the case of quantized measurements, it is very helpful in the analysis to employ $\mathsf{q}(z_k)$ in the control law for agent i and $-\mathsf{q}(z_k)$ in the one for agent j. Moreover, there might be cases in which the agent is equipped with a coarse sensor that will deliver the agent the quantized measurement $\mathsf{q}(z_k)$ rather than z_k. The proposed control law also covers this scenario.

(iii) In the quantization-based transmission and control protocol described above, the solution of the closed-loop system is not prevented from evolving along a discontinuity surface. In practice, due to delays in the transmission and in the implementation of the control law, this could result in chattering, which is of course undesirable in the present context, since it would require fast information transmission. Nevertheless, in [7] a new class of hybrid quantizers have been considered which prevent the occurrence of chattering. This class could also be used for the problem at hand in this paper, but this is not pursued further for the sake of brevity.

The implementation described above requires that each agent identifies the neighbors toward which information is transmitted. Other phenomena which can be introduced by the communication channel such as delays and packet loss and which can degrade the behavior of the system are neglected in this paper.

3.2.2. A notion of solution. The system (3.6) has a discontinuous right-hand side due to the presence of the quantization functions, and its analysis requires a suitable notion of solution. In this paper we adopt Krasowskii solutions. In fact, it was shown in [7] that Carathéodory solutions may not exist for agreement problems. Moreover, Krasowskii solutions include Carathéodory solutions and the results we derive for the former also hold for the latter in case they exist.

Denoted by $\dot{X}(t) = F(t, X)$ the system (3.6), a function $X(\cdot)$ defined on an interval $I \subset \mathbb{R}$ is a Krasowskii solution to the system on I if it is absolutely continuous and satisfies the differential inclusion [19]

$$\dot{X}(t) \in \mathcal{K}(F(t,X)) := \bigcap_{\delta>0} \overline{\text{co}}\,(F(t, B(X,\delta))) \tag{3.7}$$

for almost every $t \in I$. The operator $\overline{\text{co}}(S)$ denotes the convex closure of S, i.e., the smallest closed set containing the convex hull of S. Since the right-hand side of (3.6) is locally bounded, local existence of Krasowskii solutions is guaranteed [19].

The differential inclusion corresponding to the system (3.6) can be written explicitly. More precisely, for every $k \in \{1, 2, \dots, M\}$ and $i \in \{1, 2, \dots, p\}$, we observe that $\mathcal{K}\mathsf{q}(r)$ is given by

$$\mathcal{K}\mathsf{q}(r) = \begin{cases} m\Delta, & r \in \left((m-\frac{1}{2})\Delta, (m+\frac{1}{2})\Delta\right), m \in \mathbb{Z}, \\ [m\Delta, (m+1)\Delta]\,, & r = (m+\frac{1}{2})\Delta, m \in \mathbb{Z}. \end{cases}$$

Using $\mathcal{K}\mathsf{q}$, the differential inclusion (3.7) for (3.6) can be written as

$$\begin{aligned} \dot{x} &= h(\xi) + \mathbf{1}_N \otimes v, \\ \dot{\xi} &\in f(\xi) + g(\xi)(-(D \otimes I_p)\mathcal{K}\mathsf{q}(z)), \end{aligned} \tag{3.8}$$

where $\mathcal{K}\mathsf{q}(z) := \times_{k=1}^{M} \mathcal{K}\mathsf{q}(z_k)$, $\mathcal{K}\mathsf{q}(z_k) := \times_{j=1}^{p} \mathcal{K}\mathsf{q}(z_{kj})$. Note that we have used the calculus rule for the set-valued map $\mathcal{K}$, i.e., $\mathcal{K}\big[g(\xi)(-(D \otimes I_p)\mathsf{q}(z))\big] = g(\xi)(-(D \otimes I_p)\mathcal{K}\mathsf{q}(z))$ (see also [13], [29, Theorem 1]). The Krasowskii solutions to (3.6) are also Filippov solutions as it follows from [19, Lemma 2.8] for a piecewise continuous vector field F. Since every Carathéodory solution to (3.6) is also a Krasowskii solution to (3.6), the stability properties of the Krasowskii solutions are also inherited by the classical Carathéodory solutions [19] in case the latter exist.

3.2.3. Analysis. Recalling that $(D^T \otimes I_p)(\mathbf{1}_N \otimes v) = \mathbf{0}$ and bearing in mind (2.12), the system (3.6) in the coordinates (z, ξ) writes as

$$\begin{aligned} \dot{z} &= (D^T \otimes I_p)h(\xi), \\ \dot{\xi} &= f(\xi) + g(\xi)(-(D \otimes I_p)\mathsf{q}(z)). \end{aligned} \tag{3.9}$$

Even the system above is discontinuous and again its solutions must be intended in the Krasowskii sense. It is straightforward to verify that, given any Krasowskii solution (x, ξ) to (3.6), the function $(z, \xi) = ((D^T \otimes I_p)x, \xi)$ is a Krasowskii solution to (3.9). The differential inclusion corresponding to (3.9) is easily understood from (3.8). In what follows we investigate the asymptotic properties of the Krasowskii solutions to (3.9) and infer stability properties of (3.6).

The first fact we notice is the following.

LEMMA 3.1. *Let Assumption* 2.1 *hold and let the communication graph G be undirected and connected. Then any Krasowskii solution to* (3.9) *converges to the set of Krasowskii equilibria:*

$$\{(z, \xi) \,:\, \xi = \mathbf{0},\ \mathbf{0} \in (D \otimes I_p)\mathcal{K}\mathsf{q}(z)\}. \tag{3.10}$$

Proof. To analyze the system (3.9) we consider the Lyapunov function

$$V(z, \xi) = \underbrace{\sum_{i=1}^{N} S_i(\xi_i)}_{S(\xi)} + \underbrace{\sum_{k=1}^{M} P_k(z_k)}_{P(z)} = \sum_{i=1}^{N} S_i(\xi_i) + \sum_{k=1}^{M}\sum_{j=1}^{p} \int_0^{z_{kj}} \mathsf{q}(s)ds. \tag{3.11}$$

The function is a locally Lipschitz and regular function. In fact, each term $\int_0^{z_{kj}} \mathsf{q}(s)ds$ is convex and as such it is regular [10, Proposition 2.3.6], [13]. Then the sums $P_k(z_k)$ and $\sum_{k=1}^{M} P_k(z_k)$ are also regular. The function $V(z, \xi)$ is nonnegative and vanishes on the set of points such that $\xi = \mathbf{0}$ and $z_{kj} \in [-a, a]$ for all $k \in \{1, 2, \ldots, M\}$ and all $j \in \{1, 2, \ldots, p\}$.

