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*Published in:*  
IEEE Transactions on Automatic Control

*DOI:*  
[10.1109/TAC.2018.2811787](https://doi.org/10.1109/TAC.2018.2811787)

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*Document Version*  
Final author's version (accepted by publisher, after peer review)

*Publication date:*  
2018

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*  
Kawano, Y., & Scherpen, J. M. A. (2018). Structure Preserving Truncation of Nonlinear Port Hamiltonian Systems. *IEEE Transactions on Automatic Control*, 63(12), 4286-4293.  
<https://doi.org/10.1109/TAC.2018.2811787>

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# Structure Preserving Truncation of Nonlinear Port Hamiltonian Systems

Yu Kawano, *Member*, and Jacquelin M.A. Scherpen, *Senior Member*

**Abstract**—In this paper, we present a novel balancing method for nonlinear port Hamiltonian systems based on the Hamiltonian and the controllability function. This corresponding balanced truncation method results in a reduced order model that is still in port Hamiltonian form in contrast to the traditional balanced truncation method based on the controllability and observability functions.

## I. INTRODUCTION

Port Hamiltonian systems (PHSs) [2]–[4] form an important class of passive state-space systems, and many physical systems such as electro-mechanical systems can be represented as PHSs. Furthermore, various control methods are developed for this type of systems, relying on the fact that interconnecting Hamiltonian systems with each other, results in a closed loop PHS again. Besides, the corresponding control methods are generally physically interpretable and intuitively it provides a clear framework. However, models of physical systems can easily be high dimensional, e.g., large scale circuits, discretized PDE models, such as from flexible beams, etc., which makes analysis and control difficult. Therefore, it is important to study model reduction methods. To benefit from the structure of a PHS, it is natural to develop a model reduction method preserving the PH structure. Structure preserving model reduction methods for PHSs have been studied in several ways such as through Kalman decomposition [5], [6], interpolation [7], moment matching [8]–[10], and modified balanced truncation for limited subclasses [11], [12]. In particular, [6], [9], [11], [12] study nonlinear PHSs.

In contrast to the above methods, the traditional nonlinear balanced truncation method proposed by [13] and developed further by [14], [15] does not preserve the PH structure. The paper [11] studies balanced truncation via a supply and storage function and shows that the structure is preserved if these two functions satisfy specific conditions. The paper [12] studies balanced truncation of the controllability and observability functions and shows that the structure is preserved if the Hamiltonian is identical to a weighted controllability or observability function. Therefore, these methods are applicable only for specific PHSs.

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This work of Y. Kawano was partly supported by JST CREST Grant Number JPMJCR15K2, Japan.

Some preliminary results for the linear case of this paper were presented at the 22th international symposium on mathematical theory of networks and systems, July 2016 [1].

In this paper, we establish a balancing procedure for PHSs based on the controllability function and the internal energy given by the Hamiltonian, i.e., a combination of balancing and modal analysis. That is, for standard balancing for stable systems, [16], nonlinear one, i.e., [13] the controllability and observability Gramians are used, whereas in modal analysis the eigen modes of the system related to the internal energy are considered, [17]. To the best of our knowledge, this has not been done elsewhere. When using such procedure for truncation, the port Hamiltonian structure is preserved naturally, whereas this is not the case in any other balancing procedure. In [1] we have obtained some preliminary result solely focusing on the linear case. Here we show that in the linear case our method can be modified for gradient systems [18], another class of systems arising from physics. In fact, for a class of linear gradient systems our method gives the same balanced realization as the traditional balanced realization based on controllability and observability Gramians. Furthermore, the above mentioned modification of our method results in balancing for all linear gradient systems. Based on this fact, we show that for a special class of passive gradient systems such as RL networks, our method for PHSs is equivalent to traditional balancing. From this fact, our method for linear systems can be viewed as an extension of the traditional balancing method to preserve the PH structure.

The remainder of this paper is organized as follows. Section II shows the nonlinear PHS and summarizes the results on nonlinear controllability function in [13]. Section III presents our PH structure preserving truncation method based on the Hamiltonian and controllability function. In Section IV, we proceed with further analysis of our method in the linear case. We establish a connection between our method and traditional balanced truncation for specific passive gradient systems. Finally, Section V summarizes this paper.

## II. PRELIMINARIES

### A. Port Hamiltonian Systems

Consider a nonlinear PHS [4] with states  $x \in \mathbb{R}^n$ , and inputs and outputs  $u, y \in \mathbb{R}^m$ , respectively, as follows.

$$\begin{cases} \dot{x} = (J(x) - R(x)) \frac{\partial^T H(x)}{\partial x} + g(x)u, \\ y = g^T(x) \frac{\partial^T H(x)}{\partial x}, \end{cases} \quad (1)$$

where  $J(x) = -J^T(x)$  is the interconnection matrix,  $R(x) = R^T(x) \geq 0$  is the damping matrix, and  $H : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $H(x) > 0$  is the Hamiltonian representing the total energy in the system. Suppose that  $\partial H(0)/\partial x = 0$ ,  $\partial^2 H(0)/\partial x^2 > 0$ , and the system is asymptotically stable at the origin.

An important property of a PHS is that it is a passive system with the Hamiltonian as storage function. It can be seen by computing  $\frac{dH(x)}{dt} = u^T y - \frac{\partial^T H(x)}{\partial x} R(x) \frac{\partial H(x)}{\partial x} \leq u^T y$ . Therefore, when applying model reduction, preserving the PH structure implies preserving passivity.

### B. Controllability Function

In this paper, we consider a truncation based on the Hamiltonian and controllability function. Since PHS (1) is an input-affine system, the results of [13] about the controllability function are applicable. In order to be self-contained, we summarize these results.

The controllability function for system (1) is defined as follows [13].

$$L_C(x_0) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty) = 0, x(0) = x_0}} \frac{1}{2} \int_{-\infty}^0 \|u(t)\|^2 dt.$$

Under the assumption that  $L_C$  exists and is smooth around the origin, and that system (1) is asymptotically stable at the origin, it follows that  $L_C$  is the (local) unique anti-stabilizing solution to the following Hamilton-Jacobi equation (HJE) [13]:

$$\frac{\partial L_C(x)}{\partial x} (J(x) - R(x)) \frac{\partial^T H(x)}{\partial x} + \frac{1}{2} \frac{\partial L_C(x)}{\partial x} g(x) g^T(x) \frac{\partial^T L_C(x)}{\partial x} = 0. \quad (2)$$

Anti-stabilizing means that the inverse-time system given by

$$\dot{x}^- = - (J(x^-) - R(x^-)) \frac{\partial^T H(x^-)}{\partial x^-} - g(x^-) g^T(x^-) \frac{\partial^T L_C(x^-)}{\partial x^-} \quad (3)$$

is asymptotically stable at the origin. It follows that  $L_C(x) > 0$  is equivalent to asymptotic stability of (3) [13]. In [10], asymptotic stability of the inverse-time system (3) is called (local) asymptotic reachability of PHS (1), and it is shown that asymptotic reachability is a sufficient condition for (local strong) accessibility [19] of (1).

