Recurrence for quenched random Lorentz tubes

Giampaolo Cristadoro, Marco Lenci, and Marcello Seri

A Lorentz tube (LT) is a system of a particle (or, from a statistical viewpoint, many noninteracting particles) freely moving in a domain extended in one direction and performing elastic collisions with randomly placed obstacles. These kinds of “extended billiards” are, on the one hand, paradigms of systems where some transport properties can be studied in a rigorous mathematical way and, on the other hand, reliable models for real situations, such as transport in nanotubes, heat diffusion and molecular dynamics in wires or other disordered tubular settings, etc. The primary interest in their study lies on such properties as recurrence, diffusivity, and transmission rates. Unfortunately, few rigorous results are available and their proofs typically rely on some periodic structure. In this paper a more realistic situation is taken into account: the so-called quenched disorder. Recurrence is proved for almost every realization of the configuration of obstacles, implying strong chaotic properties for these types of systems.

I. INTRODUCTION

This paper concerns the dynamics of a particle in certain two-dimensional systems which are infinitely extended in one dimension. More precisely, we will study dynamical systems in which a point particle moves in a strip (or similar set) $T \subset \mathbb{R}^2$, which contains a countable number of convex scatterers, see the example in Fig. 1. The motion of the particle is free until it collides with either the boundary of $T$ or a scatterer, both of which are thought to have infinite mass. The collisions are totally elastic, so they obey the usual Fresnel law: the angle of reflection equals the angle of incidence.

In the taxonomy of dynamical systems, these models belong to the class of semidispersing planar billiards. In particular, they are extended semidispersing billiards, which very much resemble a Lorentz gas. We thus call them essentially one-dimensional Lorentz gases or, more concisely, LTs.

Systems like these (especially their three-dimensional counterparts, cf. last paragraph of this section) find application in the sciences as models for the dynamics of particles (e.g., gas molecules) in narrow tubes (e.g., carbon nanotubes). A very minimal list of references, from the more experimental to the more mathematical, includes Refs. 10, 3, 15, 5, 2, 9, and 8. (See further references in those papers.) An interesting fact is that both experimentalists and theoreticians seem to have a primary interest—sometimes for different reasons—in the same question, namely, the diffusion properties of these gases. As we discuss below, this is our case as well, although the results we present in this note must be considered preliminary in this respect.

From a mathematical viewpoint, LTs are interesting because they are among the very few extended dynamical systems, with a certain degree of realism, that mathematicians can prove something about. By the ill-defined expression extended dynamical system we generally mean a dynamical system on a noncompact phase space whose physically relevant (invariant) measure is infinite. For such systems, the very fundamentals of ordinary ergodic theory do not work: for example, the Poincaré recurrence theorem fails to hold and one does not know whether the system is totally recurrent (almost every point returns arbitrarily close to its initial condition), totally transient (almost every point escapes to infinity), or mixed.

In fact, as it turns out, recurrence is not just the most basic property one wants to establish in order to even consider studying the chaotic features of an extended dynamical system (it is sometimes said that if ergodicity is the first of a whole hierarchy of stochastic properties that a dynamical system can possess, recurrence is the zeroth property); for a Lorentz gas at least, a number of stronger ergodic properties follow from recurrence: for example, ergodicity of the extended dynamical system, $K$-mixing of the first-return map to a given scatterer, etc.

Let us briefly explain our model. We consider the connected set $T \subset \mathbb{R}^2$ tessellated by the repetition, under the action of $\mathbb{Z}$, of a given fundamental domain $C$, which we assume to be a polygon. In each copy of $C$, henceforth referred to as cell, we place a random configuration of convex scatterers, according to some rule that we specify later. Given a
fusive properties of these gases, cf. also Ref. 6. 

provides another strong motivation for the study of the di- 

tion. Almost sure recurrence means that almost every LT in 

one-dimensional 

presented elsewhere.

important generalization to the higher-dimensional case will be 

presented elsewhere.
The following are our assumptions on the geometry of the LT:

(A2) There exist a positive integer $K$ such that, for II-almost every realization $\ell \in \Omega^2$, $\partial Q_{n,j}$ is made up of at most $K$ compact connected $C^3$ pieces, which may intersect only at their end points. These points will be referred to as vertices.