In order to apply LaSalle's invariance principle for differential inclusions [2, 12, 13], we analyze the set-valued derivative V with respect to (3.9) as follows.

Define $\dot{\overline{V}}(z, \xi) = \{a \in \mathbb{R} \,:\, \exists w \in \mathcal{K}\tilde{F}(z, \xi) \text{ s.t. } a = \langle p, w\rangle \text{ for all } p \in \partial V(z, \xi)\}$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product and $\tilde{F}(z, \xi)$ the right-hand side of (3.9). We first observe that by the definition of $V(z, \xi)$, $\partial V(z, \xi)$ and $\partial P(z)$, $p \in \partial V(z, \xi)$ implies the existence of $p_z \in \partial P(z)$ such that $p = \left(\begin{smallmatrix} p_z \\ \nabla S(\xi) \end{smallmatrix}\right)$. Moreover, if $w \in \mathcal{K}\tilde{F}(z, \xi)$, then there exists $w_z \in \mathcal{K}\mathsf{q}(z)$ [19], [29] such that

$$w = \begin{pmatrix} (D^T \otimes I_p)h(\xi) \\ f(\xi) \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ g(\xi) \end{pmatrix}(-D \otimes I_p)w_z.$$

Now let $p \in \partial V(z, \xi)$ and $w \in \mathcal{K}\tilde{F}(z, \xi)$ and write

$$\begin{aligned} \langle p, w\rangle &= \langle \nabla S(\xi), f(\xi) + g(\xi)(-D \otimes I_p)w_z\rangle + \langle p_z, (D^T \otimes I_p)h(\xi)\rangle \\ &\le -\sum_{i=1}^{N} W_i(\xi_i) + \langle h(\xi), (-D \otimes I_p)w_z\rangle + \langle p_z, (D^T \otimes I_p)h(\xi)\rangle, \end{aligned} \tag{3.12}$$

where the inequality is a consequence of (2.2). Suppose now that for some (z, ξ), $\dot{\overline{V}}(z, \xi) \neq \emptyset$. Then, for every $a \in \dot{\overline{V}}(z, \xi)$ and for every $p \in \partial V(z, \xi)$, there exists

$w \in \mathcal{K}\tilde{F}(z,\xi)$ such that $a = \langle p, w\rangle$. By the definition of $\mathsf{q}(z)$ and $P(z)$, $\partial P(z) = \mathcal{K}\mathsf{q}(z)$ [19], [29]. It implies that $a = \langle p, w\rangle$ holds when $p = \left(\begin{smallmatrix} p_z \\ \nabla S(\xi)\end{smallmatrix}\right) = \left(\begin{smallmatrix} w_z \\ \nabla S(\xi)\end{smallmatrix}\right)$ with $w_z \in \mathcal{K}\mathsf{q}(z)$. Thus (3.12) becomes $\langle p, w\rangle \le -\sum_{i=1}^N W_i(\xi_i)$. Hence, for all (z,ξ) such that $\dot{\overline{V}}(z,\xi) \neq \emptyset$, we have that

$$\dot{\overline{V}}(z,\xi) = \left\{ a \in \mathbb{R} \,:\, a \le -\sum_{i=1}^N W_i(\xi_i) \right\}. \tag{3.13}$$

Since $\frac{d}{dt}V(z(t),\xi(t)) \in \dot{\overline{V}}(\xi(t),z(t)) \subseteq (-\infty, 0]$ for almost every t, $V(z(t),\xi(t))$ cannot increase, and any Krasowskii solution $(z(t),\xi(t))$ is bounded. Hence, $(z(t),\xi(t))$ exists for all t.

Given any initial condition $(z(0),\xi(0))$, the set $\mathcal{S}$ such that $V(z,\xi) \le V(z(0),\xi(0))$ is a strongly invariant set for (3.9) which contains the initial condition. An application of the nonsmooth LaSalle invariance principle [2, 12, 13] shows that any Krasowskii solution converges to the largest weakly invariant set contained in $\mathcal{S} \cap \{(z,\xi) \,:\, \mathbf{0} \in \dot{\overline{V}}(z,\xi)\}$.

Moreover, in view of (3.13), the set Z of points (z,ξ) such that $0 \in \dot{\overline{V}}(z,\xi)$ is contained in the set of points such that $\xi = \mathbf{0}$. Hence, any point of the largest weakly invariant set contained in $\mathcal{S} \cap Z$ is such that $\xi = \mathbf{0}$. Pick a point $(z, \mathbf{0})$ on this invariant set. Then in order for a Krasowskii solution to (3.9) starting from this point to remain in the invariant set, it must be true that $\mathbf{0} \in f(\mathbf{0}) + g(\mathbf{0})(-(D \otimes I_p)\mathcal{K}\mathsf{q}(z)) = g(\mathbf{0})(-(D \otimes I_p)\mathcal{K}\mathsf{q}(z))$. Since the matrix $g(\mathbf{0})$ is full-column rank (recall that each $g_i(\mathbf{0})$ is full-column rank), the inclusion above requires the existence of $w_z \in \mathcal{K}\mathsf{q}(z)$ such that $(D \otimes I_p)w_z = \mathbf{0}$. In other words, the largest weakly invariant set included in $\mathcal{S} \cap Z$ is contained in the set (3.10). Finally, observe that, taking any point in the set (3.10) as an initial condition for (3.9), at least a Krasowskii solution $(z(t),\xi(t))$ originating from this point must coincide with the trivial solution, i.e., $(z(t),\xi(t)) = (\mathbf{0},\mathbf{0})$ for all t. Hence, any point in (3.10) is a Krasowskii equilibrium for (3.9). ∎

It is now possible to prove the following.

THEOREM 3.2. *Let Assumption* 2.1 *hold and let the communication graph G be undirected and connected. Let $v : \mathbb{R}_+ \to \mathbb{R}^p$ be a bounded and piecewise continuous function and Δ be a positive number. Then any Krasowskii solution to* (3.6) *converges to the set*

$$\{(x,\xi) : \xi = \mathbf{0},\ z \in (\mathcal{A}_1 \times \cdots \times \mathcal{A}_M), z = (D^T \otimes I_p)x\}, \tag{3.14}$$

where the sets $\mathcal{A}_k$'s are defined in (3.1) *with $a = \Delta/2$. Moreover,* $\lim_{t\to+\infty}[\dot{x}(t) - \mathbf{1}_N \otimes v(t)] = \mathbf{0}$.

