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Brief paper

Power-based control of physical systems[☆]

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ABSTRACT

It is well known that energy-balancing control is stymied by the presence of pervasive dissipation. To overcome this problem in electrical circuits, the alternative paradigm of *power shaping* was introduced in Ortega, Jeltsema, and Scherpen (2003)—where, as suggested by its name, stabilization is achieved shaping a function akin to the power instead of the energy function. In this paper we extend this technique to general nonlinear systems. The method relies on the solution of a PDE, which identifies the open-loop storage function. We show through some physical examples, that the power-shaping methodology yields storage functions which have units of power. To motivate the application of this control technique we illustrate the procedure with two case studies: a tunnel diode circuit and a two-tanks system.

1. Introduction and background material

The main idea behind passivity-based control (PBC) is to modify the energy function of the system, by assigning a minimum at the desired equilibrium point. A step called energy shaping, which, combined with damping injection, constitute the two main stages of PBC (Ortega, van der Schaft, Mareels, & Maschke, 2001; van der Schaft, 2000). Among the several ways to achieve energy shaping, we can mention the controlled Lagrangian approach (Auckly, Kapitanski, & White, 2003; Bloch, Leonhard, & Marsden, 2000), the interconnection and damping assignment (IDA) (Ortega, van der Schaft, Maschke, & Escobar, 2002), the control by interconnection (van der Schaft, 2000) and the so-called *energy-balancing* PBC method (Ortega et al., 2001).

In the particular case of energy-balancing PBC, the energy function assigned to the closed-loop system is the difference between the total energy of the system and the energy supplied by the controller, hence the name energy balancing. To put our contribution in perspective, let us briefly recall the principles of energy-balancing control (Ortega et al., 2001). Consider a system whose state space representation is given by[1]

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}, \tag{1a}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}), \tag{1b}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{u}, \boldsymbol{y} \in \mathbb{R}^m$ are the input and output vectors, respectively. We assume that the system (1) satisfies a cyclo-passive inequality, that is, along all trajectories compatible with $u : [0, t] \to \mathbb{R}^m$,

$$H(\boldsymbol{x}(t)) - H(\boldsymbol{x}(0)) \leq \int_0^t \boldsymbol{u}^\top(\tau)\boldsymbol{y}(\tau)\mathrm{d}\tau, \tag{2}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is the storage function. Inequality (2) represents a universal property of physical systems, where typically, $\boldsymbol{u}$, $\boldsymbol{y}$ are conjugated variables, in the sense that their product has units of power, and $H(\boldsymbol{x})$ is the total stored *energy* in the system. Notice that no assumption of non-negativity on $H(\boldsymbol{x})$ is imposed.

We recall that a system is passive if (2) holds and $H(\boldsymbol{x})$ is bounded from below. Because of this additional restriction, every passive system is cyclo-passive, but the converse is not true. In terms of energy exchange, cyclo-passive systems exhibit a net

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1 All vectors defined in the paper are *column* vectors, including the gradient of a scalar function that we denote by the operator $\nabla = \partial/\partial\boldsymbol{x}$. Differentiation of functions with scalar arguments is denoted by $(\cdot)'$.

absorption of energy along *closed* trajectories (Hill & Moylan, 1980), while passive systems absorb energy along *any* trajectory that starts from a state of minimal energy $\boldsymbol{x}(0) = \arg\min H(\boldsymbol{x})$.

Usually, the point where the storage function has a minimum is not the operating point of interest, and we would rather stabilize another admissible equilibrium point $\boldsymbol{x}^\star$. Thus, in energy-balancing control, we look for a control law such that the energy supplied by the controller, that we denote by H_a, can be expressed as a function of the state. Indeed, from (2) we see that for any function $\widehat{\boldsymbol{u}} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$-\int_0^t \widehat{\boldsymbol{u}}^\top(\boldsymbol{x}(\tau))\boldsymbol{h}(\boldsymbol{x}(\tau))\mathrm{d}\tau = H_a(\boldsymbol{x}(t)) - H_a(\boldsymbol{x}(0)) \tag{3}$$

for some function $H_a : \mathbb{R}^n \to \mathbb{R}$, the control $\boldsymbol{u} = \widehat{\boldsymbol{u}}(\boldsymbol{x}) + \boldsymbol{v}$ will ensure that the closed-loop system satisfies

$$H_d(\boldsymbol{x}(t)) - H_d(\boldsymbol{x}(0)) \le \int_0^t \boldsymbol{v}^\top(\tau)\boldsymbol{y}(\tau)\mathrm{d}\tau,$$

where $H_d(\boldsymbol{x}) = H(\boldsymbol{x}) + H_a(\boldsymbol{x})$ is the new total energy function. If, furthermore, $\boldsymbol{x}^\star = \arg\min H_d(\boldsymbol{x})$, then an appropriate feedback $\hat{\boldsymbol{u}}(\boldsymbol{x})$ will ensure $\boldsymbol{x}^\star$ is a stable equilibrium of the closed-loop system (with the Lyapunov function being the difference between the stored and the supplied energies $H_d(\boldsymbol{x})$).