Denoting, as we will do throughout the paper, $x:=(q,v)$, let $\gamma(x)$ be the first time at which the point with initial conditions $x$ hits a non-flat part of the boundary (so this is not exactly the usual free flight function!). Also, if $q$ is a smooth point of $\partial Q$, let $k(q)$ be the curvature of $\partial Q$ at $q$. We have the following assumption.

(A3) There exist two positive constants $\gamma_a \leq \gamma_M$ such that, for almost every $\ell$ and all $x=(q,v)$ with $q \in \partial Q$,

$$\gamma_a \leq \gamma(x) \leq \gamma_M.$$

(A4) There exists a positive constant $k_m$ such that, for almost every $\ell$, given a smooth point $q$ of the boundary, either $\partial Q$ is totally flat at $q$ or $k(q) \geq k_m$.

In the language of billiards, a singular trajectory is a trajectory which, at some time, hits the boundary of the table tangentially or in a vertex. It follows that a finite segment of a nonsingular trajectory depends continuously on its initial condition. Also notice that by (A2), the set of all singular trajectories is a countable union of smooth curves in $Q \times S^1$ and thus has measure zero. The next assumption is meant to exclude pathological situations:

(A5) For almost every $\ell$ and all $i,j \in \{1,2\}$, there is a nonsingular trajectory entering $C_0$ through $G^i$ and leaving it through $G^j$.

A convenient way to represent a continuous-time dynamical system is to select a suitable Poincaré section and consider the first-return map there. For billiards, this is customarily taken to be the set of all pairs $(q,v) \in \partial Q \times S^1$, where $v$ is a postcollisional unit vector at $q$ (hence an inner vector relative to $Q$). Here we slightly modify this choice.

For $n \in \mathbb{Z}$ and $j \in \{1,2\}$, denote by $G^j_n:=\gamma^j(G^j)$ the side of $C_n$ corresponding to $G^j$ in $C_0$ ($G^1_n$ and $G^2_n$ may be called the gates of $C_n$, whence the notation). Let $o_j$ be the inner normal to $G^j_n$, relative to $C_n$. Notice that under our hypotheses, $o_2=-o_1$. Define

$$\mathcal{N}_n^j := \{(q,v) \in G^j_n \times S^1 | |v \cdot o_j| > 0\}.$$  \hspace{1cm} (2.1)

The cross section we use is

$$\mathcal{M} := \bigcup_{n \in \mathbb{Z}} \bigcup_{j=1,2} \mathcal{N}_n^j,$$  \hspace{1cm} (2.2)

whose corresponding Poincaré map we denote $T=T_{\ell}$. In other words, we only consider those times at which the particle crosses one of the gates. In the lingo of billiards, cross sections like these are sometimes called “transparent walls.”

The Liouville measure for the flow induces on a transparent wall an invariant measure given by $d\mu(q,v)=|v \cdot o_q| dq dv$, where $o_q$ is the normal to the section at $q$, directed toward the outgoing side of $(q,v)$ (Ref. 7) (in our case, $o_q = o_1$ whenever $q \in N^j_0$).

So we end up with the dynamical system $(\mathcal{M}, T_{\ell}, \mu)$, whose invariant measure is infinite and $\sigma$-finite. Notice that by design, the only object that depends on the random configuration is the map $T_{\ell}$.

In order to discuss the hyperbolic properties of this system, we need to introduce its local stable and unstable manifolds (LSMs). Since our exposition does not require a rigorous definition of these objects, we shall refrain from providing one, and point the interested reader to the existing literature, e.g., Ref. 7. Here we just mention that in our system, a local stable manifold (LSM) $W^s(x)$ is a smooth curve containing $x$ and whose main property is that, for all $y \in W^s(x)$, $\lim_{n \to \pm\infty} \text{dist}(T^n x, T^n y)=0$, where $\text{dist}$ is the natural Riemannian distance in $\mathcal{M}$ [with the convention that if $x$ and $y$ belong to different connected components of $\mathcal{M}$, $\lim_{n \to \pm \infty} \text{dist}(x,y)=\infty$]. A local unstable manifold (LUM) $W^u(x)$ has the analogous property for the limit $n \to -\infty$.