Proof. Consider any Krasowskii solution $(x(t),\xi(t))$ to (3.6), whose existence is guaranteed locally. It can also be extended for all $t \in [0,+\infty)$. In fact suppose by contradiction this is not true, i.e., $(x(t),\xi(t))$ is defined on the interval $[0,t_f)$ with $t_f < +\infty$. Define $(z(t),\xi(t)) = ((D^T \otimes I_p)x(t),\xi(t))$ which is a Krasowskii solution to (3.9). As proven before, such a solution is bounded on its domain of definition. Since by (3.6) $\dot{x}(t) = h(\xi(t)) + \mathbf{1}_N \otimes v(t)$ and both terms on the right-hand side are bounded, then $x(t)$ grows linearly in t and therefore it must be bounded on the maximal interval of definition, i.e., $t_f = +\infty$. Hence both $(x(t),\xi(t))$ and $(z(t),\xi(t)) = ((D^T \otimes I_p)x(t),\xi(t))$ are defined for all t. Moreover, by Lemma 3.1, $z(t) = (D^T \otimes I_p)x(t)$ converges to the set of points (3.10), i.e., to

$$\{(x,\xi) \,:\, \xi = \mathbf{0},\ \mathbf{0} \in (D \otimes I_p)\mathcal{K}\mathsf{q}(z),\ z = (D^T \otimes I_p)x\}. \tag{3.15}$$

Let $(x, \mathbf{0})$ belong to the set (3.15). Then $z = (D^T \otimes I_p)x$, i.e., z belongs to the span of $D^T \otimes I_p$ and there exists $w_z \in \mathcal{K}\mathsf{q}(z)$ such that $(D \otimes I_p)w_z = \mathbf{0}$. The two conditions imply that $\langle w_z, z\rangle = 0$. We claim that then necessarily $z \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_M$ with the sets $\mathcal{A}_i$ given in (3.1). In fact, if this is not true, then there must exist a pair of indices j, k such that $|z_{kj}| > a$. This implies that the entry $k + j$ of the vector w_z is different from zero and also $w_{z,k+j} \cdot z_{kj} > 0$. Moreover, since $w_z \in \mathcal{K}\mathsf{q}(z)$, for any pair of indices i, ℓ such that $i \neq k$ or $\ell \neq j$, $w_{z,i+\ell} \cdot z_{i\ell} \geq 0$. This contradicts that $\langle w_z, z\rangle = 0$. Then we have proved that the set (3.15) is included in the set

$$\{(x, \xi) \,:\, \xi = \mathbf{0},\ z \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_M,\ z = (D^T \otimes I_p)x\}. \tag{3.16}$$

Hence, any Krasowskii solution $(x(t), \xi(t))$ to (3.6) converges to a subset of (3.16).

As for the second part of the statement, any Krasowskii solution to (3.6) is such that $\dot{x}(t) - \mathbf{1}_N \otimes v(t) = h(\xi(t))$, and since we have proved that $\xi(t) \to \mathbf{0}$ as $t \to \infty$, we have also proved that $\lim_{t\to+\infty}[\dot{x}(t) - \mathbf{1}_N \otimes v(t)] = \mathbf{0}$. □

3.3. Examples. We provide two examples of application of the quantized agreement result described above.

Agreement of double integrators by quantized measurements. Consider the case of N agents modeled as

$$\ddot{x}_i = f_i\,, \quad i = 1, 2, \ldots, N, \tag{3.17}$$

with $x_i, f_i \in \mathbb{R}^2$, for which we want to solve the agreement problem with quantized measurements. This means that all the agents should practically converge toward the same position and also asymptotically evolve with the same velocity v. The preliminary feedback [1]

$$f_i = -K_i(\dot{x}_i - v) + \dot{v} + u_i\,, \quad K_i = K_i^T, \tag{3.18}$$

with u_i to design, and the change of variables $\xi_i = \dot{x}_i - v$, makes the closed-loop system

$$\begin{aligned} \dot{x}_i &= \xi_i + v, \\ \dot{\xi}_i &= -K_i\xi_i + u_i, \\ y_i &= \xi_i \end{aligned}$$

passive with storage function $S_i(\xi_i) = \frac{1}{2}\xi_i^T\xi_i$ and $W_i(\xi_i) = -K_i\xi_i^T\xi_i$. The system above is in the form (2.1). Theorem 3.2 guarantees that the Krasowskii solutions of (3.17), (3.18), (3.5) converge asymptotically to the set (3.14) and that all the agents' velocities converge to v. In other words, the formation achieves practical position agreement and convergence to the prescribed velocity.

The case of unknown reference velocity. If the reference velocity v is not available to all the agents, then [4, 3] suggest replacing it with an estimate which is generated by each agent on the basis of the current available measurements. Here we examine this control scheme when the measurements are quantized. We consider the special case in which the unknown reference velocity is constant. Then each agent i, with the exception of one which acts as a leader and can access the prescribed reference velocity v, uses an estimated version of v, namely, $\hat{v}_i$, that has to be generated online starting from the available local measurements. The agent's dynamics (2.1) become

$$\begin{aligned} \dot{x}_i &= y_i + \hat{v}_i, \\ \dot{\xi}_i &= f_i(\xi_i) + g_i(\xi_i)u_i, \\ y_i &= h_i(\xi_i), \qquad\qquad i = 1, 2, \ldots, N, \end{aligned}$$

with $\hat{v}_i = v$ if $i = 1$ (without loss of generality agent 1 is taken as the leader) and otherwise generated by $\dot{\hat{v}}_i = \Lambda_i u_i$ with $\Lambda_i = \Lambda_i^T > 0$ and u_i as in (2.10). Observe that in this case, the estimated velocity is updated via quantized measurements. Consider the closed-loop system

$$\begin{aligned} \dot{x} &= h(\xi) + \mathbf{1}_N \otimes \hat{v}, \\ \dot{\xi} &= f(\xi) - g(\xi)(D \otimes I_p)\mathsf{q}(z), \\ \dot{\hat{v}} &= -\Lambda(D \otimes I_p)\mathsf{q}(z), \end{aligned} \tag{3.19}$$

where $\Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_N)$ and $z = (D^T \otimes I_p)x$. Let

$$\hat{v}_i(t) = v + \hat{v}_i(t) - v = v + (\hat{v}_i(t) - v) =: v + \tilde{v}_i(t),$$

where $\tilde{v}_1 = \mathbf{0}$. Rewrite the system using the coordinates z and $\tilde{v}$ and obtain

$$\begin{aligned} \dot{z} &= (D^T \otimes I_p)[h(\xi) + \tilde{v}], \\ \dot{\xi} &= f(\xi) - g(\xi)(D \otimes I_p)\mathsf{q}(z), \\ \dot{\tilde{v}} &= \Lambda(D \otimes I_p)\mathsf{q}(z). \end{aligned} \tag{3.20}$$