Unfortunately, as shown in Ortega et al. (2001), energy-balancing control is stymied by the existence of pervasive dissipation—a term which refers to the existence of dissipative elements whose power does not vanish at the desired equilibrium point. In mechanical systems, where the velocities are driven to zero, pervasive dissipation is not present as the dissipated power equals the product between dissipative forces and the velocities. However, this is no longer the case for most electrical or electromechanical systems where power involves the product of voltages and currents and the latter may be nonzero for nonzero equilibria. In other words, (3) holds if and only if the PDE

$$\underbrace{\nabla H_a^\top(\boldsymbol{x})[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\widehat{\boldsymbol{u}}^\top(\boldsymbol{x})]}_{\dot{H}_a} = -\widehat{\boldsymbol{u}}^\top(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x}), \tag{4}$$

can be solved for $H_a(\boldsymbol{x})$. Since the left hand side is equal to zero at $\boldsymbol{x}^\star$, it is clear that the method is applicable only to systems verifying $\widehat{\boldsymbol{u}}^\top(\boldsymbol{x}^\star)\boldsymbol{h}(\boldsymbol{x}^\star) = 0$.

Several control methodologies have been developed to overcome the so-called *dissipation obstacle*, such as interconnection and damping assignment passivity-based control (IDA–PBC) (Ortega et al., 2002), where the stabilization problem is accomplished by endowing the closed-loop system with a desired port-Hamiltonian structure. In Maschke, Ortega, and van der Schaft (2000), the authors derive a constructive procedure to generate new storage functions for nonzero equilibria in the presence of pervasive dissipation, by modifying the interconnection structure of the closed-loop for port-Hamiltonian systems with constant input control. Additionally, in Jeltsema, Ortega, and Scherpen (2004) an alternative definition of the energy supply for port-Hamiltonian systems when the damping is pervasive is proposed. The associated energy-balancing property is then obtained via a swap of the damping terms. In Ortega, van der Schaft, Castaños, and Astolfi (2008), some extensions of the control by interconnection methodology have been recently introduced to circumvent the dissipation obstacle.

In this paper, we concentrate on the paradigm of *power shaping*, as originally introduced in Ortega et al. (2003) to overcome the dissipation obstacle in nonlinear RLC circuits. As suggested by its name, stabilization is achieved by shaping the power instead of the energy as is done in the aforementioned methodologies. The starting point of power shaping is a description of the circuit in the form (Brayton & Moser, 1964)

$$\mathbf{Q}(\boldsymbol{x})\dot{\boldsymbol{x}} = \nabla P(\boldsymbol{x}) + \mathbf{G}(\boldsymbol{x})\boldsymbol{u}, \tag{5}$$

where $\mathbf{Q} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a *full rank* matrix containing the incremental inductance and capacitance matrices and $P : \mathbb{R}^n \to \mathbb{R}$ is the circuit's *mixed-potential function*, which has units of *power*—cf. Ortega et al. (2003) for further details. A practical advantage of the Brayton–Moser equations is that they naturally describe the dynamics of the system in terms of "easily" measurable quantities, that is, the inductor currents and capacitor voltages, instead of fluxes and charges that are normally used as canonical coordinates in port-Hamiltonian systems.

We make the observation that if $\mathbf{Q}(\boldsymbol{x}) + \mathbf{Q}^\top(\boldsymbol{x}) \le 0$ then the system satisfies the *power balance inequality*

$$P(\boldsymbol{x}(t)) - P(\boldsymbol{x}(0)) \le \int_0^t \boldsymbol{u}^\top(\tau)\tilde{\boldsymbol{y}}(\tau)\mathrm{d}\tau, \tag{6}$$

with $\widetilde{\boldsymbol{y}} = \widetilde{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u})$ and

$$\widetilde{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u}) := -\mathbf{G}^\top(\boldsymbol{x})\mathbf{Q}^{-1}(\boldsymbol{x})[\nabla P(\boldsymbol{x}) + \mathbf{G}(\boldsymbol{x})\boldsymbol{u}]. \tag{7}$$