*Remark 2.1:* Asymptotic reachability of the asymptotically stable system implies controllability of the linearized system. This immediately follows from the fact that the inverse-time system, i.e., an unstable system is stabilizable by continuous feedback.  $\triangleleft$

## III. MAIN RESULTS

Nonlinear balanced truncation for asymptotically stable systems is based on the controllability and observability functions and preserves stability, controllability, and observability [14], [15] but not the PH structure. Here, we present a balanced truncation method based on the Hamiltonian and controllability functions. We use the following assumptions.

*Assumption 3.1:* For PHS (1) with  $J(x) = -J^T(x)$ ,  $R(x) = R^T(x) \geq 0$ , and  $H(x) > 0$ , assume that

- 1)  $\partial H(0)/\partial x = 0$  and  $\partial^2 H(0)/\partial x^2 > 0$ .
- 2) Its linearized system at the origin is asymptotically stable.
- 3)  $L_C(x) > 0$  exists and is smooth around the origin.  $\triangleleft$

*Remark 3.2:* The inverse of the controllability Gramian of the linearized system  $\partial^2 L_C(0)/\partial x^2$  is assumed to be positive

definite for nonlinear balancing [13], [15]. However, when the linearized system is asymptotically stable, and  $L_C(x) > 0$ , it follows from Remark 2.1 that the linearized systems is controllable, and consequently  $\partial^2 L_C(0)/\partial x^2 > 0$ .  $\triangleleft$

### A. A Semi-Balanced Realization

In [13]–[15], the input normal form is used for setting up a nonlinear balancing procedure, which is a realization such that the controllability function  $L_C(x)$  and observability function  $L_o(x)$  respectively become  $L_C(x) = x^T x/2$  and  $L_o(x) = x^T \Lambda(x)x/2$  with a diagonal  $\Lambda(x)$ . In this paper, we follow a similar procedure, but we now use the controllability function and the Hamiltonian, and the Hamiltonian is normalized. If in addition the Hamiltonian and the controllability function are simultaneously brought into a diagonal form we call this a *semi-balanced* realization.

As the first step for PH structure preserving semi-balanced truncation, we transform the Hamiltonian into the normalized form  $H(x) = x^T x/2$ . Note that the PH structure is preserved under a coordinate transformation. After the coordinate transformation by diffeomorphic  $z = \varphi(x)$  ( $\varphi(0) = 0$ ), system (1) becomes

$$\begin{cases} \dot{z} = (J_z(z) - R_z(z)) \frac{\partial^T H_z(z)}{\partial z} + g_z(z)u, \\ y = g_z^T(z) \frac{\partial^T H_z(z)}{\partial z}, \end{cases}$$

where

$$\begin{aligned} H_z(\varphi(x)) &:= H(x), \quad J_z(\varphi(x)) := \frac{\partial \varphi(x)}{\partial x} J(x) \frac{\partial^T \varphi(x)}{\partial x}, \\ R_z(\varphi(x)) &:= \frac{\partial \varphi(x)}{\partial x} R(x) \frac{\partial^T \varphi(x)}{\partial x}, \quad g_z(\varphi(x)) := \frac{\partial \varphi(x)}{\partial x} g(x). \end{aligned}$$

Note that  $J_z(z) = -J_z^T(z)$ ,  $R_z(z) = R_z^T(z) \geq 0$ ,  $H_z(z) > 0$ ,  $\partial H_z(0)/\partial z = 0$ , and  $\partial^2 H_z(0)/\partial z^2 > 0$ . Because of the assumption  $\partial^2 H(0)/\partial x^2 > 0$ , Morse's lemma [20] implies that there exists a local coordinates  $z$  with smooth  $z = \varphi(x)$  ( $\varphi(0) = 0$ ) such that  $H_z(z) = \frac{1}{2} z^T z$ . In such a coordinate, the PHS becomes

$$\begin{cases} \dot{z} = (J_z(z) - R_z(z))z + g_z(z)u, \\ y = g_z^T(z)z. \end{cases} \quad (4)$$

For representation (4), the controllability function is given by  $L_{C_z}(z) := L_C(\varphi^{-1}(z))$ . Since  $z = \varphi(x)$  ( $\varphi(0) = 0$ ) is smooth, the controllability function  $L_{C_z}(z)$  in the  $z$ -coordinates is again smooth and positive definite, and  $\partial^2 L_{C_z}(0)/\partial z^2 > 0$ . In [13], it is shown that for any controllability function  $L_{C_z}(z)$  satisfying  $\partial^2 L_{C_z}(0)/\partial z^2 > 0$ , there exists a local coordinate transformation

$$z = \psi(z_H) := T_H(z)z_H, \quad T_H(z)T_H^T(z) = I_n \quad (5)$$

such that

$$\begin{aligned} L_{CH}(z_H) &:= L_{C_z}(\psi(z_H)) = (1/2)z_H^T \Sigma^{-1}(z_H)z_H, \quad (6) \\ \Sigma(z_H) &:= \text{diag}\{\sigma_1(z_H), \dots, \sigma_n(z_H)\}, \end{aligned}$$

where  $\sigma_i(z_H)$  is smooth for all  $i$ , and  $\sigma_1(z_H) \geq \dots \geq \sigma_n(z_H) > 0$  around the origin. In general this order depends on state space regions as for traditional balancing found in [14]. This could be further “decoupled” per coordinates as in [15]. The paper [21] gives a similar but computationally

different method for obtaining the form (6), which can be used in stead of the method in [13].

In the  $z_H$ -coordinates, we notice the following.

*Lemma 3.3:* In the  $z_H$  coordinates the Hamiltonian is still in its normalized form, i.e.,  $H_{z_H} = (1/2)z_H^T z_H$ .

*Proof:* Since  $T_H(z)$  is an orthonormal matrix, it follows immediately. ■

In summary, for the local coordinate transformation  $x = \Phi(z_H) := \varphi^{-1}(\psi(z_H))$ , we have the following theorem.