Consider the Lyapunov function $V(z, \xi, \tilde{v}) = S(\xi) + P(z) + \frac{1}{2}\tilde{v}^T \Lambda^{-1} \tilde{v}$, where $P(z)$ is as in (3.11), and let $\tilde{F}(z, \xi, \tilde{v})$ be the right-hand side of (3.20). One can now proceed as in the proof of Lemma 3.1 and prove that $\dot{\overline{V}}(z, \xi, \tilde{v}) = \{a \in \mathbb{R} : a \leq -\sum_{i=1}^N W_i(\xi_i)\}$. Hence, any Krasowskii solution $(z(t), \xi(t), \tilde{v}(t))$ is bounded and exists for all t. Let $\mathcal{S}$ be the level set such that $V(z, \xi, \tilde{v}) \leq V(z(0), \xi(0), \tilde{v}(0))$ and Z the set of points $(z, \xi, \tilde{v})$ such that $\mathbf{0} \in \dot{\overline{V}}(z, \xi, \tilde{v})$. Then any solution $(z, \xi, \tilde{v})$ converges to the largest weakly invariant subset contained in $\mathcal{S} \cap Z$. Observe that $Z \subset \{(z, \xi, \tilde{v}) : \xi = 0\}$. Moreover, for a set in $\mathcal{S} \cap Z$ to be weakly invariant, it must be true that $\mathbf{0} \in \mathcal{K}\tilde{F}(z, \xi, \tilde{v})$ with $\tilde{F}(z, \xi, \tilde{v})$ the right-hand side of (3.20). These two facts together imply that there must exist $w_z \in \mathcal{K}\mathsf{q}(z)$ such that $(D \otimes I_p)w_z = \mathbf{0}$ and additionally $(D^T \otimes I_p)\tilde{v} = \mathbf{0}$. The latter implies that $\tilde{v} = (\mathbf{1}_N \otimes I_p)c$ for some $c \in \mathbb{R}$. Since $\tilde{v}_1 = \mathbf{0}$, then on the largest weakly invariant set contained in $S \cap Z$ it is also true that $\tilde{v} = \mathbf{0}$. Hence it follows that any Krasowskii solution to (3.20) converges to the set $\{(z, \xi, \tilde{v}) : \xi = \mathbf{0},\ \mathbf{0} \in (D \otimes I_p)\mathcal{K}\mathsf{q}(z),\ \tilde{v} = \mathbf{0}\}$.

One can then focus on the system (3.19) and follow the same arguments of Theorem 3.2 to conclude that the solutions of the closed-loop system converge to the set where all the systems evolve with the same velocity, achieve practical consensus on the position variable and the estimated velocities $\hat{v}_i$ converge to the true reference velocity v. These results are summarized in the following proposition.

PROPOSITION 3.3. *Let Assumption* 2.1 *hold and let the communication graph G be undirected and connected. Let $v \in \mathbb{R}^p$ be a constant vector and Δ a positive number. Then any Krasowskii solution to* (3.19) *converges to the set*

$$\{(x, \xi, \hat{v}) : \xi = \mathbf{0}, z \in (\mathcal{A}_1 \times \cdots \times \mathcal{A}_M), z = (D^T \otimes I_p)x, \hat{v} = \mathbf{1}_N \otimes v\},$$

where the sets $\mathcal{A}_k$ are defined in (3.1) *with $a = \Delta/2$. In particular,* $\lim_{t \to +\infty}[\dot{x}(t) - \mathbf{1}_N \otimes v] = \mathbf{0}$.

4. Quantized synchronization of passive systems. We now turn our attention to the systems in (2.13) where the control law that we consider is a static quantized output-feedback control law of the form

$$u = -(D \otimes I_p)\mathsf{q}(z) \text{ with } z = (D^T \otimes I_p)w. \tag{4.1}$$

The overall closed-loop system is

$$\begin{aligned} \dot{\xi} &= (I_N \otimes A)\xi - (I_N \otimes B)(D \otimes I_p)\mathsf{q}(z), \\ z &= (D^T \otimes I_p)w = (D^T \otimes I_p)(I_N \otimes C)\xi. \end{aligned} \tag{4.2}$$

Applications where synchronization problems under communication constraints and passivity are relevant are reviewed in [16]. Later in this section, we briefly discuss another example where the use of quantized measurements for synchronization can be useful.

To study the robustness of the synchronization algorithm to quantized measurements we need a more explicit characterization of the exponential stability of (2.15). To this purpose we introduce a different Lyapunov function which is characterized in the following lemma. As we consider time-invariant graphs, observability can be replaced by a detectability assumption.

LEMMA 4.1. *Let* (C, A) *be detectable and* $\Pi = I_N - \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}$. *The integral* $R := \int_0^{+\infty} (\Pi \otimes I_n)^T \mathrm{e}^{\tilde{A}^T s} \mathrm{e}^{\tilde{A} s} (\Pi \otimes I_n) ds$, *with* $\tilde{A}$ *as in* (2.15), *is finite and satisfies*

$$||R|| \le \int_0^{+\infty} \left\| \begin{pmatrix} \exp(A - \lambda_2 BC)s & \dots & \mathbf{0}_{n\times n} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \dots & \exp(A - \lambda_N BC)s \end{pmatrix} \right\|^2 ds. \tag{4.3}$$

Moreover, the Lyapunov function $U(\tilde{\xi}) = \tilde{\xi}^T R \tilde{\xi}$ *satisfies*

$$\begin{aligned} c_1 ||\tilde{\xi}||^2 \le U(\tilde{\xi}) \le c_2 ||\tilde{\xi}||^2, \\ \nabla U(\tilde{\xi}) \cdot \tilde{A}\tilde{\xi} \le -||\tilde{\xi}||^2 \end{aligned} \tag{4.4}$$

for each $\tilde{\xi} \in \mathbb{R}^{nN}$.

Proof. The proof is given in Appendix A. ∎

The first fact we prove about (4.2) is that the control law (4.1) achieves practical synchronization of the outputs.

PROPOSITION 4.2. *Let Assumption* 2.3 *hold and let the communication graph* G *be undirected and connected. Then any Krasowskii solution to* (4.2) *converges to the largest weakly invariant subset contained in*

$$\left\{ \xi \in R^{nN} : |z_{kj}| \le \frac{\Delta}{2} \ \forall\, k = 1, 2, \dots, M,\ j = 1, 2, \dots p \right\} \tag{4.5}$$

with $z = (D^T \otimes I_p)(I_N \otimes C)\xi$.