This property follows immediately pre-multiplying (5) by $\dot{\boldsymbol{x}}^\top$ and then integrating. The mixed-potential function is shaped with the control $\boldsymbol{u} = \widehat{\boldsymbol{u}}(\boldsymbol{x})$ where

$$\mathbf{G}(\boldsymbol{x})\widehat{\boldsymbol{u}}(\boldsymbol{x}) = \nabla P_a(\boldsymbol{x}) \tag{8}$$

for some $P_a : \mathbb{R}^n \to \mathbb{R}$. This yields the closed-loop system $\mathbf{Q}(\boldsymbol{x})\dot{\boldsymbol{x}} = \nabla P_d(\boldsymbol{x})$, with total Lyapunov function $P_d(\boldsymbol{x}) := P(\boldsymbol{x}) + P_a(\boldsymbol{x})$, and the equilibrium will be stable if $\boldsymbol{x}^\star = \arg\min P_d(\boldsymbol{x})$.

Two key observations are, first, that the resulting controller is *power–balancing*, in the sense that the power function assigned to the closed-loop system is the difference between the total power of the system and the power supplied by the controller. Indeed, from (7) and (8) we have that

$$\dot{P}_a = -\widehat{\boldsymbol{u}}^\top(\boldsymbol{x})\widetilde{\boldsymbol{h}}(\boldsymbol{x}, \widehat{\boldsymbol{u}}(\boldsymbol{x})) \tag{9}$$

which, upon integration, establishes the claimed property. Second, in contrast with energy-balancing control, power–balancing is applicable to systems with pervasive dissipation. As opposed to (4), the right hand side of (9) is – because of (5) – zero at the equilibrium, therefore, this equation may be solvable even if $\widehat{\boldsymbol{u}}^\top(\boldsymbol{x}^\star)\boldsymbol{h}(\boldsymbol{x}^\star) \neq 0$.

As indicated above, instrumental for the application of power-shaping is the description of the system in the form (5). Our main contribution is to make the procedure applicable to nonlinear systems described by (1). To this end, we apply Poincaré lemma,[2] which we quote below, to derive necessary and sufficient conditions to achieve this transformation. We prove in this way that the power-shaping problem boils down to the solution of two linear homogeneous PDEs. Despite the intrinsic difficulty of solving PDEs, we show through some physical examples, that the power-shaping procedure yields storage functions which have units of power.

Lemma 1 (*Poincaré Lemma*). *Given $\mathbf{K} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{K} \in \mathcal{C}^1$. There exists $P : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla P(\boldsymbol{x}) = \mathbf{K}(\boldsymbol{x})$ in a neighborhood of $\boldsymbol{x} = \boldsymbol{x}^\star$ if and only if $\nabla\mathbf{K}(\boldsymbol{x}) = (\nabla\mathbf{K}(\boldsymbol{x}))^\top$, where*

$$\nabla\mathbf{K} = \begin{bmatrix} \dfrac{\partial K_1}{\partial x_1} & \cdots & \dfrac{\partial K_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial K_n}{\partial x_1} & \cdots & \dfrac{\partial K_n}{\partial x_n} \end{bmatrix}.$$

[2] We refer the reader to Spivak (1990) for the Poincaré lemma in the differential form language, and to the recent work Yap (2009) for an inductive development using partial differential equations.

2. Power-shaping control of nonlinear systems

The main contribution of this paper is contained in the following proposition.

Proposition 1. *Consider the general nonlinear system (1). Given an equilibrium point $\boldsymbol{x}^\star \in X^\star \subset \mathbb{R}^n$, where $X^\star := \{\bar{\boldsymbol{x}} \in \mathbb{R}^n \mid \boldsymbol{g}^\perp(\bar{\boldsymbol{x}})\boldsymbol{f}(\bar{\boldsymbol{x}}) = \boldsymbol{0}\}$, and $\boldsymbol{g}^\perp(\boldsymbol{x})$ is a full-rank left annihilator of $\boldsymbol{g}(\boldsymbol{x})$.*[3] *Assume*

A.1 *There exists a nonsingular matrix $\mathbf{Q} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ that*
 (i) *solves the partial differential equation*

$$\nabla(\mathbf{Q}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})) = [\nabla(\mathbf{Q}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}))]^\top, \tag{10}$$

 (ii) *and verifies $\mathbf{Q}(\boldsymbol{x}) + \mathbf{Q}^\top(\boldsymbol{x}) \le 0$ in a neighborhood of $\boldsymbol{x}^\star$.*

A.2 *There exists a scalar function $P_a : \mathbb{R}^n \to \mathbb{R}$, (locally) positive definite in a neighborhood of $\boldsymbol{x}^\star$, that verifies*
 (iii) $\boldsymbol{g}^\perp(\boldsymbol{x})\mathbf{Q}^{-1}(\boldsymbol{x})\nabla P_a(\boldsymbol{x}) = \boldsymbol{0}$,
 (iv) $\nabla P_d(\boldsymbol{x}^\star) = 0$, $\nabla^2 P_d(\boldsymbol{x}^\star) > 0$, *with*

$$P_d(\boldsymbol{x}) := P(\boldsymbol{x}) + P_a(\boldsymbol{x}), \tag{11}$$

 and $P(\boldsymbol{x})$ satisfies $\nabla P(\boldsymbol{x}) = \mathbf{Q}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$.