The system has a hyperbolic structure à la Pesin in the following sense:

**Theorem 2.1:** For $\mu$-almost every $x \in \mathcal{M}$ there is a LSM $W^s(x)$ and a LUM $W^u(x)$. The corresponding two foliations—more correctly, laminations—can be chosen invariant, namely, $TW^s(x) \subset W^s(Tx)$ and $T^{-1}W^u(x) \subset W^u(T^{-1}x)$. Also, when endowed with a Lebesgue-equivalent one-dimensional transversal measure, they are absolutely continuous with respect to $\mu$.

The next theorem is the core technical result for all the proofs that follow. It is not by chance that in the field of hyperbolic billiards, this is called the fundamental theorem.

**Theorem 2.2:** Given $n \in \mathbb{Z}$, $j \in \{1,2\}$ and a full-measure $A \subset N^j_n$, there exists a full-measure $B \subset N^j_n$ such that all pairs $x,y \in B$ are connected via a polyline of alternating LSMs and whose vertices lie in $A$. This means that for $x,y \in B$, there is a finite collection of LSMs, $W^s(x_1), W^s(x_2), W^s(x_3), \ldots, W^s(x_n)$, with $x_1=x$, $x_n=y$, and such that each LSM intersects the next transversely in a point of $A$.

The above theorems are proved in Ref. 14 for Lorentz gases that are effectively two-dimensional and whose scatterers are smooth, i.e., $K=1$ in (A2). The first of the two differences is absolutely inconsequential. The second affects the singularity set of $T$, that is, the set of all $x \in \mathcal{M}$ whose trajectory, up to the next crossing of a transparent wall, is singular. It is a well-known and easily derivable fact that in each component $N^j_n$ of the cross section, the singularity set is a union of smooth curves, each of which is associated to a specific source of singularity within the cell $C_n$ (a tangential scattering, a vertex, the end point of a gate) and an itinerary of visited scatterers before that. Since both the number of scatterers in each cell and the number of vertices per scatterer are bounded, there can only be a finite number of singularity lines in each $N^j_n$. With this provision, the proofs of Ref. 14 work in this case as well. [In truth, the actual proofs are found in Ref. 11, where the existence of a hyperbolic structure and the fundamental theorem are shown for the standard billiard cross section. In Ref. 14 these are extended to the transparent cross section. The idea behind the results of Ref. 11 is this: assumptions (A2)–(A4) guarantee that the
geometric features of the LT are “uniformly good.” Then a refinement of a standard trick ensures that most orbits of the system do not approach the singularity set too fast, so that, in the construction of the hyperbolic structure, one can practically neglect them. As for the fundamental theorem, all the local arguments in the classical proofs of Sinai and followers for compact billiards apply—notice that we have uniform hyperbolicity and no cusps, namely, zero-angle corners. The global arguments have to do essentially with controlling the neighborhoods of certain portions of the singularity set, which can be done with the above-mentioned trick.]

III. RECURRENCE

We are interested in the recurrence and ergodic properties of the LTs defined earlier. To this goal, let us recall some definitions that may not be obvious for infinite-measure dynamical systems.

**Definition 3.1:** The measure-preserving dynamical system \((M, T, \mu)\) is called (Poincaré) recurrent if, for every measurable \(A \subseteq M\), the orbit of \(\mu\)-almost every \(x \in A\) returns to \(A\) at least once (and thus infinitely many times due to the invariance of \(\mu\)).

**Definition 3.2:** The measure-preserving dynamical system \((M, T, \mu)\) is called ergodic if every \(A \subseteq M\) measurable and invariant mod \(\mu\) [that is \(\mu(T^{-1}A \Delta A) = 0\)] has either zero measure or full measure [that is \(\mu(M \setminus A) = 0\)].

If the system in question is a LT, as introduced in Sec. II (\(T = T_\ell\) for some \(\ell \in \Omega^2\)), it is proved in Refs. 11 and 14 the following theorem.