Proof. Any Krasowskii solution to (4.2) satisfies the differential inclusion

$$\dot{\xi} \in (I_N \otimes A)\xi - (I_N \otimes B)(D \otimes I_p)\mathcal{K}\mathsf{q}(z).$$

Consider the Lyapunov function $V(\xi) = \xi^T (I_N \otimes P)\xi$. Then, for any $\xi \in R^{Nn}$ and any $\nu \in \mathcal{K}\mathsf{q}(z)$, with $z = (D^T \otimes I_p)(I_N \otimes C)\xi$, we have

$$\begin{aligned} \dot{V}(\xi) &:= \nabla V(\xi) \cdot [(I_N \otimes A)\xi - (I_N \otimes B)(D \otimes I_p)\nu] \\ &= 2\xi^T (I_N \otimes PA)\xi - 2\xi^T (I_N \otimes PB)(D \otimes I_p)\nu. \end{aligned}$$

Using Assumption 2.3 and the definition of z we further obtain that for all $\nu \in \mathcal{K}\mathsf{q}(z)$,

$$\dot{V}(\xi) \le -2\xi^T (I_N \otimes C^T)(D \otimes I_p)\nu = -2z^T \nu \le 0. \tag{4.6}$$

This shows that $V(\xi(t))$ cannot increase and that $\xi(t)$ is bounded. Moreover, by LaSalle's invariance principle for differential inclusions, any Krasowskii solution converges to the largest weakly invariant subset contained in

$$\left\{\xi \in \mathbb{R}^{Nn} : \exists\, \nu \in \mathcal{K}\mathsf{q}(z)\,\text{s.t.}\, \nabla V(\xi)\cdot[(I_N\otimes A)\xi - (I_N\otimes B)(D\otimes I_p)\nu] = 0\right\}.$$

In view of (4.6), any point ξ in this set is such that $z_{kj}\,\nu_{kj} = 0$ for all $k = 1, 2, \ldots, M$ and for all $j = 1, 2, \ldots, p$. Since $\nu_{kj} \in \mathcal{K}\mathsf{q}(z_{kj})$, then $z_{kj}\,\nu_{kj} = 0$ implies that $|z_{kj}| \le \frac{\Delta}{2}$. This ends the proof. ∎

Remark 4.3 (practical output synchronization). A consequence of the previous statement is that any two outputs w_i, w_j practically asymptotically synchronize. Namely, considered any Krasowskii solution $\xi(t)$ and the corresponding output $w(t) = (I_N \otimes C)\xi(t)$ for each $\ell = 1, 2, \ldots, n$ and each $t \ge 0$, the difference $|w_{i\ell}(t) - w_{j\ell}(t)|$ is upper bounded by a quantity which asymptotically converges to $d\frac{\Delta}{2}$ with d the diameter of the graph.

In the special case of the consensus problem, i.e., with $n = p$, $A = \mathbf{0}$, and $C = I_n$, each entry of $\xi_i - \xi_j$ is in magnitude bounded by a quantity which converges to $d\frac{\Delta}{2}$. The result can be compared with Theorem 4 in [14]. One difference is that while trees are considered in [14], connected graphs are considered here. Moreover, in [14] the scalar states are guaranteed to converge to a ball of radius $\frac{||D^TD||\sqrt{M}}{\lambda_{\min}(D^TD)}\Delta$. Hence, denoting by ρ the ratio $\frac{||D^TD||}{\lambda_{\min}(D^TD)}$ and considering the bound $M \le N-1$, any two states ξ_i, ξ_j may differ for $2\rho\Delta\sqrt{N-1}$. The passivity approach considered here yields that they differ for not more than $d\frac{\Delta}{2}$, where d grows as $O(\rho\log(N))$ [9] for not complete and regular graphs (graphs with all the nodes having the same degree), thus leading to a smaller region of convergence, the quantizer resolution Δ being the same.

The proof of the proposition above clearly does not rely on the linearity of the systems but rather on the passivity property. Hence, if one considers nonlinear passive systems, that is, systems for which a positive definite continuously differentiable storage function $V_i(\xi_i)$ exists such that $\nabla V_i \cdot f_i(\xi_i, u_i) \le w_i^T \cdot u_i$, with $w_i = h_i(\xi_i)$, then for the closed-loop system $\dot{\xi}_i = f_i(\xi_i, u_i)$, with u_i given in (4.1) and $i = 1, 2, \ldots, N$, it is still true that the overall storage function $V(\xi) = \sum_{i=1}^N V_i(\xi_i)$ satisfies the inequality $\dot{V}(\xi) \le -2z^T\nu$ for all $z = (D^T \otimes I_p)h(\xi)$ and all $\nu \in \mathcal{K}\mathsf{q}(z)$. Hence, the following holds.

PROPOSITION 4.4. *Let Assumption* 2.1 *hold and let the communication graph G be undirected and connected. Then any Krasowskii solution to the systems* (2.1) *in closed-loop with $u = -(D \otimes I_p)\mathsf{q}(z)$ and $z = (D^T \otimes I_p)h(\xi)$ converges to the largest weakly invariant subset contained in the set* (4.5).

The next lemma states a property of the average of the solutions to (4.2) which helps to better characterize the region where the solutions converge.

LEMMA 4.5. *Let Assumption* 2.3 *hold and let the communication graph G be undirected and connected. Any Krasowskii solution $\xi(t)$ to* (4.2) *satisfies*

$$(\mathbf{1}_N^T \otimes I_n)\xi(t) = \mathrm{e}^{At}(\mathbf{1}_N^T \otimes I_n)\xi(0)$$

for all $t \ge 0$.