Then, the control law

$$\boldsymbol{u} = \left[\boldsymbol{g}^\top(\boldsymbol{x})\mathbf{Q}^\top(\boldsymbol{x})\mathbf{Q}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})\right]^{-1}\boldsymbol{g}^\top(\boldsymbol{x})\mathbf{Q}^\top(\boldsymbol{x})\nabla P_a(\boldsymbol{x}) \tag{12}$$

ensures $\boldsymbol{x}^\star$ is a (locally) stable equilibrium with Lyapunov function $P_d(\boldsymbol{x})$. Assume, in addition,

A.3 *$\boldsymbol{x}^\star$ is an isolated minimum of $P_d(\boldsymbol{x})$ and the largest invariant set contained in the set*

$$\{\boldsymbol{x} \in \mathbb{R}^n | \nabla^\top P_d(\boldsymbol{x})\left[\mathbf{Q}^{-1}(\boldsymbol{x}) + \mathbf{Q}^{-\top}(\boldsymbol{x})\right]\nabla P_d(\boldsymbol{x}) = 0\}$$

 equals $\{\boldsymbol{x}^\star\}$.

Then, $\boldsymbol{x}^\star$ is an asymptotically stable equilibrium and an estimate of its domain of attraction is given by the largest bounded level set $\{\boldsymbol{x} \in \mathbb{R}^n \mid P_d(\boldsymbol{x}) \le c\}$.

Proof. The first part of the proof consist of showing that, under Assumption A.1, system (1) can be written in the form (5).[4] To this end, invoking Poincaré lemma, we have that (10) is *equivalent* to the existence of $P : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathbf{Q}\boldsymbol{f} = \nabla P. \tag{13}$$

Substituting (1) in the above equation and taking into account the full-rank property of $\mathbf{Q}$ in A.1, we get (5) with $\mathbf{G} := \mathbf{Q}\boldsymbol{g}$. To prove the stability claim, we proceed as follows. Define $\mathbf{G}^\perp := \boldsymbol{g}^\perp\mathbf{Q}^{-1}$, i.e., a full-rank left annihilator of $\mathbf{G}$, and the full-rank matrix

$$\begin{bmatrix}\mathbf{G}^\perp \\ \mathbf{G}^\top\end{bmatrix}. \tag{14}$$

Left-multiplying Eq. (5) by (14) yields

$$\begin{bmatrix}\mathbf{G}^\perp \\ \mathbf{G}^\top\end{bmatrix}\mathbf{Q}\dot{\boldsymbol{x}} = \begin{bmatrix}\mathbf{G}^\perp\nabla P \\ \mathbf{G}^\top\nabla P + \mathbf{G}^\top\mathbf{G}\boldsymbol{u}\end{bmatrix}. \tag{15}$$

Noticing from (11) that $P = P_d - P_a$, Eq. (15) becomes

$$\begin{bmatrix}\mathbf{G}^\perp \\ \mathbf{G}^\top\end{bmatrix}\mathbf{Q}\dot{\boldsymbol{x}} = \begin{bmatrix}\mathbf{G}^\perp(\nabla P_d - \nabla P_a) \\ \mathbf{G}^\top(\nabla P_d - \nabla P_a) + \mathbf{G}^\top\mathbf{G}\boldsymbol{u}\end{bmatrix}.$$

Now, substituting the control action (12) and (iii) of A.2, we finally get the closed-loop dynamics $\mathbf{Q}\dot{\boldsymbol{x}} = \nabla P_d$. Condition iv) of A.2 clearly implies that $\boldsymbol{x}^\star$ is a local minimum point of $P_d(\boldsymbol{x})$. Taking the time derivative of P_d along the closed-loop dynamics, we have $\dot{P}_d = \frac{1}{2}\dot{\boldsymbol{x}}^\top(\mathbf{Q}+\mathbf{Q}^\top)\dot{\boldsymbol{x}}$. Because of (ii) of Assumption A.1, $\dot{P}_d \le 0$ and P_d qualifies as a Lyapunov function. From the closed-loop equation we have $\dot{\boldsymbol{x}} = \mathbf{Q}^{-1}(\boldsymbol{x})\nabla P_d$, hence $\dot{P}_d$ can be rewritten as

$$\dot{P}_d = \frac{1}{2}\nabla^\top P_d\mathbf{Q}^{-\top}(\mathbf{Q}+\mathbf{Q}^\top)\mathbf{Q}^{-1}\nabla P_d. \tag{16}$$

[3] That is, $\boldsymbol{g}^\perp(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x}) = 0$, and $rank(\boldsymbol{g}^\perp(\boldsymbol{x})) = n - m$.