*Theorem 3.4:* Under assumption 3.1, there exists a local coordinate transformation  $x = \Phi(z_H)$  ( $\Phi(0) = 0$ ) such that

$$\begin{cases} \dot{z}_H = (J_{z_H}(z_H) - R_{z_H}(z_H))z_H + g_{z_H}(z_H)u, \\ y = g_{z_H}^T(z_H)z_H. \end{cases}, \quad (7)$$

is a PHS, and

$$L_{CH}(z_H) := L_C(\Phi(z_H)) = (1/2)z_H^T \Sigma^{-1}(z_H)z_H$$

with diagonal  $\Sigma(z_H)$  as in (6). ◁

In the  $z_H$ -coordinates, the controllability function and the internal energy (Hamiltonian) are brought in an almost semi-balanced form (for a fully semi-balanced form, a transformation of the form  $z_{B,i} = \sigma_i(z_H)^{1/4}z_{H,i}$  is necessary). In particular, they are brought into an “energy-normal/input balanced” form. This means that a small  $\sigma_i(z_H)$  implies that the state  $z_{H,i}$  is badly controllable, and hardly contributes to the internal energy captured in the Hamiltonian.

### B. Structure Preserving Truncation

Suppose that  $\sigma_k \gg \sigma_{k+1}$  for  $k < n$ . Then,  $z_{H,k}$  is more important than  $z_{H,k+1}$  in terms of the balance between the Hamiltonian and the controllability function. We partition the system in the  $z_H$ -coordinates as follows:

$$\begin{aligned} z_H &= \begin{bmatrix} z_{H,a} \\ z_{H,b} \end{bmatrix}, \quad g_{z_H} = \begin{bmatrix} g_{z_H,a}(z_{H,a}, z_{H,b}) \\ g_{z_H,b}(z_{H,a}, z_{H,b}) \end{bmatrix}, \\ J_{z_H} &= \begin{bmatrix} J_{z_H,a}(z_{H,a}, z_{H,b}) & J_{z_H,ab}(z_{H,a}, z_{H,b}) \\ -J_{z_H,ab}^T(z_{H,a}, z_{H,b}) & J_{z_H,b}(z_{H,a}, z_{H,b}) \end{bmatrix}, \\ R_{z_H} &= \begin{bmatrix} R_{z_H,a}(z_{H,a}, z_{H,b}) & R_{z_H,ab}(z_{H,a}, z_{H,b}) \\ R_{z_H,ab}^T(z_{H,a}, z_{H,b}) & R_{z_H,b}(z_{H,a}, z_{H,b}) \end{bmatrix}, \end{aligned}$$

where both  $J_{H,a}(z_{H,a}, z_{H,b})$  and  $J_{H,b}(z_{H,a}, z_{H,b})$  are skew symmetric, and both  $R_{H,a}(z_{H,a}, z_{H,b})$  and  $R_{H,b}(z_{H,a}, z_{H,b})$  are symmetric positive semidefinite.

A possibility to reduce the number of states is by truncation, i.e., to put  $z_{H,k+1} = 0, \dots, z_{H,n} = 0$ , i.e.,  $z_{H,b} = 0$ . This model reduction step is structure preserving, i.e.,

*Theorem 3.5:* A reduced order model of PHS (7) given by

$$\begin{aligned} \dot{z}_r &= (J_{z_H,a}(z_r, 0) - R_{z_H,a}(z_r, 0))z_r + g_{z_H,a}(z_r, 0)u, \\ y_r &= g_{z_H,a}^T(z_r, 0)z_r \end{aligned} \quad (8)$$

is again a PHS.

*Proof:* This is clear because we have  $J_{z_H,a}(z_r, 0) = -J_{z_H,a}^T(z_r, 0)$  and  $R_{z_H,a}(z_r, 0) = R_{z_H,a}^T(z_r, 0) \geq 0$ . ■

Our truncation method preserves the PH structure, and thus also the passivity of the system. It is however unclear if properties like controllability, observability or stability are preserved under the proposed truncation method. In fact,

we do not require observability even for the original PHS. Nevertheless, if the reduced order model is not observable, we can reduce the system further while preserving the PHS structure to the observable subsystem, [5].

*Remark 3.6:* It is clear that the reduced order model is in PHS form again with Hamiltonian  $H_a = \frac{1}{2}z_a^T z_a$ . However, the “diagonal” structure of the controllability function may not be preserved, thus the “energy-normal/input balanced” form is not necessarily preserved. This can only be proven to be preserved under additional assumptions on the structure of the system, similarly to [13]. ◁

*Remark 3.7:* For stability, if  $R(z) > 0$  then PHS (1) is asymptotically stable with the Hamiltonian as Lyapunov function. From the Schur complement,  $R_{z_H}(z_H) > 0$  implies  $R_{z_H,a}(z_a, 0) > 0$ . Therefore, the reduced order model (8) is asymptotically stable. The case when  $R(z) \geq 0$  but PHS (1) is asymptotically stable is discussed in Section IV-B. ◁

*Remark 3.8:* One could also consider to develop this method for a combination of the Hamiltonian and the observability function. However, in contrast to the controllability function, the observability function is obtained integrating over future time, thus resulting in another type of transformation, and thus not interpretable as balancing. This point is clarified further for the linear case treated in the next section. ◁

In general, the reduced order model does not have the original physical interpretation, and a mass-spring-damper or RLC interpretation may not be possible for the reduced order model as for other PH structure preserving methods in [5]–[12]. The main reason for this is that we balance the internal energy (the Hamiltonian) which is like modal analysis, [17], with the controllability function. The controllability function is related to the inputs and thus not related to the internal physics. However, we do preserve the interpretation that the Hamiltonian presents the internal energy. Furthermore, a new interconnection and damping matrix are obtained.

### C. Examples

*Example 3.9:* Consider the following mass-spring-damper system represented as a PHS,

$$\begin{aligned} x &:= [\dot{\xi}_1 \quad \dot{\xi}_2 \quad \xi_2]^T, \quad H(x) := x^T x / 2, \\ J(x) &:= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad g(x) := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ R(x) &:= \begin{bmatrix} D(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad D(x) := \begin{bmatrix} 1 & -1 \\ -1 & 2 + x_2^2 \end{bmatrix}. \end{aligned}$$

This system is already in the form (4), i.e. here  $x = z$ .