Proof. Observe that for almost every t,

$$\begin{aligned}\frac{d}{dt}(\mathbf{1}_N^T \otimes I_n)\xi(t) &= (\mathbf{1}_N^T \otimes I_n)\frac{d}{dt}\xi(t)\\ &\in (\mathbf{1}_N^T \otimes I_n)(I_N \otimes A)\xi(t) - (\mathbf{1}_N^T \otimes I_n)(I_N \otimes B)(D \otimes I_p)\mathcal{K}\mathsf{q}(z).\end{aligned}$$

Bearing in mind that for matrices $F \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{p\times q}$, the property of the Kronecker product, $F\otimes G = (F\otimes I_p)(I_n\otimes G) = (I_m\otimes G)(F\otimes I_q)$ holds, one can further show that

$$\begin{aligned}\frac{d}{dt}(\mathbf{1}_N^T\otimes I_n)\xi(t) &\in (\mathbf{1}_N^T\otimes I_n)(I_N\otimes A)\xi(t) - B(\mathbf{1}_N^T D\otimes I_p)\mathcal{K}\mathsf{q}(z)\\ &= (\mathbf{1}_N^T\otimes I_n)(I_N\otimes A)\xi(t) = A(\mathbf{1}_N^T\otimes I_n)\xi(t),\end{aligned}\tag{4.7}$$

where the equality before the last one exploited the fact that $\mathbf{1}_N^T D = \mathbf{0}_M^T$, which holds by definition of the incidence matrix D. Hence, any Krasowskii solution $\xi(t)$ is such that the average $(\mathbf{1}_N^T\otimes I_n)\xi(t)$ satisfies $(\mathbf{1}_N^T\otimes I_n)\xi(t) = \mathrm{e}^{At}(\mathbf{1}_N^T\otimes I_n)\xi(0)$. ∎

The following result provides an estimate of the region where the solutions converge and shows practical synchronization under quantized relative measurements.

THEOREM 4.6. *Let Assumption* 2.3 *hold and let the communication graph G be undirected and connected. Assume that (C, A) is detectable. Then for any Krasowskii solution $\xi(t)$ to*

$$\dot\xi = (I_N\otimes A)\xi - (I_N\otimes B)(D\otimes I_p)\mathsf{q}((D^T\otimes I_p)(I_N\otimes C)\xi)\tag{4.8}$$

there exists a finite time T such that $\xi(t)$ satisfies

$$\frac{1}{\sqrt{pM}}\left\|\xi(t) - (\mathbf{1}_N\otimes I_n)\frac{(\mathbf{1}_N^T\otimes I_n)\xi(t)}{N}\right\| \le 2\sqrt{\frac{c_2}{c_1}}||R||\,||B||\,||D\otimes I_p||\Delta\tag{4.9}$$

for all $t\ge T$, where $c_1, c_2, ||R||$ are defined in (4.3), (4.4). *Moreover, $\frac{\mathbf{1}_N^T\otimes I_n}{N}\xi(t) = \xi_0(t)$, where $\xi_0(t)$ is the solution of $\dot\xi_0(t) = A\xi_0(t)$ with the initial condition $\xi_0(0) = \frac{\mathbf{1}_N^T\otimes I_n}{N}\xi(0)$.*

Proof. By definition, any Krasowskii solution ξ to (4.8) is such that $\tilde\xi = (\Pi\otimes I_n)\xi$, with $\Pi = I_N - \frac{\mathbf{1}_N\mathbf{1}_N^T}{N}$, satisfies

$$\dot{\tilde\xi} \in (I_N\otimes A)\tilde\xi - (I_N\otimes B)(D\otimes I_p)\mathcal{K}\mathsf{q}((D^T\otimes I_p)(I_N\otimes C)\xi),$$

where similar manipulations as in (4.7) were used. Moreover, any $\nu\in\mathcal{K}\mathsf{q}((D^T\otimes I_p)(I_N\otimes C)\xi)$ is such that $||\nu - (D^T\otimes I_p)(I_N\otimes C)\xi|| \le \sqrt{pM}\frac{\Delta}{2}$. Under the assumption on the detectability of (C, A), we can consider the Lyapunov function $U(\tilde\xi)$ introduced in Lemma 4.1. For any ξ and any $\nu\in\mathcal{K}\mathsf{q}((D^T\otimes I_p)(I_N\otimes C)\xi)$,

$$\begin{aligned}&\nabla U(\tilde\xi)[(I_N\otimes A)\tilde\xi - (I_N\otimes B)(D\otimes I_p)\nu]\\ &\quad= \nabla U(\tilde\xi)[(I_N\otimes A) - (I_N\otimes B)(DD^T\otimes I_p)(I_N\otimes C)]\tilde\xi\\ &\qquad+ \nabla U(\tilde\xi)(I_N\otimes B)(D\otimes I_p)[(D^T\otimes I_p)(I_N\otimes C)\tilde\xi - \nu]\\ &\quad\le -||\tilde\xi||(||\tilde\xi|| - ||R||\,||B||\,||D\otimes I_p||\sqrt{pM}\Delta).\end{aligned}$$

Hence, for $||\tilde\xi|| > \frac{1}{2}||R||\,||B||\,||D\otimes I_p||\sqrt{pM}\Delta$,

$$\nabla U(\tilde\xi)[(I_N\otimes A)\xi - (I_N\otimes B)(D\otimes I_p)\nu] \le -\frac{||\tilde\xi||^2}{2} \le -\frac{1}{2c_2}U(\tilde\xi).$$

It follows that any Krasowskii solution converges in finite time to the set of points $\tilde{\xi}$ such that

$$||\tilde{\xi}|| \leq 2\sqrt{\frac{c_2}{c_1}}||R||\,||B||\,||D\otimes I_p||\sqrt{pM}\Delta$$

from which the thesis is proved by definition of $\tilde{\xi}$.

The proof of the final claim follows from the fact that by Lemma 4.5, for all $t \geq 0$,

$$\tilde{\xi}(t) = \xi(t) - (\mathbf{1}_N \otimes I_n)\frac{(\mathbf{1}_N^T \otimes I_n)\xi(t)}{N} = \xi(t) - (\mathbf{1}_N \otimes I_n)\frac{\mathrm{e}^{At}(\mathbf{1}_N^T \otimes I_n)\xi(0)}{N}. \qquad \square$$

Remark 4.7 (role of $||R||$). In the case $A = 0$, $B = C = 1$, the bound on R reduces to $||R|| \leq \frac{1}{2\lambda_2}$, where λ_2 is the algebraic connectivity of the graph. In this case, the size of the region of convergence in (4.9) resembles the estimate given in Theorem 1 and Corollary 1 in [7] for quantized consensus of single integrators. Theorem 4.6 can be viewed as the extension of the results in [7] to the problem of synchronization of linear multivariable passive systems by quantized output feedback.

4.1. Examples. In the following examples, we discuss how synchronization with quantized measurements can play a role in a decentralized output regulation problem in which heterogeneous systems asymptotically agree on the trajectory to track.