[4] To simplify the expressions, the arguments of all functions will be omitted.

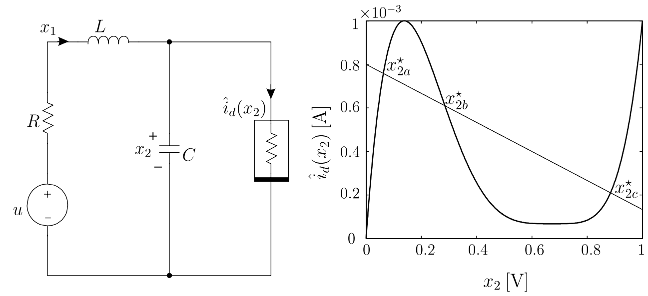

Fig. 1. Tunnel diode circuit and tunnel diode characteristic $\hat{i}_d(x_2)$ with the equilibrium points.

Asymptotic stability follows immediately, with Assumption A.3 and invoking LaSalle's invariance principle. This completes the proof. □

Remark 1. Assumption A.1 of Proposition 1 involves the solution of the PDE (10) subject to the sign constraint (ii)—which may be difficult to satisfy. In Ortega et al. (2003), a constructive procedure is proposed, starting from a pair $\{\mathbf{Q}, P\}$ describing the dynamics (5), explicitly generate alternative pairs $\{\tilde{\mathbf{Q}}, \tilde{P}\}$ that also describe the dynamics, i.e., $\tilde{\mathbf{Q}}(\boldsymbol{x})\dot{\boldsymbol{x}} = \nabla\tilde{P}(\boldsymbol{x}) + \tilde{\mathbf{G}}(\boldsymbol{x})\boldsymbol{u}$, where $\tilde{\mathbf{G}}(\boldsymbol{x}) = \tilde{\mathbf{Q}}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})$.

Remark 2. Clearly, the power-shaping stage of the procedure – after transforming (1) into the form (5) – coincides with the one proposed in Ortega et al. (2002) for energy-shaping using interconnection and damping assignment passivity-based control (IDA–PBC). Additional remarks on the relation between these techniques may be found in Blankenstein (2005); Jeltsema et al. (2004) and in the recent work by Ortega et al. (2008). Indeed, for port-Hamiltonian (pH) systems (van der Schaft, 2000)

$$\dot{\boldsymbol{x}} = [\mathbf{J}(\boldsymbol{x}) - \mathbf{R}(\boldsymbol{x})]\nabla H(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u},$$
$$\boldsymbol{y} = \boldsymbol{g}^\top(\boldsymbol{x})\nabla H(\boldsymbol{x}),$$

with *full-rank* matrix $\mathbf{J}(\boldsymbol{x}) - \mathbf{R}(\boldsymbol{x})$, a trivial solution of (10) is obtained by setting $\mathbf{Q}(\boldsymbol{x}) = [\mathbf{J}(\boldsymbol{x}) - \mathbf{R}(\boldsymbol{x})]^{-1}$. However, in such a case the associated potential function is not modified and remains the total stored energy instead of the power as is desired.

3. Case study I: The tunnel diode

In García-Canseco, Ortega, Scherpen, and Jeltsema (2007), we have applied the power-shaping methodology to stabilize the tunnel diode circuit (Khalil, 1996). The resulting control law is a simple linear (partial) state feedback controller that ensures (robust) global asymptotic stability of the desired equilibrium point. For the sake of illustration we sketch here the main result of García-Canseco et al. (2007).

Consider the circuit depicted in Fig. 1, which represents the approximate behavior of a tunnel diode (Khalil, 1996). The dynamics of the circuit is given by

$$\dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{u}{L}, \tag{17a}$$

$$\dot{x}_2 = \frac{1}{C}x_1 - \frac{1}{C}\hat{i}_d(x_2), \tag{17b}$$

where x_1 is the current through the inductor L and x_2 the voltage across the capacitor C. The function $\hat{i}_d : \mathbb{R} \to \mathbb{R}$ represents the characteristic curve of the tunnel diode depicted in Fig. 1. The assignable equilibrium points of the circuit are determined by $x_1^\star = \hat{i}_d(x_2^\star)$, with the corresponding constant control $u^\star = R\hat{i}_d(x_2^\star) + x_2^\star$.