**Theorem 3.1:** \((M, T_\ell, \mu)\) is ergodic if and only if it is recurrent.

Understandably, proving recurrence (and thus ergodicity) of every system in the quenched random ensemble might be a daunting task. It is possible, however, to prove it for a typical system. This will be achieved via a general result by Schmidt \(16\) on the recurrence of commutative cocycles over finite-measure dynamical systems. We state it momentarily.

**Definition 3.3:** Let \((\Sigma, F, \lambda)\) be a probability-preserving dynamical system, and \(f\) a measurable function \(\Sigma \to \mathbb{Z}^p\). The family of functions \(\{S_n\}_{n \in \mathbb{N}}\), defined by \(S_0(\xi) = 0\) and, for \(n \geq 1\),

\[
S_n(\xi) := \sum_{k=0}^{n-1} (f \circ F^k)(\xi),
\]

is called the cocycle of \(f\). Any such family is generically called commutative, \(p\)-dimensional, discrete cocycle.

**Theorem 3.2:** Assume that \((\Sigma, F, \lambda)\) is ergodic and denote by \(Q_\alpha\) the distribution of \(S_n/n^{1/p}\) relative to \(\lambda\), i.e., the distribution on \(\mathbb{R}^p\) defined by

\[
Q_\alpha(A) := \lambda\left(\left\{ \xi \in \Sigma \left| \frac{S_n(\xi)}{n^{1/p}} \in A \right. \right\}\right),
\]

where \(A\) is any measurable set of \(\mathbb{R}^p\). If there exists a positive-density sequence \(\{n_k\}_{k \in \mathbb{N}}\) and a constant \(\kappa > 0\) such that

\[
Q_{\kappa n_k}(B(0, \rho)) \geq \kappa \rho^p
\]

for all sufficiently small balls \(B(0, \rho) \subset \mathbb{R}^p\) (of center 0 and radius \(\rho\)), then the cocycle \([S_n]\) is recurrent, namely, for \(\lambda\)-almost every \(\xi \in \Sigma\), there exists a subsequence \(\{n_j\}_{j \in \mathbb{N}}\) such that

\[
S_{n_j}(\xi) = 0, \quad \forall j \in \mathbb{N}.
\]

The above result is a slight weakening of the original theorem by Schmidt, whose proof can be found in Ref. 16. (In truth, the original formulation required \(F\) to be invertible mod \(\lambda\). The generalization to noninvertible measure-preserving maps is an easy exercise which can be found, e.g., in Ref. 13, Appendix A, Sec. 2.)

In the following we will introduce a suitable probability-preserving dynamical system and a one-dimensional cocycle with the property that the recurrence of the latter is equivalent to the Poincaré recurrence of \(\lambda\)-almost every LT \(\ell\) (we call this situation almost sure recurrence of the quenched random LT; details in Sec. IV). Observe that for \(p = 1\), the quantity \(S_{n_j}/n^{1/p}\) is precisely the Birkhoff average of \(f\). Thus, the ergodicity of \((\Sigma, f, \lambda)\), which implies the law of large numbers for \(\{S_n\}\), is enough to apply Theorem 3.2.

IV. THE POINT OF VIEW OF THE PARTICLE

For \(j \in \{1, 2\}\), let us consider \(N_0^3\) as defined in Eq. (2.1), and rename it \(N^0\) for short. In this section we will work extensively with the cross-section \(N := N^0 \cup N^2\).

Let us call \(\mu_0\) the standard billiard measure on \(N\), normalized to 1. If \(\omega \in \Omega\) determines the configuration of scatterers in \(C_0\), we can define a map \(R_\omega: N \to N\) as usual (cf. Fig. 3). Trace the forward trajectory of \(x = (q, v) \in N\) until it crosses \(G^1\) or \(G^2\) for the first time (almost all trajectories do). This occurs at a point \(q_1\) with velocity \(v_1\). If, for \(\epsilon \in [-1, +1]\), \(C_\epsilon\) is the cell that the particle enters upon leaving \(C_0\), define

\[
R_{\omega x} = R_\omega(q, v) := (\tau^\epsilon(q_1), v_1) \in N, \quad (4.1)
\]

\[
e(x, \omega) := \epsilon. \quad (4.2)
\]

We name \(e\) the exit function. From our earlier discussion on the transparent cross sections, \(R_\omega\) preserves \(\mu_0\).