Output synchronization for heterogeneous linear systems. In [38] (see also [3, section 3.6]) the following problem is investigated. Given N heterogeneous linear systems

$$\begin{aligned} \dot{x}_i &= F_i x_i + G_i u_i,\\ y_i &= H_i x_i, \qquad\qquad i = 1,2,\ldots,N, \end{aligned} \tag{4.10}$$

with (F_i, G_i) stabilizable and (H_i, F_i) detectable, and a graph G (which here, as usual in this paper, we assume to be static undirected and connected), find a feedback control law u_i for each system i (i) which uses relative measurements concerning only the systems which are connected to the system i via the graph G and (ii) such that output synchronization is achieved, i.e., $\lim_{t\to\infty} ||y_i(t) - y_j(t)|| = 0$ for all $i, j \in \{1, 2, \ldots, N\}$. Excluding the trivial case in which the closed-loop system has an attractive set of equilibria where the outputs are all zero, the authors of [38] show that the output synchronization problem for N heterogeneous systems is solvable if and only if there exist matrices S, R such that $\lim_{t\to\infty} ||y_i(t) - Re^{-St}w_0|| = 0$ for each $i \in \{1, 2, \ldots, N\}$, for some w_0. Moreover, provided that $\sigma(S) \subset j\mathbb{R}$, the controllers which solve the regulation problem are

$$\begin{aligned} \dot{\hat{x}}_i &= F_i\hat{x}_i + G_i u_i + L_i(\hat{y}_i - C_i x_i),\\ \hat{y}_i &= H_i\hat{x}_i,\\ u_i &= K_i(\hat{x}_i - \Pi_i\xi_i) + \Gamma_i\xi_i, \end{aligned} \tag{4.11}$$

where $\xi_i \in \mathbb{R}^p$ are the exosystem states that synchronize via communication channels and are described by

$$\begin{aligned} \dot{\xi} &= (I_N \otimes S)\xi - (I_N \otimes B)(D \otimes I_p)z,\\ z &= (D^T \otimes I_p)(I_N \otimes C)\xi, \end{aligned} \tag{4.12}$$

where D is the incidence matrix associated to the graph, the pair (C, S) is detectable, the matrices L_i, K_i are such that $F_i + G_iK_i, F_i + L_iH_i$ are Hurwitz, and Π_i, Γ_i are matrices which solve the regulator equations

$$F_i\Pi_i + G_i\Gamma_i = \Pi_i S,$$
$$H_i\Pi_i = R.$$

The controllers (4.11)–(4.12) are a modified form of the ones in [38, equation (10)], where in the latter, the local controller communicates the entire exosystem state ξ_i to its connecting nodes. When the relative measurement z_k is transmitted via a digital communication line, this information is quantized and the variable z in the controller (4.11)–(4.12) is replaced by its quantized form $\mathtt{q}(z)$.

Let the eigenvalues of S have in addition multiplicity of one in the minimal polynomial, so that we can restrict S to be skew-symmetric without loss of generality and $B = C^T$. Then the exosystems

$$\begin{aligned} \dot{\xi}_i &= S\xi_i + Bu_i, \\ w_i &= C\xi_i, \qquad i = 1, 2, \ldots, N, \end{aligned} \tag{4.13}$$

trivially satisfy Assumption 2.3. Then Theorem 4.6 applies and the solutions ξ_i, $i = 1, 2, \ldots, N$, of (4.12) practically synchronize under the quantization of z. It is then possible to see that the closed-loop system of (4.10) and the controllers (4.11)–(4.12) with z replaced by $\mathtt{q}(z)$ achieve practical output synchronization. This follows from similar arguments in [38, Theorem 5], where [38, Theorem 1], which is used in the proof of the theorem, is replaced by Theorem 4.6.

Before ending the section, we remark that Theorem 4.6 also holds under a slightly different set of conditions which do not require passivity.

Assumption 4.8. Let (A, B, C) be stabilizable and detectable, and assume that

$$[I_p + \lambda_N \mathbf{G}][I_p + \lambda_2 \mathbf{G}]^{-1}$$

is strictly positive real, where $\mathbf{G}(s) = C(sI - A)^{-1}B$ is the transfer function of (2.13) and λ_N is the largest eigenvalue of L.

Under Assumption 4.8, the results in Theorem 4.6 still hold *mutatis mutandis.* Indeed, by the multivariable circle criterion in [21, Theorem 3.4], $(A - \lambda_i BC)$ is Hurwitz for every nonzero eigenvalue λ_i of L. This implies that (2.15) is exponentially stable (this is evident from the proof of Lemma 4.1—see the appendix) and Lemma 4.1 and 4.5 continue to hold. As a consequence the proof of Theorem 4.6 holds word by word under the assumption that (A, B, C) is minimal and Assumption 4.8 holds.

The case of output synchronization with filtered and quantized signals. As a concrete example to the case of exosystems satisfying Assumption 4.8, we consider again the closed-loop systems in the previous example where the heterogenous linear systems (4.10) are interconnected with the controllers (4.11)–(4.12) with

$$S = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & a & -a \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

The system (S, B, C) can be considered as a cascade interconnection of a second-order oscillator with frequency ω and a low-pass filter with a cut-off frequency a, and its transfer function is given by

$$G(s) = \frac{as}{(s^2 + \omega^2)(s + a)}.$$

Using the above (S, B, C), the interconnected exosystems (4.12) with quantized measurement $\mathsf{q}(z)$ resemble a network of oscillators where the relative measurements z_k are filtered and quantized. In the limiting case $a \to \infty$, the exosystems are given by (4.13), where $A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, and it satisfies Assumption 2.3. A direct application of Theorem 4.6 shows that (4.9) holds with

$$\|R\| \leq \int_0^\infty \left\| \begin{pmatrix} \exp\begin{pmatrix} 0 & \omega \\ -\omega & -\lambda_2 \end{pmatrix} s & \cdots & \mathbf{0}_{n\times n} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \cdots & \exp\begin{pmatrix} 0 & \omega \\ -\omega & -\lambda_N \end{pmatrix} s \end{pmatrix} \right\| \mathrm{d}s.$$

In particular, if $\lambda_2 > 4\omega^2$, then $\|R\| \leq \frac{1}{\lambda_2 - \sqrt{\lambda_2^2 - 4\omega^2}}$.

On the other hand, if $0 < a < \infty$, i.e., when the low-pass filter is used, then it can be checked that

$$\inf_\nu \mathrm{Re}\left(\frac{1+\lambda_N G(i\nu)}{1+\lambda_2 G(i\nu)}\right) \geq 0,$$
$$\Leftrightarrow \inf_\nu (a\omega^2 - a\nu^2)^2 + ((\omega^2 + \lambda_N a)\nu - \nu^3)((\omega^2 + \lambda_2 a)\nu - \nu^3) \geq 0.$$

Note that for a sufficiently large $a > 0$, the above condition holds. Thus, the cut-off frequency a can be designed based only on the knowledge of λ_2, λ_N, and ω such that the exosystems (4.13) satisfy Assumption 4.8.