The fourth-order Taylor approximation of the solution to the HJE (2) for the controllability function can be computed as

$$L_{Cz}(z) = \frac{1}{2}z^T \begin{bmatrix} 6 + z_2^2 & -12 - 5z_2^2 & -4 - z_2^2 \\ -12 - 5z_2^2 & 34 + 21z_2^2 & 8 + 4z_2^2 \\ -4 - z_2^2 & 8 + 4z_2^2 & 6 + 2z_2^2 \end{bmatrix} z.$$

A third-order Taylor approximation of  $z_H = T_H(z)z$ , which diagonalize  $L_{Cz}$ , is

$$T_H = \begin{bmatrix} -0.3419 - 0.06660z_2^2 & 0.2482 - 0.03257z_2^2 \\ 0.9064 - 0.03404z_2^2 & 0.3419 - 0.06663z_2^2 \\ 0.2482 - 0.03259z_2^2 & -0.9064 + 0.034052z_2^2 \end{bmatrix}$$

$$\begin{bmatrix} 0.9064 + 0.034038z_2^2 \\ 0.2482 - 0.03254z_2^2 \\ 0.3419 - 0.06662z_2^2 \end{bmatrix}.$$

After the coordinate transformation, a fourth-order Taylor approximation of the controllability function becomes

$$\begin{aligned} L_{CH}(z_H) &= (1/2)z_H^T \Sigma^{-1}(z_H)z_H, \\ \Sigma(z_H) &= \text{diag}\{20.36 + 11.28z_{H,2}^2, 2.039 + 0.6408z_{H,2}^2, \\ &\quad 0.6022 + 0.07931z_{H,2}^2\}, \end{aligned}$$

and the Hamiltonian is  $H_{z_H} = (1/2)z_H^T z_H$ .

The second-order reduced model in Theorem 3.5 is a PHS with  $H_{z_H,a} = z_r^T z_r/2$ , and

$$\begin{aligned} J_{z_H,a}(z_r) &= \begin{bmatrix} 0 \\ -0.3419 + 0.06660z_{r,2}^2 \\ 0.3419 - 0.06660z_{r,2}^2 \\ 0 \end{bmatrix}, \\ R_{z_H,a}(z_r) &:= \begin{bmatrix} 0.4948 + 0.1331z_{r,2}^2 \\ 0.02319 + 0.07337z_{r,2}^2 \\ 0.02319 + 0.07337z_{r,2}^2 \\ 0.1257 + 0.06495z_{r,2}^2 \end{bmatrix}, \\ g_{z_H,a}(z_r) &= \begin{bmatrix} 0.9064 + 0.03404z_{r,2}^2 \\ 0.2482 - 0.03257z_{r,2}^2 \end{bmatrix}. \end{aligned}$$

Therefore, the PH structure is preserved.

*Example 3.10:* Next, we apply our method for a high-dimensional system. Consider a nonlinear mass-spring-damper system consisting  $q$  masses in Fig. 1, where  $\xi_i$  is the position of the mass  $i$ . The control input is added to the first mass, and  $m_i = 1$  for all  $i = 1, \dots, q$ . The damping and spring between masses  $i$  and  $i + 1$  are  $(\xi_{i+1} - \xi_i) + (\xi_{i+1} - \xi_i)^3/4$  and  $(\xi_{i+1} - \xi_i) + (\xi_{i+1} - \xi_i)^3/4$  for  $i = 1, \dots, q - 1$ , respectively, and the damping and spring on mass  $q$  are  $\xi_q$  and  $\xi_q$ , respectively. Note that the considered cubic damping and spring naturally arises in the real modeling and analysis; for instance see [22], [23]. This system can be represented as a PHS with the Hamiltonian

$$\begin{aligned} H(x) &= \sum_{i=1}^{q-1} \left( \frac{1}{2}(\xi_i - \xi_{i+1})^2 + \frac{1}{16}(\xi_i - \xi_{i+1})^4 \right) + \frac{\xi_q^2}{2} + \sum_{i=1}^q \frac{\xi_i^2}{2} \end{aligned}$$

in the coordinates  $x = [\xi_1 \ \dots \ \xi_q \ \dot{\xi}_1 \ \dots \ \dot{\xi}_q]$ . For instance, when  $q = 2$ , the corresponding  $J(x)$ ,  $R(x)$ , and  $g(x)$  are

$$J(x) := \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}, \quad R(x) := \begin{bmatrix} 0 & 0 \\ 0 & R_{22}(x) \end{bmatrix},$$

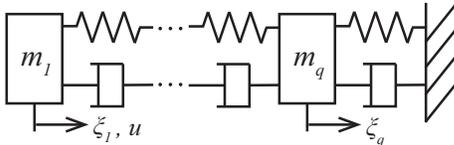


Fig. 1. Nonlinear mass-spring-damper system

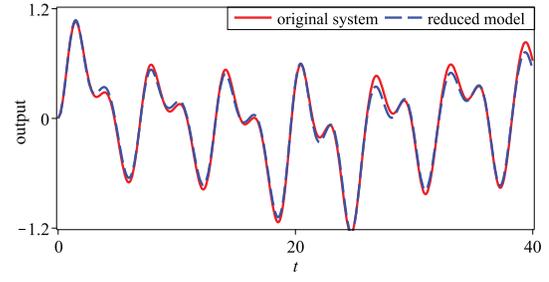


Fig. 2. Output trajectories of the 40-dimensional original system and the 6-dimensional reduced-order model of a nonlinear mass-spring-damper system

$$\begin{aligned} R_{22}(x) &:= \begin{bmatrix} d(x) & -d(x) \\ -d(x) & d(x) + 1 \end{bmatrix}, \quad g(x) := \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \\ d(x) &= 1 + (x_3^2 + x_3x_4 + x_4^2)/4. \end{aligned}$$

We take a model with  $q = 20$ , i.e.,  $n = 40$ . By using the fourth order Taylor series expansion, we compute the form (4) and its controllability function. Then, we compute a semi-balanced realization (7). Based on the obtained balanced  $\Sigma(z_H)$  as in (6), we decide the dimension of the reduced order model as  $k = 6$ . Due to the limitation of the space, it is not possible to show the reduced order model. Fig. 2 shows the output trajectories of the original system and the reduced-order model starting from zero initial states and input  $u(t) = \sin t + \sin(2t)$ .

## IV. LINEAR CASE

### A. Proposed Method for linear PH Systems

Here, we study more detailed properties of our method for the linear case. A linear PHS (1) is given by

$$\begin{cases} \dot{x} = (J - R)Qx + Bu, \\ y = B^T Qx, \end{cases} \quad (9)$$

where  $J = -J^T$ ,  $R = R^T \geq 0$ , and the Hamiltonian is given by  $H(x) = \frac{1}{2}x^T Qx$ , with  $Q > 0$ , and  $B$  is the input matrix. All matrices are of sizes corresponding to the states with  $x \in \mathbb{R}^n$ ,  $u, y \in \mathbb{R}^m$ . Suppose that the system is asymptotically stable and controllable. Then, the controllability Gramian  $W$  of PHS (9) is the symmetric positive definite solution to the following Lyapunov equation.

$$WQ(-J - R) + (J - R)QW = -BB^T. \quad (10)$$

In this section, we consider two types of balanced realizations. One is studied in the previous section. The other is given in the following theorem. Our objective is to show that these two balanced realizations yield equivalent reduced order models.