We introduce the dynamical system \((\Sigma, F, \lambda)\), where

- \(\Sigma := N \times \Omega^2\).
• $F(x,\ell):=(R_\ell x,\sigma^{(x,\ell)}(\ell))$, defining a map $\Sigma \to \Sigma$. Here $\ell_0$ is the zeroth component of $\ell$ and $\sigma$ is the left shift on $\Omega^{\mathbb{Z}}$, introduced in (A1) [therefore, $\sigma^{(x,\ell)}(\ell) = (\ell'_{\ell_n})_{n \in \mathbb{Z}}$ with $\ell'_{\ell_0} = \ell_{\ell_0+\ell}$].

• $\lambda := \mu_0 \times \Pi$. Clearly, $\lambda(\Sigma) = 1$. Also, using that $F$ is invertible, $R_\omega$ preserves $\mu_0$ for every $\omega \in \Omega$, and $\sigma$ preserves $\Pi$, it can be seen that $F$ preserves $\lambda$. (This is ultimately a consequence of the fact that every LT preserves the same measure.)

The idea behind this definition is that instead of following a given orbit from one cell to another, we every time shift the LT in the direction opposite to the orbit displacement, so that the point always lands in $C_0$. For this reason the dynamical system just introduced is called the point of view of the particle. Clearly, $F: \Sigma \to \Sigma$ encompasses the dynamics of all points on all realizations of $\Omega^{\mathbb{Z}}$.

**Proposition 4.1**: If the cocycle of the exit function $e$ is recurrent, then the quenched random LT is almost surely recurrent in the sense that for $\Pi$-almost every $\ell \in \Omega^{\mathbb{Z}}$, $(M,T,\mu)$ is recurrent.

**Proof**: Before starting the actual proof, we recall that an easy argument (Ref. 14, Proposition 2.6) shows that the extended system $(M,T,\mu)$ is either recurrent or totally dissipative (i.e., transient): no mixed situations occur. Therefore, the existence of one recurrent set (i.e., a positive-measure set $A$ such that $\mu$-almost all points of $A$ return there at some time in the future) is enough to establish the same property for all measurable sets.

Now, calling $\{S_n\}$ the cocycle of $e$, the hypothesis of Proposition 4.1 amounts to saying that for $\lambda$-almost every $(x,\ell) \in \Sigma$, there exists $n = n(x,\ell)$ such that $S_n(x,\ell) = 0$. That is, considering the LT $\ell$, $T_{\ell_n}x \in N_0$ (recall that $x \in N_0$ by construction). Let us call such a pair $(x,\ell)$ typical.

By Fubini’s theorem, $\Pi$-almost all $\ell \in \Omega^{\mathbb{Z}}$ are such that $(x,\ell)$ is typical for $\mu_0$-almost all $x \in N$. For such $\ell$, $N_0 = \mathbb{N}$ is a recurrent set of $T_\ell$; therefore, $(M,T,\mu)$ is recurrent. Q.E.D.

As it was mentioned at the end of Sec. III, the recurrence of the cocycle of $e$ is implied by ergodicity of $(\Sigma,F,\lambda)$. On the other hand, see the following theorem.

**Theorem 4.1**: Under assumptions (A1)–(A5), the dynamical system $(\Sigma,F,\lambda)$ defined above is ergodic.

**Proof**: The proof can be divided in three steps:

1. Every ergodic component of $(\Sigma,F,\lambda)$ is of the form $\bigcup_{j=1}^2 N^{\mathbb{Z}} \times B_j \mod \lambda$, where $B_j$ is a measurable set of $\Omega^{\mathbb{Z}}$.

2. $\Pi(B_j) \in \{0,1\}$.

3. There is only one ergodic component.

We now describe each step separately.