Proposition 2 (*García-Canseco et al., 2007*). *Consider the dynamic equations of the tunnel diode circuit (17). Assume:*

A.4 $\min_{x_2} \hat{i}'_d(x_2) > -\frac{RC}{L}$.

Then, the power-shaping procedure of Proposition 1 yields the linear (partial) state feedback control

$$u = -k(x_2 - x_2^\star) + u^\star, \tag{18}$$

which globally asymptotically stabilizes the equilibrium point $\boldsymbol{x}^\star$ with Lyapunov function

$$P_d(\boldsymbol{x}) = \int_0^{x_2} \hat{i}_d(\tau)\mathrm{d}\tau + \frac{L}{2RC}(x_1 - \hat{i}_d(x_2))^2 + \frac{k}{2R}(x_2 - x_2^\star)^2 + \frac{1}{2R}(x_2 - u^\star)^2, \tag{19}$$

provided k satisfies

$$k > -[1 + R\hat{i}_d'(x_2^\star)] > 0. \tag{20}$$

Proof (*Sketch*). It can be shown that a solution matrix $\mathbf{Q}(\boldsymbol{x})$ to the PDE (10) is given by

$$\mathbf{Q}(x_2) = \begin{bmatrix} 0 & \frac{L}{R} \\ -\frac{L}{R} & -C - \frac{L}{R}\hat{i}_d'(x_2) \end{bmatrix}, \tag{21}$$

which is invertible for all x_2 and, under Assumption A.4,[5] verifies $\mathbf{Q} + \mathbf{Q}^\top \leq 0$ for all x_2.

Condition (iii) of Proposition 1 becomes $\frac{\partial P_a}{\partial x_1} = 0$, indicating that P_a cannot be a function of x_1. Hence, we fix $P_a = \Psi(x_2)$, where $\Psi : \mathbb{R} \to \mathbb{R}$ is an arbitrary differentiable function that must be chosen so that $P_d(\boldsymbol{x}) = P(\boldsymbol{x}) + P_a(x_2)$ has a minimum at $\boldsymbol{x}^\star$. To obtain $P_d(\boldsymbol{x})$ from (11) we need to compute $P(\boldsymbol{x})$ from $\nabla P(\boldsymbol{x}) = \mathbf{Q}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$. By virtue of the gradient vector field pre-supposed by Poincaré Lemma, the path of integration is free and we get $P_d(\boldsymbol{x})$ as

$$P_d(\boldsymbol{x}) = \int_0^{x_2} \hat{i}_d(\tau)\mathrm{d}\tau + \frac{L}{2RC}(x_1 - \hat{i}_d(x_2))^2 + \frac{1}{2R}x_2^2 + \Psi(x_2), \tag{22}$$

where we have used $\hat{i}_d(0) = 0$. Notice that the above equation has indeed units of power. By choosing the simple quadratic function

$$\Psi(x_2) = \frac{k}{2R}(x_2 - x_2^\star)^2 - \frac{x_2 u^\star}{R},$$

we have that condition (iv) of Assumption A.2 is satisfied provided (20) holds. Thus, the resulting Lyapunov function $P_d(\boldsymbol{x})$ is given, after completing the square, by (19). $P_d(\boldsymbol{x})$ has a global minimum at $\boldsymbol{x}^\star$. From (12) we obtain the simple linear state feedback (18). This completes the proof. □

Remark 3. The simplicity of the controller (18), which results from the effective exploitation of the physical structure of the system, should be contrasted with the daunting complexity of the "solution" proposed in Rodriguez and Boyd (2005), or the design based on linear approximations Khalil (1996). This linear controller should be also compared with the control law obtained by following the IDA–PBC methodology (Ortega et al., 2002), since the application of IDA–PBC without a priori knowledge of the Lyapunov function P_d is not evident.

4. Case study II: Two-tank system

Consider the two-tank system depicted in Fig. 2. Using Torricelli's law, the dynamics of the system can be written as (Johnsen & Allgower, 2006)

5 Assumption A.4 is made for simplicity and can be easily replaced by the knowledge of a lower bound on $\hat{i}_d'(x_2)$. We notice that Assumption A.4 coincides with the constraint given in Moser (1960) to exclude the appearance of limit cycles in these kinds of circuits.