*Theorem 4.1:* If PHS (9) is asymptotically stable and controllable, then there exist coordinates  $z_W$  such that  $Q = W = \Sigma_W := \text{diag}\{\sigma_{W1}, \dots, \sigma_{Wn}\}$  ( $\sigma_{W1} \geq \dots \geq \sigma_{Wn} > 0$ ).

*Proof:* After the coordinate transformation  $z_W = T_W x$ , the PHS (9) and the Lyapunov equation (10) respectively become

$$\begin{cases} \dot{z}_W = (T_W J T_W^T - T_W R T_W^T) T_W^{-T} Q T_W^{-1} z_W + T_W B u, \\ y = (T_W B)^T T_W^{-T} Q T_W^{-1} z_W, \end{cases}$$

and

$$\begin{aligned} & (T_W W T_W^T)(T_W^{-T} Q T_W^{-1})(-T_W J T_W^T - T_W R T_W^T) \\ & + (T_W J T_W^T - T_W R T_W^T)(T_W^{-T} Q T_W^{-1})(T_W W T_W^T) \\ & = -T_W B B^T T_W^T. \end{aligned}$$

Therefore, in the  $z_W$ -coordinates, the Hamiltonian and controllability Gramian are  $\frac{1}{2}z^T T_W^{-T} Q T_W^{-1} z$  and  $T_W W T_W^T$ , respectively. Similar to obtaining a balanced realizations via the controllability and observability Gramians, [16], [24], it can be shown that there exists a  $T_W$  such that  $T_W W T_W^T = T_W^{-T} Q T_W^{-1} = \Sigma_W$ . ■

*Remark 4.2:* As mentioned in Remark 3.8, replacing the controllability Gramian by the observability Gramian in the balancing procedure with the Hamiltonian does not result in coordinates that are balanced between observability and internal energy. This can be seen by checking a coordinate transformation  $z = Tx$ . The observability Gramian  $M$  then transforms into  $T^{-T} M T^{-1}$ , in the same way as the Hamiltonian. In general, there is no  $T$  nor diagonal matrix  $\bar{\Sigma}$  such that  $T^{-T} M T^{-1} = T^{-T} Q T^{-1} = \bar{\Sigma}$ . In contrast, the controllability Gramian corresponds to the past input energy function, i.e.,  $L_C(x) = \frac{1}{2}x^T W^{-1}x$ , and thus  $W$  transforms into  $T W T^T$  for the  $z$ -coordinates, resulting in a balancing transformation that allows for  $T W T^T = T^{-T} Q T^{-1} = \bar{\Sigma}$ . ◀

Theorem 4.1 implies that the  $z_W$ -coordinates are semi-balanced coordinates. In these coordinates, PHS (9) and the Lyapunov equation for controllability Gramian respectively become

$$\begin{cases} \dot{z}_W = (J_W - R_W)\Sigma_W z_W + B_W u, \\ y = B_W^T \Sigma_W z_W, \end{cases} \quad (11)$$

$$\Sigma_W^2 (-J_W - R_W) + (J_W - R_W)\Sigma_W^2 = -B_W B_W^T. \quad (12)$$

Now, we have two types of coordinates, i.e., the  $z_W$ - and  $z_H$ -coordinates. Recall that in the  $z_H$  coordinates the system is in energy normal/input diagonal form, i.e., this means that  $H_{z_H} = \frac{1}{2}z_H^T z_H$  and  $L_{CH} = \frac{1}{2}z_H^T \Sigma^{-1} z_H$ . It follows immediately that  $\Sigma = \Sigma_W^2$  with  $z_H = \Sigma_W^{1/2} z_W$ .

We call the realization in coordinates  $z_H$  the Hamiltonian normal form. It follows straightforwardly that the reduced order model based on  $\Sigma_W$  and  $\Sigma$  are equivalent.

### B. Properties of Truncated Systems

It is not theoretically guaranteed that the reduced order model obtained by our semi-balancing is asymptotically stable. In this subsection, we proceed with stability analysis for linear PHSs first after which we briefly analyze the nonlinear case. The  $k$ -dimensional reduced order model of a linear PHS (9) is denoted by

$$\begin{cases} \dot{z}_r = (J_a - R_a)z_r + B_a u, \\ y_r = B_a^T z_r, \end{cases} \quad (13)$$

where  $J_a = -J_a^T$  and  $R_a = R_a^T \geq 0$ ; see Theorem 3.5. Note that this satisfies the following Lyapunov equation

$$(J_a - R_a)\Sigma_a + \Sigma_a(-J_a - R_a) = -B_a B_a^T, \quad (14)$$

for diagonal and positive definite  $\Sigma_a$ . It is well known that if there exists  $\Sigma_a > 0$  satisfying (14) then the non-asymptotically

stable mode is not controllable [25]. For the PHS, we have a stronger statement.

*Lemma 4.3:* If the PHS (13) is not asymptotically stable, every eigenvalue of  $J_a - R_a$  not being in the open left half plane is the zero. Moreover, let  $V$  be a basis matrix of the eigenspace of  $J_a - R_a$  corresponding to the zero eigenvalue. Then,  $J_a V = 0$ ,  $R_a V = 0$ , and  $V^T B_a = 0$ . Furthermore,  $V$  can be chosen such that  $\Sigma_a V = V \bar{\Sigma}_a$  for some diagonal  $\bar{\Sigma}_a$ .

*Proof:* Since the PHS has a positive definite solution  $\Sigma_a$  to (14), each eigenvalue of  $J_a - R_a$  that is not in the open left half plane is on the imaginary axis [25]. Consider an eigenvalue on the imaginary axis, which is denoted by  $j\omega$ ,  $\omega \in \mathbb{R}$ . Let  $W$  be a basis matrix for the right null space of  $J_a - R_a - j\omega I_n$ . Then, we have

$$(J_a - R_a - j\omega I_n)W = 0. \quad (15)$$

By pre-multiplying with  $W^*$ , we obtain

$$W^*(J_a - R_a - j\omega I_n)W = 0. \quad (16)$$

The sum of (16) and its conjugate transpose yields  $-2W^* R_a W = 0$ , and thus  $R_a W = 0$ .