1. For a fixed $\ell$, consider the extended dynamical system $(M,T,\mu)$, for which Theorem 2.1 holds. Through the obvious isomorphism, copy those LSUMs of the extended system which are included in $N_0$ onto $N \times \{\ell\}$. These may be called LSUMs for the fiber $N \times \{\ell\}$ [although $(\Sigma,F,\lambda)$ cannot be regarded as a bona fide hyperbolic dynamical system]. By Theorem 2.2, in each connected component of $N \times \{\ell\}$, namely, $N^2 \times \{\ell\}$ and $N^2 \times \{\ell\}$, almost every pair of points can be connected through a sequence of LSUMs for the fiber, intersecting at typical points. Hence, via the usual Hopf argument, the whole $N^2 \times \{\ell\}$ lies the same ergodic component, at least for almost every $\ell$. Therefore, an $F$-invariant set in $\Sigma$ can only come in the form $I = \bigcup_{j=1}^2 N^{\mathbb{Z}} \times B_j$. That $B_j$ is measurable is a consequence of Lemma A.1 in Ref. 14.

2. If $I$ as written above is $F$-invariant, then $N \times B_1$ is $F_1$-invariant, where $F_1$ is the first-return map of $F$ onto $N \times \Omega^{\mathbb{Z}}$. Consider a typical $\ell \in B_1$ in the following sense: for $\mu_0$-almost every $x \in N^1$, the $F_1$-orbit of $(x,\ell)$ is entirely included in $N \times B_1$; also, looking at (A5), the LT $\ell$ possesses a positive-measure set of trajectories entering $C_0$ through $G^1$ and leaving it through $G^2$. This implies that there exists an $x \in N^1$ such that $F(x,\ell) \in N \times B_1$ and $F(x,\ell) = (x',\sigma(\ell))$ for some $x'$. Hence, $\sigma(\ell) \in B_1$. Considering that this happens for $\Pi$-almost all $\ell \in B_1$, we obtain $\sigma(B_1) \subseteq B_1 \mod \Pi$. (A1) then implies that $\Pi(B_1) \in \{0,1\}$. The analogous assertion for $B_2$ can be proved by using $F_2$, the first-return map onto $N^2 \times \Omega^{\mathbb{Z}}$; the existence of a nonsingular trajectory going from $G^2$ to $G^1$, and $\sigma^1$ instead of $\sigma$.

3. It cannot happen that $N^2 \times \Omega^{\mathbb{Z}}$ and $N^2 \times \Omega^{\mathbb{Z}}$ are different ergodic components because via (A5), for $\Pi$-almost every $\ell \in \Omega^{\mathbb{Z}}$, there is a positive $\mu_0$-measure of points $x \in N^1$ for which $F(x,\ell) \in N^2 \times \Omega^{\mathbb{Z}}$. Q.E.D.

As explained in the last paragraph of Sec. III, Proposition 4.1 and Theorem 4.1 yield our main result:

**Theorem 4.2**: Under assumptions (A1)–(A5), $(M,T,\mu)$ is recurrent for $\Pi$-almost every $\ell \in \Omega^{\mathbb{Z}}$.

V. EXTENSIONS

If we look at the proof of Theorem 4.1, it is apparent that its key argument is that each horizontal fiber $N \times \Omega^{\mathbb{Z}}$ is part of the same ergodic component. Once that is known, one simply uses (A5) to show that a given ergodic component invades the whole phase space, first for the map $F_1$ and then for the map $F$ itself. The details of the dynamics are not relevant for this argument.

By Theorem 3.2, the ergodicity of the point of view of the particle implies the recurrence of our cocycle, because the cocycle is one-dimensional. Thus, as long as we deal with systems in which the position of the particle can be described, in a discrete sense, by a one-dimensional cocycle, the foregoing arguments can be used to prove the almost sure recurrence of a more general class of LTs.

In the present section we sketch the construction of some of these extensions.

A. Same gates, different cells

There is no reason why all the cells $C_n$ should be the same polygon. One can easily consider random cells $C_n$ in which the border too depends on the random parameter $\ell_n$. This can be devised by putting extra flat scatterers in a sufficiently large cell in order to produce any desired shape; see
As long as each cell has two opposite congruent gates and (A1)–(A5) are verified, all the previous results continue to hold.

In fact, one can allow for the distance between the gates to vary with $n$ as well [in Eq. (4.1) simply replace $\tau^x$ with the cell-dependent local translation $\tau^x_{\ell_n}$]. An example of this type of LT is shown in Fig. 5.

