Abstract

We give a construction of a tree in which the contact process with any positive infection rate survives but, if a certain privileged edge $e^*$ is removed, one obtains two subtrees in which the contact process with infection rate smaller than $1/4$ dies out.

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1 Introduction

In this paper, we present an example of interest to the discussion of how the behaviour of interacting particle systems can be affected by local changes in the graph on which they are defined.

The contact process on a locally finite and connected graph $G = (V, E)$ with rate $\lambda \geq 0$ is a continuous-time Markov process $(\xi_t)_{t \geq 0}$ with state space $\{0, 1\}^V$ and generator

$$L f(\xi) = \sum_{x \in V, \xi(x) = 1} \left[ f(\xi^0\to x) - f(\xi) + \lambda \sum_{y \in V : \{x, y\} \in E} (f(\xi^1\to y) - f(\xi)) \right],$$

where $f$ is a local function, $\xi \in \{0, 1\}^V$ and, for $i \in \{0, 1\}$ and $z \in V$,

$$\xi^{i\to z}(w) = \begin{cases} i, & \text{if } w = z; \\ \xi(w), & \text{otherwise}, \end{cases} \quad w \in V.$$
We say that two points \( D \) we regard each vertex set and edge set of \( G \) Combining (1.2) and (1.3), it is seen that the probability has the distribution of the contact process with initial configuration \( \{0, \infty\} \). Let \( x \) transitive if, for any two vertices \( \lambda \) and \( V \) connected graphs with the same vertex set \( P \). Assume \( G = (V, E) \) and \( G' = (V, E') \) are two connected graphs with the same vertex set \( V \) and \( E' = E \cup \{\{x, y\}\} \), with \( x, y \in V \). Then, \( \lambda_c(G) = \lambda_c(G') \). Jung [3] proved this conjecture for vertex-transitive graphs (a graph \( G \) is vertex transitive if, for any two vertices \( x \) and \( y \), there exists an automorphism in \( G \) that maps \( x \) into \( y \)). Proving the conjecture in full generality is still an open problem. Rather than making progress on this problem, we consider a slightly different line of inquiry. Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs with disjoint vertex sets \( V_1 \) and
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Let \( x \in V_1, y \in V_2 \) and define \( G = (V,E) \) with \( V = V_1 \cup V_2 \) and \( E = E_1 \cup E_2 \cup \{ \{x,y\} \} \)
(that is, we connect the two graphs using the edge \( \{x,y\} \)). It follows from (1.2) that \( \lambda_c(G) \leq \min(\lambda_c(G_1), \lambda_c(G_2)) \), and it is natural to ask whether or not the inequality can be strict. Strict inequality would mean that, for some \( \lambda < \min(\lambda_c(G_1), \lambda_c(G_2)) \), the contact process with rate \( \lambda \) on \( G \) survives. This leads to a curious situation: since the process is subcritical on both \( G_1 \) and \( G_2 \), under \( P^\lambda_G \) there are almost surely no infinite infection paths entirely contained either in \( G_1 \) or in \( G_2 \), so any infinite infection path in \( G \) needs to traverse the edge \( \{x,y\} \) infinitely many times.

We present an example of a graph in which this situation indeed occurs.

**Theorem 1.2.** There exists a tree \( G = (V,E) \) with a privileged edge \( e^* \) so that

\[
\lambda_c(G) = 0
\]

and, letting \( G_1, G_2 \) be the two subgraphs of \( G \) obtained by removing \( e^* \), we have

\[
\lambda_c(G_1), \lambda_c(G_2) \geq \frac{1}{4}.
\]

We end this Introduction discussing some related works in the interacting particle systems literature (apart from the already mentioned [3]). In [6], Madras, Schinazi and Schonmann considered the contact process on \( Z \) in deterministic inhomogeneous environments – for them, this means that the recovery rates (that is, the rates of transition from state 1 to state 0) are vertex-dependent and deterministic, while the infection rate is the same everywhere. Among other results, they showed that if the recovery rate is equal to 1 everywhere except for a sufficiently sparse set \( S \subseteq Z \), where it is equal to some other value \( b \in (0,1) \), then the critical infection rate \( \lambda_c \) is the same as that of the original process on \( Z \). In [8], Newman and Volchan studied a contact process on \( Z \) in an environment in which the recovery rates are chosen randomly, independently among the vertices (the infection rate is again constant). They give a condition for the recovery rate distribution under which the process survives for any value of the infection rate (similarly to what happens to our graph \( G \) of Theorem 1.2). In [2], Handjani exhibited a modified version of the voter model (which is another class of interacting particle system) in which modifications of the flip mechanism in a single site can change the probability of survival of the set of 1’s from zero to positive.