Next, we show  $\omega = 0$ . By pre-multiplying (15) with  $W^T$ , we have

$$W^T(J_a - j\omega I_n)W = 0 \quad (17)$$

The sum of (17) and its transpose yields  $2j\omega W^T W = 0$ , and consequently  $\omega = 0$ . Therefore, every eigenvalue of  $J_a - R_a$  not being in the open left half plane is zero, and  $W = V$ . From (15),  $J_a V = 0$  follows. Moreover, by pre-and post-multiplying (14) with  $V^T$  and  $V$  respectively, we have  $-V^T B_a B_a^T V = 0$ , and consequently  $V^T B_a = 0$ .

To address the last statement, post-multiply (14) with  $V$ . Then,  $(J_a - R_a)\Sigma_a V = 0$ . Therefore, in a similar manner as in [25], one can find  $V$  satisfying the statement. ■

From its definition,  $V$  in Lemma 4.3 gives the non-asymptotically stable subspace of (14). Let  $\ell$  be the number of zero eigenvalues of  $J_a - R_a$  (i.e., multiplicity  $\ell$ ). It is possible to choose  $U \in \mathbb{R}^{k \times (k-\ell)}$  such that  $U^T U = I_{k-\ell}$ ,  $U^T V = 0$  and  $[U \ V] \in \mathbb{R}^{n \times n}$  is nonsingular. Define  $T := [U \ V(V^T V)^{-1/2}]$ . Then,  $T^T T = I_k$ . Now, we have the following theorem.

*Theorem 4.4:* There exists an orthogonal coordinate transformation  $z_r = \bar{T} \xi_r$  such that (13) becomes

$$\dot{\xi}_r = \begin{bmatrix} \bar{U}^T(J_a - R_a)\bar{U} & 0 \\ 0 & 0 \end{bmatrix} \xi_r + \begin{bmatrix} \bar{U}^T B_a \\ 0 \end{bmatrix} u \quad (18)$$

for some  $\bar{U} \in \mathbb{R}^{k \times (k-\ell)}$ . Moreover,  $\bar{T}^T \Sigma_a \bar{T}$  is diagonal.

*Proof:* In the  $\eta_r$  coordinates with  $z_r = T \eta_r$ , a solution to (14) becomes

$$T^T \Sigma_a T = \begin{bmatrix} U^T \Sigma_a U & 0 \\ 0 & \bar{\Sigma}_a \end{bmatrix},$$

where we use Lemma 4.3 and  $U^T V = 0$ . The first block can be diagonalized by the orthogonal matrix  $\bar{T}_1$ . Define  $\bar{T} := T \text{diag}\{\bar{T}_1, I_\ell\}$ . Then,  $\bar{T}^T \Sigma_a \bar{T}$  is diagonal. Finally, one can confirm that in the  $\xi_r$  coordinates, the reduced order model becomes (18) for  $\bar{U} := U \bar{T}_1$ . ■

This theorem implies that if the reduced order model is not asymptotically stable while the original one is, it can be fully decoupled as (18). Then, we can pick up its first subsystem

$$\dot{\xi}_{rr} = \bar{U}^T(J_a - R_a)\bar{U}\xi_{rr} + \bar{U}^T B_a u. \quad (19)$$

It is clear that the two PHSs (18), i.e., (13) and (19) have the same output response for any initial state and input. Since  $\bar{U}^T(J_a - R_a)\bar{U}$  is non-singular, the PHS (19) is asymptotically stable. Also, the corresponding controllability Gramian  $\bar{T}_1^T U^T \Sigma_a U \bar{T}_1 = \bar{U}^T \Sigma_a \bar{U}$  is diagonal and positive definite, and thus it is also controllable. Moreover, since  $\bar{T}$  is orthogonal, both diagonal matrices  $\Sigma_a$  and

$$\bar{T}^T \Sigma_a \bar{T} = \text{diag}\{\bar{U}^T \Sigma_a \bar{U}, \bar{\Sigma}_a\}$$

have the same set of eigenvalues. Therefore, if  $\sigma_k \gg \sigma_{k+1}$  then every diagonal element of  $\bar{U}^T \Sigma_a \bar{U}$  is much larger than  $\sigma_{k+1}$ .

Next, we consider the nonlinear reduced order model (8). Suppose that the reduced order model is not asymptotically stable at the origin. From Lasalle's invariance principle [23],  $J_{z_H,a}(0,0) - R_{z_H,a}(0,0)$  is singular at the origin. Define  $J_a := J_{z_H,a}(0,0)$ ,  $R_a := R_{z_H,a}(0,0)$ , and  $B_a := g_{z_H,a}(0,0)$ . Then, the linearization of the reduced order model at the origin is (13), and  $\Sigma_a := \text{diag}\{\sigma_1(0), \dots, \sigma_k(0)\}$  satisfies (14). Since the linearized model is marginally stable, there exists  $\bar{T}$  in Theorem 4.4. After the coordinate transformation  $z_r = \bar{T}\xi_r$ , the reduced order nonlinear model becomes

$$\begin{aligned} \dot{\xi}_r &= \begin{bmatrix} \bar{U}^T \\ \bar{V}^T \end{bmatrix} (J_{z_H,a}(z_r,0) - R_{z_H,a}(z_r,0)) \begin{bmatrix} \bar{U} & \bar{V} \end{bmatrix} \xi_r \\ &\quad + \begin{bmatrix} \bar{U} & \bar{V} \end{bmatrix}^T g_{z_H,a}(z_r,0)u, \\ y_r &= g_{z_H,a}^T(z_r,0) \begin{bmatrix} \bar{U} & \bar{V} \end{bmatrix}^T \xi_r. \end{aligned} \quad (20)$$

Its linearization is (18). The first subsystem of (20),

$$\begin{aligned} \dot{\xi}_{rr} &= \bar{U}^T(J_{z_H,a}(z_r,0) - R_{z_H,a}(z_r,0))\bar{U}\xi_{rr} + \bar{U}g_{z_H,a}(z_r,0)u, \\ y_r &= g_{z_H,a}^T(z_r,0)\bar{U}^T\xi_{rr} \end{aligned}$$

is again a PHS. Since its linearization (19) is asymptotically stable and controllable, this subsystem is locally stable and locally controllable at the origin.

### C. Normal Form of Gradient Systems

Similar to PHSs, gradient systems [18] arise from physics as well. In general, there is no direct connection between these two types of systems except when the gradient system is passive [26]. In contrast to PHSs, for linear systems standard balanced model reduction based on the controllability and observability Gramians preserves the gradient structure [18]. In this section, we investigate the relation between our method and the standard balancing method for gradient systems.