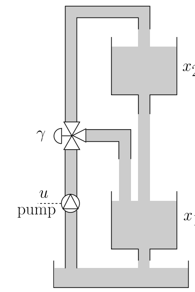

Fig. 2. Two-tank system.

$$\dot{x}_1 = -\frac{a_1\sqrt{2gx_1}}{A_1} + \frac{a_2\sqrt{2gx_2}}{A_1} + \frac{\gamma}{A_1}u \tag{23a}$$

$$\dot{x}_2 = -\frac{a_2\sqrt{2gx_2}}{A_2} + \frac{1-\gamma}{A_2}u, \tag{23b}$$

where the state variables $x_1 > 0$ and $x_2 > 0$ represent the water level in the lower and upper tank, respectively. The system parameters are all positive constants, where g is the gravitational constant and, A_i and a_i, with $i = 1, 2$, are the cross sections of the tanks and the outlet holes, respectively. The valve parameter is the constant $\gamma \in [0, 1]$, with $\gamma = 0$ if the valve is fully open, i.e., all the water is directed to the upper tank, and $\gamma = 1$ if the valve is closed.

For $\gamma \in [0, 1)$, the assignable equilibrium points of the system are restricted to the line $x_1^\star = (a_2^2 x_2^\star)/(a_1^2(\gamma - 1)^2)$, with the corresponding constant control $u^\star = (a_2\sqrt{2gx_2^\star})/(1-\gamma)$. For $\gamma = 1$, the equilibrium points are $x_2^\star = 0$, $u^\star = a_1\sqrt{2gx_1^\star}$. The control objective is to stabilize a given equilibrium point $\boldsymbol{x}^\star = [x_1^\star \; x_2^\star]^\top$.

Following the power-shaping procedure outlined in Section 1, we have the following result for $\gamma \in [0, 1)$.

Proposition 3. *Consider the two-tank system (23) in a closed-loop with the linear state feedback controller*

$$u = -k_1(x_1 - x_1^\star) - k_2(x_2 - x_2^\star) + u^\star. \tag{24}$$

If the tuning parameters k_1 and k_2 satisfy

$$k_1 > 0, \quad k_2 > \frac{(1-\gamma)A_2}{4A_1}k_1, \tag{25}$$

then for all $\gamma \in [0, 1)$, $\boldsymbol{x}^\star$ is a globally asymptotically stable equilibrium of the closed-loop system with the Lyapunov function

$$P_d(\boldsymbol{x}) = \frac{2a_1k_1\sqrt{2g}}{3A_1}x_1^{\frac{3}{2}} + \frac{2a_2k_2\sqrt{2g}}{3(1-\gamma)A_1}x_2^{\frac{3}{2}} + \frac{1}{2A_1}\left[k_1(x_1 - x_1^\star) + k_2(x_2 - x_2^\star)\right]^2 - \frac{u^\star}{A_1}\left[k_1(x_1 - x_1^\star) + k_2(x_2 - x_2^\star)\right]. \tag{26}$$

Proof. Fixing the matrix $\mathbf{Q}$ constant, i.e., $\mathbf{Q} = \{q_{ij}\}$, with $i, j = 1, 2$, a solution to the PDE (10) yields $q_{12} = \frac{A_2 q_{11}}{A_1}$, $q_{21} = 0$, with $q_{11} \neq 0$ and $q_{22} \neq 0$ free parameters. Hence, $\mathbf{Q}$ is invertible. Moreover, $\mathbf{Q}$ verifies $\mathbf{Q} + \mathbf{Q}^\top < 0$ if and only if $q_{11} < 0$ and $q_{22} < \frac{A_2^2 q_{11}}{4A_1^2}$. To simplify the computations, let $q_{11} = -k_1$, $q_{22} = -\frac{A_2 k_2}{A_1(1-\gamma)}$, where k_1 and k_2 are positive constants.

The resulting function $P(\boldsymbol{x})$,

$$P(\boldsymbol{x}) = \frac{2a_1k_1\sqrt{2g}}{3A_1}x_1^{\frac{3}{2}} + \frac{2a_2k_2\sqrt{2g}}{3(1-\gamma)A_1}x_2^{\frac{3}{2}}, \tag{27}$$

can be seen as a power-like function. Indeed, by Torricelli's law, we know that the terms $\sqrt{2gx_1}$ and $\sqrt{2gx_2}$ have the units of velocity, hence we define $v_1 := \sqrt{2gx_1}$ and $v_2 := \sqrt{2gx_2}$.