## 2 Notation and preliminary results

Given a set \( A \), the indicator function of \( A \) is denoted \( \mathbb{1}_A \) and the cardinality of \( A \) is denoted \( |A| \).

Given a graph \( G = (V,E) \), the degree of \( x \in V \) is denoted \( \text{deg}_G(x) \), the graph distance between \( x, y \in V \) is \( \text{dist}_G(x,y) \) and the ball of radius \( R \) with center \( x \) is \( B_G(x,R) \). We omit \( G \) from the notation when it is clear from the context. We sometimes abuse notation and associate \( G \) with its set of vertices (so that, for example, \( |G| \) denotes the number of vertices of \( G \)). A star graph \( S \) with hub \( o \) on \( n \) vertices is a tree with one internal node (\( o \)) and \( n-1 \) leaves.

We always assume that the contact process is constructed from the graphical construction. Given \( A, B \subseteq V, J_1, J_2 \subseteq [0,\infty) \), we write \( A \times J_1 \leftrightarrow B \times J_2 \) if \( (x,t_1) \leftrightarrow (y,t_2) \) holds for some \( x \in A, y \in B, t_1 \in J_1 \) and \( t_2 \in J_2 \).

In the remaining part of this section we will describe five preliminary results that will be needed in the proof of Theorem 1.2. We start with the following.

**Lemma 2.1.** For any \( \lambda \leq \frac{1}{2}, \) letting \( I_n = \{1, \ldots, n\} \), we have

\[
P^\lambda_Z (\xi^{I_n}_t \subseteq I_n \ \forall t) \geq \frac{1}{2}.
\]
Proof. Define
\[ L_t = \inf \{ x : \xi_t^L(x) = 1 \}, \quad R_t = \sup \{ x : \xi_t^L(x) = 1 \}, \quad t \geq 0, \]
with \( \inf \emptyset = \infty \) and \( \sup \emptyset = -\infty \). It is readily seen that \( R_t \) is stochastically smaller than the continuous-time Markov chain \( (X_t) \) on \( Z \) with \( X_0 = n \) which jumps one unit to the right with rate \( \lambda \) and jumps one unit to the left with rate 1. Hence,
\[ P_Z^\lambda (R_t < n + 1 \forall t) \geq P (X_t < n + 1 \forall t) \geq \frac{3}{4}, \]
by an elementary computation for biased random walk on \( Z \). Similarly \( P_Z^\lambda (L_t > 0 \forall t) \geq \frac{3}{4} \), so
\[ P_Z^\lambda (\xi_t^L \subseteq I_n \forall t) = P_Z^\lambda (R_t < n + 1 \text{ and } L_t > 0 \forall t) \geq \frac{1}{2}. \]

Our remaining four preliminary results are taken from [7]. The following shows that the contact process survives on a large star graph \( S \) for a time that is exponential in \( \lambda^2 |S| \). It is a refinement of the first result to this effect that appeared in [1] in Lemma 5.3.

**Lemma 2.2.** ([7], Lemma 3.1) There exists \( \bar{c} > 0 \) such that, if \( \lambda < 1 \), \( S \) is a star with hub \( o \) so that \( \deg(o) > 64e^2 \cdot \frac{1}{\lambda^2} \) and \( |\xi_0| > \frac{1}{16e} \cdot \lambda \deg(o) \), then
\[ P^\lambda_S (\xi_{c\lambda^2 \deg(o)} \neq \emptyset) \geq 1 - e^{-\bar{c} \lambda^2 \deg(o)}. \]  

(2.2)

In [7], this lemma was applied to guarantee that in a connected graph \( G \) an infection around a vertex with sufficiently high degree is maintained long enough to produce an infection path that reaches another vertex at a certain distance.