A gradient system, e.g., [18], is given as follows.

$$\begin{cases} G\dot{x} = -Px + Bu, \\ y = B^T x, \end{cases} \quad (21)$$

where  $G, P \in \mathbb{R}^{n \times n}$  are symmetric and in addition  $G$  is non-singular.  $G$  represents a "pseudo-metric", and a gradient system is a symmetric system, i.e., the transfer function

$K(s)$  fulfills  $K(s) = K(s)^T$ . After coordinate transformation  $z = Gx$ , we have

$$\begin{cases} \dot{z} = -PG^{-1}z + Bu, \\ y = B^T G^{-1}z. \end{cases}$$

If  $P$  is positive semidefinite and  $G$  is positive definite, this can be seen as a PHS with  $J = 0$ ,  $P$  the damping, and  $H = \frac{1}{2}z^T G^{-1}z$ . However, in general both  $G$  and  $P$  are indefinite. Although  $G$  is indefinite, we can compute a variation of the normal form of  $G$  with the controllability Gramian, where the variation of the normal form means a state space representation such that  $G$  becomes a signature matrix  $\hat{G} = \text{diag}\{\pm 1, \dots, \pm 1\}$ .

Suppose that gradient system (21) is controllable and asymptotically stable at the origin. Then, the Lyapunov equation for the controllability Gramian  $W \in \mathbb{R}^{n \times n}$

$$-WPG^{-1} - G^{-1}PW = -G^{-1}BB^T G^{-1}$$

has the symmetric and positive definite solution.

There exists a coordinate transformation  $z_{GW} = T_{GW}x$  which transforms  $G$  and  $W$  into a signature matrix  $\hat{G}$  and a diagonal matrix  $\Sigma_G := \text{diag}\{\sigma_{G1}, \dots, \sigma_{Gn}\}$ , respectively [18]. Interestingly, it is established in [18] that the system in  $z_{GW}$  coordinates is in the classical balanced form, i.e.,

*Theorem 4.5:* [18] In the  $z_{GW}$ -coordinates, the observability Gramian is  $\Sigma_G$ .  $\triangleleft$

### D. Normal Forms of PHSs and Gradient Systems

In this subsection, we show that for passive gradient systems with a positive definite metric  $G$ , our normal forms of PHSs and gradient systems are equivalent.

A passive gradient system can always be represented as follows with suitable matrices [26].

$$\begin{aligned} \begin{bmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{bmatrix} \dot{z} &= - \begin{bmatrix} P_1 & P_c \\ P_c^T & P_2 \end{bmatrix} z + \begin{bmatrix} C_1^T \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} C_1 & 0 \end{bmatrix} z, \end{aligned}$$

where  $k_1, k_2 \geq 0$  and  $k_1 + k_2 = n$ ;  $P_1$  and  $P_2$  are positive and negative semidefinite, respectively. It can be confirmed that by premultiplying  $-I_{k_2}$  by the second equation, we obtain a PH form with  $Q = I_n$ .

Let  $W$  be the controllability Gramian of this system. In the previous subsections, we provided two normal forms. One is the Hamiltonian normal form, i.e.,  $T_H W T_H^T = \Sigma_H$  and  $T_H^{-T} Q T_H^{-1} = T_H^{-T} T_H^{-1} = I_n$ . The other is the variation of the normal form, in particular a type of "pseudo normal form", i.e.  $T_G W T_G^T = \Sigma_G$  and

$$T_G^{-T} \begin{bmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{bmatrix} T_G^{-1} = \begin{bmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{bmatrix}$$

We already noted above that the latter pseudo normal form is nothing but a traditional balanced realization based on controllability and observability Gramians.

In the specific case when the metric is positive definite, i.e.,  $G > 0$ ,  $k_2 = 0$ , i.e., we deal with PHSs with  $J = 0$ . Then, we have  $T_G = T_H$ , i.e., the Hamiltonian normal form is equivalent to the traditional balanced realization. Therefore, traditional

balanced truncation and our PH structure preserving truncation give the same reduced order model, which has preserved both the gradient and the PH structure.

*Example 4.6:* Consider an RL electric network with 1000 nodes in Fig. 3, where  $x_i \in \mathbb{R}$  is the voltage at node  $i = 1, 2, \dots, 1000$ ,  $u \in \mathbb{R}$  is the source current, and  $y = x_1$ . Its state space representation in the gradient form is

$$G = I_{1000}, \quad P = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is a passive gradient system with  $k_1 = 1000$  and  $k_2 = 0$ , i.e., a PHS with  $Q = I_{1000}$ ,  $J = 0$ , and  $R = P$ . Then, the normal Hamiltonian form is the traditional balanced realization with controllability and observability Gramian. In this example, our objective is to demonstrate the passivity preservation of a gradient system. Due to the space limitation, we only show the first 6 singular values and the 3-dimensional reduced order model.

$$\begin{aligned} \sigma_1 &= 0.358, \quad \sigma_2 = 0.0865, \quad \sigma_3 = 0.0303, \quad \sigma_4 = 0.0127, \\ \sigma_5 &= 0.00592, \quad \sigma_6 = 0.00297, \\ \dot{x}_r &= A_r x_r + B_r u, \quad y_r = B_r^T x_r, \end{aligned}$$

$$A_r = \begin{bmatrix} -1.13 & 0.730 & -0.442 \\ 0.730 & -0.751 & -0.589 \\ -0.442 & -0.589 & -0.600 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0.9 \\ 0.361 \\ 0.191 \end{bmatrix}.$$

This results in standard error bounds given by  $0.0127 \leq \|G - G_r\|_{\mathcal{H}_\infty} \leq 0.0249$ , where  $G$  and  $G_r$  are the transfer functions of the original and reduced order models, respectively. This reduced order model is again both a gradient system with  $G = I_3$ , and  $P = A_r$ , and PHS with  $Q = I_3$ ,  $J = 0$ , and  $R = A_r$ . Therefore, passivity of the gradient system is preserved under traditional balanced truncation, i.e., the gradient structure is preserved by our PH structure preserving truncation.  $\triangleleft$

### E. Example

We now treat an example of a system that can be expressed as a gradient system with non-definite  $G$ , and as a PHS. We

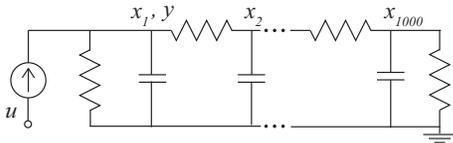


Fig. 3. Linear electrical circuit

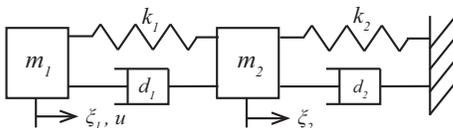


Fig. 4. Mass-spring-damper system

first apply our Hamiltonian balancing method as presented in Section IV-A, and then the traditional balancing method which corresponds to a pseudo normal Hamiltonian, i.e. normal Form of gradient systems, as presented in Section IV-C.