In both cases, practical output synchronization of the closed-loop systems (4.10)–(4.12) with quantized $\mathsf{q}(z)$ is obtained.

5. Conclusions. The passivity approach to coordinated control problems presents several interesting features, such as the possibility to deal with agents which have complex and high-dimensional dynamics. In this paper we have shown how it also lends itself to taking into account the presence of quantized measurements. Using the passivity framework along with appropriate tools from nonsmooth control theory and differential inclusions, we have shown that many of the results of [1, 33] continue to hold in an appropriate sense in the presence of quantized information. We believe that the results presented in the paper are a promising addition to the existing literature on continuous-time consensus and coordinated control under quantization [14, 7, 25, 17].

Many additional aspects deserve attention in future work on the topic. The approach to quantized coordinated control pursued in this paper appears to be suitable to tackling more complex formation control problems such as those considered, e.g., in section II.C of [1], [14, section 4], and [36]. These possible extensions can also benefit from the results of [4].

The paper did not discuss whether the use of quantized measurements yields sliding modes. Sliding modes were shown to occur in problems of quantized consensus for single integrators [7] and hysteretic quantizers were introduced to overcome the problem. A similar device could prove useful in quantized coordination problems.

The literature on synchronization and coordination problems which exploit passivity is rich (see, e.g., [30, 33, 8, 37] and references therein) and the problems presented there could be reconsidered in the presence of quantized measurements. The book [3] provides many other results of cooperative control within the passivity approach. These results are all potentially extendible to the case in which quantized measurements are in use.

Appendix A. Proof of Lemma 4.1. Following [28, Theorem 3] (see also [15]), we introduce the $N \times N$ nonsingular matrices $T = (\mathbf{1}_N/\sqrt{N}\ v_2 \ldots v_N)$, $T^{-1} = \begin{pmatrix}\mathbf{1}_N/\sqrt{N} & w_2 & \ldots & w_N\end{pmatrix}^T$, where the columns of T form an orthonormal basis of $\mathbb{R}^N$ and T, T^{-1} transform the Laplacian matrix $L = DD^T$ into its diagonal form, and notice the following:

$$\mathrm{e}^{\tilde{A}s}(\Pi \otimes I_n) = (T \otimes I_n) \left(\begin{array}{c|ccc} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \ldots & \mathbf{0}_{n\times n} \\ \hline \mathbf{0}_{n\times n} & \exp(A - \lambda_2 BC)s & \ldots & \mathbf{0}_{n\times n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \ldots & \exp(A - \lambda_N BC)s \end{array}\right)$$
$$\cdot\, (T^{-1} \otimes I_n)(\Pi \otimes I_n),$$

where $0 < \lambda_2 < \cdots < \lambda_N$ are the nonzero eigenvalues of DD^T. Since (A, B, C) is passive and (C, A) is detectable, the matrices $A - \lambda_i BC$, $i = 2, \ldots, N$, are Hurwitz. This implies that only exponentially stable modes are present in $\mathrm{e}^{\tilde{A}s}(\Pi \otimes I_n)$, and therefore the integral which defines R exists and is finite.

Using the transformation matrix T, a routine computation shows that

$$\begin{aligned}
&(\Pi \otimes I_n)^T \mathrm{e}^{\tilde{A}^T s}\mathrm{e}^{\tilde{A}s}(\Pi \otimes I_n) \\
&= (\Pi T \otimes I_n)^T \left(\begin{array}{c|ccc} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \ldots & \mathbf{0}_{n\times n} \\ \hline \mathbf{0}_{n\times n} & \exp(A - \lambda_2 BC)^T s & \ldots & \mathbf{0}_{n\times n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \ldots & \exp(A - \lambda_N BC)^T s \end{array}\right) (T^{-1} \otimes I_n) \\
&\quad \cdot (T \otimes I_n) \left(\begin{array}{c|ccc} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \ldots & \mathbf{0}_{n\times n} \\ \hline \mathbf{0}_{n\times n} & \exp(A - \lambda_2 BC)s & \ldots & \mathbf{0}_{n\times n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \ldots & \exp(A - \lambda_N BC)s \end{array}\right) (T^{-1}\Pi \otimes I_n) \\
&= ([v_2 \ldots v_N] \otimes I_n)^T \begin{pmatrix} \exp(A - \lambda_2 BC)^T s & \ldots & \mathbf{0}_{n\times n} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \ldots & \exp(A - \lambda_N BC)^T s \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} \exp(A - \lambda_2 BC)s & \ldots & \mathbf{0}_{n\times n} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \ldots & \exp(A - \lambda_N BC)s \end{pmatrix} \left(\begin{bmatrix} w_2^T \\ \vdots \\ w_N^T \end{bmatrix} \otimes I_n \right).
\end{aligned}$$

Taking the norm of the matrix,

$$||(\Pi \otimes I_n)^T \mathrm{e}^{\tilde{A}^T s}\mathrm{e}^{\tilde{A}s}(\Pi \otimes I_n)|| \leq \left\| \begin{pmatrix} \exp(A - \lambda_2 BC)s & \ldots & \mathbf{0}_{n\times n} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \ldots & \exp(A - \lambda_N BC)s \end{pmatrix} \right\|^2$$

from which (4.3) follows.

Rewrite the function $U(\tilde{\xi})$ as

$$\int_t^{+\infty} \tilde{\xi}^T (\Pi \otimes I_n)^T \mathrm{e}^{\tilde{A}^T(\tau - t)}\mathrm{e}^{\tilde{A}(\tau - t)}(\Pi \otimes I_n)\tilde{\xi}\, d\tau = \int_t^{+\infty} ||\tilde{\xi}(\tau; \tilde{\xi}, t)||^2 d\tau,$$

where $\tilde{\xi}(\tau;\tilde{\xi},t)$ is the solution to (2.15) at time τ starting from the initial condition $\tilde{\xi}$ at time t. Following standard converse Lyapunov theorem arguments (see, e.g., Khalil [23, Theorem 4.12]) one easily proves that $c_1||\tilde{\xi}||^2 \le U(\tilde{\xi}) \le c_2||\tilde{\xi}||^2$. Moreover,

$$\nabla U(\tilde{\xi})\tilde{A}\tilde{\xi} = \left[\tilde{\xi}^T(\Pi\otimes I_n)^T \mathrm{e}^{\tilde{A}^T s}\mathrm{e}^{\tilde{A}s}(\Pi\otimes I_n)\tilde{\xi}\right]_{s=0}^{s=+\infty} = -||\tilde{\xi}||^2. \qquad \square$$

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