**Lemma 2.3.** ([7], Lemma 3.2) There exists \( \lambda_0 > 0 \) such that, if \( 0 < \lambda < \lambda_0 \), the following holds. If \( G \) is a connected graph and \( x, y \) are distinct vertices of \( G \) with
\[ \deg(x) > \frac{7}{\bar{c} \lambda^2} \log \left( \frac{1}{\lambda} \right) \cdot \text{dist}_G(x, y) \text{ and } \frac{|\xi_0 \cap B(x, 1)|}{\lambda |B(x, 1)|} > \frac{1}{16e}, \]
then
\[ P^\lambda_G \left( \exists t : \frac{|\xi_t \cap B(y, 1)|}{\lambda |B(y, 1)|} > \frac{1}{16e} \right) > 1 - 2e^{-\bar{c} \lambda^2 \deg(x)}. \]  

(2.3)

(Note that \( \bar{c} \) is the same constant that appeared Lemma 2.2).

In the opposite direction as Lemma 2.2, the following result bounds from below the probability that the infection disappears from a star graph within time \( 3 \log(1/\lambda) \).

**Lemma 2.4.** ([7], Lemma 5.2) If \( \lambda < \frac{1}{2} \) and \( S \) is a star, then
\[ P^\lambda_S (\xi_{C \lambda^2 \log(\frac{1}{\lambda})} = \emptyset) \geq \frac{1}{4} e^{-16\lambda^2 |S|}. \]  

(2.4)

Using the Markov property and (1.2), for any \( t \geq 0 \) we have
\[ P^\lambda_S (\xi_t^S \neq \emptyset) \leq P^\lambda_S (\xi_{C \lambda^2 \log(\frac{1}{\lambda})} \neq \emptyset) \cdot \left[ \frac{1}{16e} \right] \left( 1 - \frac{1}{4} e^{-16\lambda^2 |S|} \right)^{C \lambda^2 |S|} \]
This implies that, if \( C > 16 \) and \( |S| \) is large enough, then the infection will with high probability disappear before time \( \exp\{C \lambda^2 |S|\} \). This estimate matches (except for the value of the constant in the exponential) the one that follows from Lemma 2.2.

It follows from Lemma 2.2 that vertices of degree much larger than \( \frac{1}{C \lambda^2} \) will sustain the infection for a long time. The last preliminary lemma in our list deals with tree graphs in which such big vertices are absent; in this case, it is unlikely that the infection spreads.
We clearly have

\[ E \text{ is as follows.} \]

\[ G_3.1 \text{ Construction of } G \]

3.1 Construction of \( G \)

Our graph \( G \) will be equal to the one-dimensional lattice \( \mathbb{Z} \) with the modification that a few vertices, denoted \( o_1, o_2, \ldots \), are given extra neighbors, so that their degrees become increasingly large. The extra neighbors are vertices which we add to the graph as leaves.

We start defining sequences of integers \( (o_i)_{i \geq 1}, (d_i)_{i \geq 1} \) satisfying

\[ 0 > o_1 > o_3 > \cdots, \quad 0 < o_2 < o_4 < \cdots, \quad 0 < d_1 < d_2 < \cdots. \]

The definition will be inductive. We set \( d_1 = 1 \), \( o_1 = -1 \) and \( o_2 = 2 \). Assume we have already defined \( o_1, \ldots, o_i \) and \( d_1, \ldots, d_{i-1} \). Then, set

\[
o_{i+1} = \begin{cases} o_{i-1} + i \cdot \sum_{j=1}^{(i-1)/2} d_{2j} & \text{if } i \text{ is odd,} \\ o_{i-1} - i \cdot \sum_{j=0}^{(i-2)/2} d_{2j+1} & \text{if } i \text{ is even,} \end{cases} \quad d_i = i \cdot |o_i - o_{i+1}|. \tag{3.1}
\]

We clearly have

\[ d_i > i! \] \tag{3.2}

Now, for each \( i \geq 1 \), let \( \{x_1^i, \ldots, x_{d_i}^i\} \) be a set with \( d_i \) distinct elements (for distinct values of \( i \), these sets are assumed to be disjoint). Then let

\[ G = (V, E), \quad V = \mathbb{Z} \cup \bigcup_{i=1}^{\infty} \{x_1^i, \ldots, x_{d_i}^i\}, \quad E = E(\mathbb{Z}) \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{d_i} \{\{o_i, x_j^i\}\}, \]

where \( E(\mathbb{Z}) \) is the set of edges of \( \mathbb{Z} \). The construction is illustrated on Figure 1. We let \( e^* \) be the edge \( \{0, 1\} \). When \( e^* \) is removed, \( G \) is split into two subgraphs: we let \( G_- \) denote the one associated to the negative half-line, and \( G_+ \) the one associated to the positive half-line.