Consider a mass-damper-spring system in Fig. 4.

$$\begin{aligned} & \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \end{bmatrix} + \begin{bmatrix} d_1 & -d_1 \\ -d_1 & d_1 + d_2 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} \\ & + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ & y = \dot{\xi}_1. \end{aligned}$$

For the sake of simplicity, we choose all parameters as 1. This system can be represented as a gradient system with

$$\begin{aligned} x &= [x_1^T \quad x_2^T]^T, \\ x_1 &:= [\dot{\xi}_1 \quad \dot{\xi}_2]^T, \quad x_2 := K [\xi_1 \quad \xi_2]^T, \\ G &:= \begin{bmatrix} M & 0 \\ 0 & -K^{-1} \end{bmatrix}, \quad P := \begin{bmatrix} D & I_2 \\ I_2 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ M &:= I_2, \quad K := D := \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad B_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

After coordinate transformation  $z = [x_1^T (K^{-1/2} x_2)^T]^T$ , we obtain a port Hamiltonian representation with Hamiltonian  $H(z) = \frac{1}{2} z^T z$  as follows

$$\begin{cases} \dot{z} = \left( \begin{bmatrix} 0 & -K^{1/2} \\ K^{1/2} & 0 \end{bmatrix} - \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right) z + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \\ y = [B_1^T \quad 0] z. \end{cases}$$

Note that this is not the standard port-Hamiltonian form, since the spring constants usually are part of the energy (Hamiltonian). However, for the sake of simplicity, we have chosen to already put the system in the energy-normal form. Therefore, this system can be realized as both a port Hamiltonian and a gradient system.

For this system, the controllability function is given by

$$L_{Cz} = \frac{1}{2} z^T \begin{bmatrix} 0.773 & 0.364 & 0.0407 & -0.122 \\ 0.364 & 0.227 & 0.0813 & -0.0407 \\ 0.0407 & 0.0813 & 0.809 & 0.346 \\ -0.122 & -0.0407 & 0.346 & 0.191 \end{bmatrix} z,$$

and its eigenvalues are

$$\sigma_1 = 0.987, \quad \sigma_2 = 0.957, \quad \sigma_3 = 0.0430, \quad \sigma_4 = 0.0127.$$

When applying our method from Section IV-A, we are ready to truncate the states corresponding to  $\sigma_3$  and  $\sigma_4$ , since  $\sigma_3 \ll \sigma_2$ . The 2-dimensional reduced order model obtained by is now given by

$$\begin{aligned} \dot{z}_r &= \begin{bmatrix} -0.149 & 0.819 \\ -0.452 & -0.252 \end{bmatrix} z_r + \begin{bmatrix} 0.533 \\ -0.705 \end{bmatrix} u, \\ y_r &= [0.533 \quad -0.705] z_r. \end{aligned}$$

The reduced order model can be represented as

$$\begin{aligned} \dot{z}_r &= (J_r - R_r) z_r + B_r u, \quad y_r = B_r^T z_r, \\ J_r &= \begin{bmatrix} 0 & 0.6359 \\ -0.6359 & 0 \end{bmatrix}, \end{aligned}$$

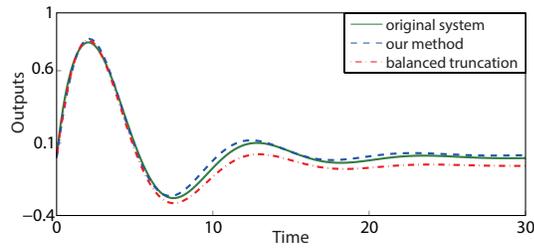


Fig. 5. Step responses of original system and reduced-order models

$$R_r = \begin{bmatrix} 0.1486 & -0.1836 \\ -0.1836 & 0.2520 \end{bmatrix}, B_r = \begin{bmatrix} 0.533 \\ -0.705 \end{bmatrix},$$

which is nothing but a port Hamiltonian system. This system can also be written in the gradient system form as follows:

$$G_r \dot{z}_r = -P_r z_r + B_r u, \quad y_r = B_r^T z_r,$$

$$G_r = \begin{bmatrix} -0.2720 & -0.9617 \\ -0.9617 & 0.2729 \end{bmatrix},$$

$$P_r = \begin{bmatrix} -0.4754 & -0.01945 \\ -0.01945 & 0.8568 \end{bmatrix}.$$

Therefore, our method of Section IV-A preserves both the PH structure and a gradient system can be built from it.

Next, we apply the method of Section IV-C based on a pseudo normal Hamiltonian equivalent to traditional balanced truncation. Then, the obtained Hankel singular values and the 2-dimensional reduced order model are given by

$$\sigma_1 = 0.960, \quad \sigma_2 = 0.933, \quad \sigma_3 = 0.0416, \quad \sigma_4 = 0.0139,$$

$$\dot{\bar{z}}_r = \begin{bmatrix} 0 & 0.615 \\ -0.615 & -0.440 \end{bmatrix} \bar{z}_r + \begin{bmatrix} -0.0188 \\ -0.905 \end{bmatrix} u,$$

$$\bar{y}_r = \begin{bmatrix} 0.0188 & -0.905 \end{bmatrix} \bar{z}_r,$$

Note that the Hankel singular values are slightly different than the singular values obtained from the Hamiltonian normal method. It now follows that the error bounds are given by  $0.0416 \leq \|G - G_r\|_{\mathcal{H}_\infty} \leq 0.0555$ . The reduced order model is again a gradient system with respect to the pseudo metric  $G = \text{diag}\{-1, 1\}$ .

Fig. 5 shows step responses of the original system and two-dimensional reduced-order models by our method and the balanced truncation. It can be observed that the response of the reduced order model by our method follows the trajectory of the original model somewhat better than the reduced order model obtained by balanced truncation.

## V. CONCLUSION

In this paper, we propose a PH structure preserving truncation method based on the Hamiltonian and the controllability function. First, we provide this method for nonlinear PHS. Then, we focus on the linear case, where we show a relation with traditional balancing for specific passive gradient systems. From this fact, we may conclude that our method is an extension of the traditional balancing method to preserve the PH structure. Future work includes analysis on how the reduced order model approximates the original system from a physical point of view, which has not been studied by any paper of the PH structure preserving model reduction yet.

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