If we fix the units of k_1 and k_2 to kg/s^2 so that the terms $k_1x_1 =: f_1$ and $k_2x_2 =: f_2$ have units of force, and defining the unitless constants $\beta_1 = \frac{2a_1}{3A_1}$, $\beta_2 = \frac{2a_2}{3(1-\gamma)A_2}$, the mixed-potential function (27) can be recast into

$$P(\cdot) = \beta_1 f_1 v_1 + \beta_2 f_2 v_2,$$

which clearly exhibits the products force $\times$ velocity. Furthermore, by choosing $\mathbf{g}^{\perp} = [-(1-\gamma)/A_2 \quad \gamma/A_1]$, condition (iii) of Proposition 1 becomes

$$\frac{1}{k_1}\frac{\partial P_a}{\partial x_1} - \frac{1}{k_2}\frac{\partial P_a}{\partial x_2} = 0. \tag{28}$$

The solution of (28) yields $P_a(\boldsymbol{x}) = \Psi(\frac{k_1}{k_2}x_1 + x_2)$, where $\Psi : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary differentiable function that must be chosen so that $P_d(\boldsymbol{x}) = P(\boldsymbol{x}) + P_a(\boldsymbol{x})$ has a minimum at $\boldsymbol{x}^\star$. Computing $P_d(\boldsymbol{x})$ from (11), we obtain $P_d(\boldsymbol{x}) = \frac{2a_1k_1\sqrt{2g}}{3A_1}x_1^{\frac{3}{2}} + \frac{2a_2k_2\sqrt{2g}}{3(1-\gamma)A_1}x_2^{\frac{3}{2}} + \Psi\left(\frac{k_1}{k_2}x_1 + x_2\right)$, which should satisfy $\nabla P_d(\boldsymbol{x}^\star) = 0$ and $\nabla^2 P_d(\boldsymbol{x}^\star) > 0$. As in the previous example, one possibility is to select a quadratic function of the form

$$\Psi(z(x_1, x_2)) = \frac{\kappa}{2}(z - z^\star)^2 + \mu(z - z^\star),$$

where $z = \frac{k_1}{k_2}x_1 + x_2$, $z^\star = \frac{k_1}{k_2}x_1^\star + x_2^\star$, $\kappa > 0$, and μ are scalars. Some simple calculations show that the minimum is assigned, i.e., $\nabla P_d(\boldsymbol{x}^\star) = 0$, if we set $\mu = -\frac{k_2u^\star}{A_1}$. The Hessian $\nabla^2 P_d$ is calculated as

$$\nabla^2 P_d = \begin{bmatrix} \dfrac{k_1a_1\sqrt{2g}}{2A_1\sqrt{x_1}} + \dfrac{\kappa k_1^2}{k_2^2} & \dfrac{\kappa k_1}{k_2} \\ \dfrac{\kappa k_1}{k_2} & \dfrac{k_2a_2\sqrt{2g}}{2A_1(1-\gamma)\sqrt{x_2}} + \kappa \end{bmatrix},$$

which is positive definite for all positive $\boldsymbol{x}$ provided (25) holds. Setting $\kappa = \frac{k_2^2}{A_1}$ yields the Lyapunov function (26), which has a unique minimum at $\boldsymbol{x}^\star$. By virtue of the negative definiteness of $\mathbf{Q} + \mathbf{Q}^\top$, $\dot{P}_d < 0$ (cf. (9)). Thus, the simple linear state feedback (24) is globally asymptotically stabilizing. □

Remark 4. The controller (24) was also derived using the IDA–PBC methodology in Johnsen and Allgower (2006), and using a control by interconnection approach in Ortega et al. (2008). We refer to Johnsen and Allgower (2006) for simulations and experimental results.

5. Concluding remarks

We have extended the *power-shaping* control design methodology, proposed in Ortega et al. (2003) for nonlinear RLC circuits, to general nonlinear systems. The success of the method relies on the solution of a PDE, which allows us to write the original dynamics in terms of the Brayton–Moser equations. In spite of the intrinsic difficulty of solving PDEs, we have illustrated this technique with physical examples, where the power-shaping methodology yields storage functions which have units of power.

Among the issues that remain open and are currently being explored are the solvability of the PDE (10) – subject to the sign constraint ii) of Asumption A.1 – for different kinds of nonlinear systems and other applications of power-shaping, for instance, to general electro-mechanical and mechanical systems. Recently, the power-shaping methodology has been successfully applied to the set point regulation problem of a micro-electromechanical system (García-Canseco, Jeltsema, Scherpen, & Ortega, 2008), and in chemical systems to control the exothermic continuous stirred tank reactor (Favache & Dochain, 2009). In the spirit of Fujimoto and Sugie (2001); Viola, Ortega, Banavar, Acosta, and Astolfi (2007), we also want to explore the effects of a coordinate change in the solvability of the PDE.

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