![Figure 1: The graph \( G \).](http://www.imstat.org/ecp/)

The strategy in the construction of \( G \), and in particular in the choice of \( (d_i) \) and \( (o_i) \), is as follows.

- As we will prove in the next subsection, for any \( \lambda > 0 \), if \( i \) is sufficiently large, the star \( B(o_i, 1) \) is large enough to sustain the infection long enough that it reaches \( B(o_{i+2}, 1) \) with high probability. The infection is then sustained there long enough to reach \( B(o_{i+2}, 1) \) and so on. As the probability of the intersection of all these events is close to 1, there is survival. Note that, as observed in the Introduction, indeed the infection necessarily relies on infinitely many traversals of \( e^* \) in order to survive.
We will abuse notation and refer to the above sets as subgraphs of $G$. We have

\[ 0 \leq P_G \left( \alpha_{\cdot} = e \right) \leq 1 \]

where $G$ is coming from Lemma 2.2. Then, by Lemma 2.3,

\[ P_G \left( \exists t : P_G^A \left( [\xi_i^A \cap B(o_{i+1}, 1)] > 1/16e \right) > 0 \right) > 1 - 2e^{-c\lambda^2 d_i}. \]

The desired result now follows from (3.2), the Strong Markov Property and a union bound.

### 3.3 Proof of $\lambda_c(G) \geq \frac{1}{4}$

We will only carry out the proof of $\lambda_c(G) \geq \frac{1}{4}$; the proof for $G_−$ is similar. As explained in the Introduction, it is sufficient to show that, in the contact process on $G_+$ with rate

\[ \lambda = \frac{1}{4} \]  

(3.3)

and started with a single infection located at vertex 1, the infection almost surely disappears, i.e.

\[ P_{G_+}^\lambda (\xi^i t = \emptyset \forall t) = 0. \]

(3.4)

Define the sets of vertices

\[ S_j = \{o_j, x^1_j, \ldots, x^d_j\}, \quad j \in \{2, 4, \ldots\} \]

\[ H_0 = \{1\}, \]

\[ H_j = (o_j, o_{j+2}) \cap \mathbb{Z}, \quad j \in \{2, 4, \ldots\}, \]

\[ G_i = H_0 \cup \left( \bigcup_{j=1}^{i/2} (S_{2j} \cup H_{2j}) \right), \quad i \in \{2, 4, \ldots\}. \]

We will abuse notation and refer to the above sets as subgraphs of $G_+$; for instance, $H_2$ will be the subgraph with vertex set defined above and set of edges having both extremities in this vertex set. We now fix an arbitrary $i \in 2\mathbb{N}$. Define

\[ \tau = \exp \left( \frac{i}{2} (d_2 + d_4 + \cdots + d_i) \right). \]

(3.5)

We have

\[ P_{G_+}^\lambda (\xi^i t = \emptyset \forall t) \leq P_{G_+}^\lambda (\xi^i t = \emptyset) \]

\[ \leq P_{G_+}^\lambda (\xi^i t = \emptyset, \xi^i t \subseteq G_i \forall t \leq \tau) + P_{G_+}^\lambda (\xi^i t \neq \emptyset, \xi^i t \not\subseteq G_i \text{ for some } t \leq \tau) \]

\[ \leq P_{G_+}^\lambda (\xi^G \neq \emptyset) + P_{G_+}^\lambda (\{o_i\} \times [0, \tau] \not\leftrightarrow \{o_{i+2}\} \times [0, \tau]). \]

(3.6)
We bound the two terms on (3.6) separately, starting with the second:
\[
P_{G_i}^\lambda (\{0_i\} \times [0, \tau] \leftrightarrow \{0_{i+2}\} \times [0, \tau])
\]
\[
= P_{H_{0_i \cup 0_{i+2}}}^\lambda (\{0_i\} \times [0, \tau] \leftrightarrow \{0_{i+2}\} \times [0, \tau])
\]
\[
\leq (\tau + 1) \cdot (2\lambda)^{\text{dist}(0_i, 0_{i+2})}
\]
\[
\leq 2\exp \left\{ i(d_2 + d_4 + \cdots + d_i) \left( \frac{1}{2} - \log(2) \right) \right\} < \exp \{-d_i\} \quad (3.7)
\]
if \(i\) is large enough.