**B. Same cells, polygates**

One can also define $G_j$ to be the union of a finite number of sides $G_i$, with $i$ varying in some index set $I$, provided that there is a translation $\tau$ such that $\tau(G_1)=G_2$; see Fig. 6. However, in order for steps 2 and 3 of the proof of Theorem 4.1 to hold, (A5) needs to be replaced by the following.

(A5') For almost every $\ell$, all $j,j' \in \{1,2\}$ and all $i,i' \in I$, there is a nonsingular trajectory entering $C_0$ through $G_i$ and leaving it through $G_i'$. In order to implement this idea, we need to slightly change our previous notation. Let $\{G_j\}_{j=1}^p$ be a fixed ordering of the $p$ congruent sides of $C$ mentioned above. For any such $j$, let $N^j$ denote the transparent, incoming, cross section relative to $G_j$, as in Eq. (2.1). Then set $N := \cup_j N^j$.

We assume that there exist two functions $j_1,j_2: \Omega \to \{1,\ldots,p\}$ such that $j_1(\omega) \neq j_2(\omega), \forall \omega$. This is how $\omega$ specifies that $G_{j_1}$ and $G_{j_2}$ are the left and right gates, respectively, of $C$.

In lieu of $R_\omega$, cf. Eq. (4.1), we use the more general map $R_\ell : \mathcal{N} \to \mathcal{N}$ defined as follows. For $x=(q,v) \in \mathcal{N}$, let $G_l$ be the first side of its kind that the forward flow trajectory of $x$ hits within $C$, and denote by $q_1$ and $v_1$, respectively, the hitting point in $G_l$ and the precollisional velocity there (see Fig. 3).

**C. From translation to general isometry**

Another hypothesis that is not crucial is that $G_1$ is mapped onto $G_2$ via a translation. One can imagine that $Z$ acts upon the LT via a general isometry, for example, a roto-translation, as in Fig. 7.

The only problem, in this case, is that quite generally, the resulting tube will have self-intersections. One can simply do away with it by disregarding the self-intersections, e.g., by declaring that any two portions of the tube that intersect in the plane actually belong to different sheets of a Riemann surface.

**D. Random gates and random isometries**

Assume that the fundamental domain is a polygon $C$ such that $p$ of its sides ($p \geq 2$) are congruent. In this case it is possible to randomize the choice of the gates too. That is, one can let the random parameter $\ell_n$ decide which of the $p$ congruent sides of $C_n$ will play the role of the “left” and “right” gates. Moreover, $\ell_n$ can also prescribe how the right gate of $C_n$ attaches to the left gate of $C_{n+1}$; see Fig. 8.
FIG. 8. (Color online) A LT with random gates (in this case p=3, see text).

- If $j = j_2(\ell_0)$ then $R_\ell x := \xi_0^{\ell_0} \circ p_{j_2}(\ell_0) \circ \xi_1(q_1, v_1)$. Here, $p_{j_2}$ is the transformation that rigidly maps the outer pairs $(q_1, v_1)$ based in $G'$ onto the inner pairs based in $G''$ (it is a rototranslation in the $q$ variable); and $\xi_0: \mathbb{N} \to \mathbb{N}$, depending on the usual random parameter $\omega$, is either the identity or the transformation that flips all the segments $G_j$ and changes the $v$ variable accordingly. So, through $\xi_0$, $e_n$ decides whether $C_n$ and $C_{n+1}$ have the same or opposite orientations (cf. Fig. 8). In this case, the exit function is set to the value $e(x, \ell_0) := 1$.

- If $j = j_1(\ell_0)$ then, in accordance with the previous case, $R_\ell x := \xi_0^{\ell_0} \circ p_{j_1}(\ell_0) \circ \xi_1(q_1, v_1)$ (notice that $\xi_0^{\ell_0} = \xi_0$). In this case, $e(x, \ell_0) := -1$.

- For all the other $j$, $R_\ell x := (q_1, v_2)$, where $v_2 := v_1 + 2(v_1 \cdot o_j) o_j$ is the postcollisional velocity corresponding to a billiard bounce against $G'$ with incoming velocity $v_1$ ($o_j$ denoted the inner normal to $G'$). For this last case, $e(x, \ell_0) := 0$.

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