We now turn to the first term in (3.6). First define
\[
t_1 = 3\log \left( \frac{1}{\lambda} \right), \quad L = i\log (\text{dist}(0_i, 0_{i+2})) = i\log (i(d_2 + d_4 + \cdots + d_i)).
\]
We will assume that \(i\) is large enough (depending on \(\lambda\)) so that \(L > t_1\). Using the Markov property and (1.2), we have
\[
P_{G_i}^\lambda (\xi_{\tau}^{G_i} \neq \emptyset) \leq P_{G_i}^\lambda (\xi_{\tau}^{G_i} \neq \emptyset)^{\lceil \tau / L \rceil}. \quad (3.9)
\]
We then bound
\[
P_{G_i}^\lambda (\xi_{\tau}^{G_i} = \emptyset)
\]
\[
\geq \prod_{j=1}^{i/2} P_{G_i}^\lambda (\xi_{t_j}^{S_1} \subseteq S_{2j} \forall t_j, \xi_{t_1}^{S_2} = \emptyset) \cdot \prod_{j=0}^{i/2} P_{G_i}^\lambda (\xi_{t_j}^{H_1} \subseteq H_{2j} \forall t_j, \xi_{t_j}^{H_2} = \emptyset). \quad (3.10)
\]
Now, for all \(j \leq i/2\),
\[
P_{G_i}^\lambda (\xi_{t_j}^{H_1} \subseteq H_{2j} \forall t_j, \xi_{t_j}^{H_2} = \emptyset) \geq P_{G_i}^\lambda (\xi_{t_j}^{H_1} \subseteq H_{2j} \forall t_j) - P_{H_{2j}}^\lambda (\xi_{t_j}^{H_2} \neq \emptyset)
\]
\[
\geq \frac{1}{2} - |H_{2j}|^3 \cdot e^{-L/4}
\]
\[
\geq \frac{1}{2} - (\text{dist}(0_i, 0_{i+2}))^{2-i/4} \geq \frac{1}{4} \quad (3.11)
\]
if \(i\) is large enough. Again for all \(j \leq i/2\), we have
\[
P_{G_i}^\lambda (\xi_{t_j}^{S_1} \subseteq S_{2j} \forall t_j, \xi_{t_j}^{S_2} = \emptyset)
\]
\[
\geq \frac{1}{4} \exp \left\{ -16\lambda^2 |S_{2j}| \right\} \cdot \exp \{-2\lambda t_1\}
\]
\[
\geq \frac{1}{4} \frac{\lambda^{6\lambda}}{16} \exp \{-17\lambda^2 d_2 + \cdots + d_i\} \quad (3.12)
\]
Using (3.11) and (3.12) in (3.10), we get
\[
P_{G_i}^\lambda (\xi_{\tau}^{G_i} = \emptyset) \geq \left( \frac{\lambda^{6\lambda}}{16} \right)^{i/4} \cdot \exp \{-17\lambda^2 (d_2 + d_4 + \cdots + d_i)\}
\]
\[
\geq \exp \{-18\lambda^2 (d_2 + d_4 + \cdots + d_i)\}\quad (3.12)
\]
if \(i\) is large enough; using this in (3.9), we get
\[
P_{G_i}^\lambda (\xi_{\tau}^{G_i} \neq \emptyset) \leq \exp \left\{ - \exp \{-18\lambda^2 (d_2 + d_4 + \cdots + d_i)\} \cdot \exp \left\{ \frac{2}{(i-1)} (d_2 + d_4 + \cdots + d_i) \right\} \right\}
\]
\[
< \exp \{-d_i\} \quad (3.13)
\]
if \(i\) is large enough.
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We can conclude using (3.7) and (3.13) in (3.6) that $P_{G_i}^{\lambda} (\xi_t^i \neq \emptyset \forall t) < 2 \exp \{-d_i\}$ for all $i$, so (3.4) follows.

References


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