SUMMARY AND CONCLUSIONS

This dissertation dealt with the Lorenz-96 model, constructed by Lorenz (2006a) as a test model to study the predictability of the atmosphere. The main goal of the current work was to unravel the dynamical structure of the Lorenz-96 model. To achieve this, we explored and exploited the symmetry of the model and analysed the bifurcations of the stable attractors using both analytical and numerical means.

A clear overview of the transition of the stable trivial equilibrium (2.7) via different bifurcations into one or more stable periodic orbits is obtained for any dimension $n$. The spatiotemporal properties of these periodic attractors are investigated, as well as their routes to chaos for $F > 0$. These properties and routes depend very much on the dimension; therefore, our research question asks especially which bifurcations persist and which dynamical properties stabilise in the limit $n \to \infty$.

In the following, we will briefly review the major contributions of this thesis and discuss corresponding issues. Thereafter, we draw the main conclusions of our findings. We end with a list of open questions that arise from our results to motivate further investigation of the Lorenz-96 model.
6.1 Summary of Results

Symmetries First of all, the Lorenz-96 model (1.4) is in any dimension equivariant with respect to a cyclic left shift $\gamma_n$, which makes the group of symmetries of the model isomorphic to $\mathbb{Z}_n$. Every subgroup of symmetries gives rise to an invariant manifold.

An important result of chapter 2 is that these invariant manifolds allow us to extrapolate results that are proven for a certain dimension $n$ to all multiples of $n$. However, this method does not guarantee to give the complete bifurcation pattern and route to chaos for any multiple of the lowest possible dimension, but only the dynamics restricted to the corresponding invariant manifold. It is also possible that another bifurcation will take place before the phenomena extrapolated from low dimension and thus a different attractor gains stability. These phenomena are indeed observed in chapter 5. The use of invariant manifolds — together with the symmetrical nature of the Lorenz-96 model — enhances our understanding of the model and its dynamics.

Bifurcations and Waves for $F > 0$ The equilibrium $x_F$ — which has full symmetry — plays a crucial role in the dynamics for parameter values $F$ close to 0. For both positive and negative forcing we have demonstrated in chapter 3 how this equilibrium loses stability and bifurcates eventually into one or more stable periodic attractors.

Let us first discuss the case $F > 0$. For all dimensions $n \geq 4$ it has been proven that $x_F$ exhibits one or more Hopf and Hopf-Hopf bifurcations. Also, it has been shown that if the first bifurcation for $F > 0$ is not a Hopf-Hopf bifurcation, then it is a supercritical Hopf bifurcation by providing an exact formula for the first Lyapunov coefficient. This formula holds for all Hopf bifurcations of $x_F$, including the ones for negative $F$.

We have shown in chapter 4 that the stable periodic orbits, emerging from the Hopf bifurcations for $F > 0$, have the physical interpretation of travelling waves. Their spatial wave number is equal to the index of the eigenpair that crosses the imaginary
axis and increases linearly with \( n \). In contrast, their period tends to a finite limit as \( n \to \infty \).

The influence of symmetry on the dynamics of the Lorenz-96 model for \( F > 0 \) is not very large. The bifurcations mentioned above are not initiated by the symmetry. On the other hand, a periodic attractor has symmetries when its spatial wave number \( l \) has a divisor in common with the dimension of the model, i.e. \( \gcd(l,n) = g > 1 \). In that case, the periodic attractor is contained in \( \text{Fix}(G^{n/g}) \). It might be possible to find more symmetries via other equilibria than the trivial one, but it is in general nontrivial to locate them.

**Organising Centre** In chapter 3 a necessary and sufficient condition has been proven for Hopf-Hopf bifurcations of \( x_F \) to occur for \( F > 0 \). To unfold these codimension two bifurcations we introduced the two-parameter model (2.13) by adding an extra parameter \( G \) to the Lorenz-96 model via a Laplace-like diffusion term. For this particular unfolding the Hopf bifurcations of the trivial equilibrium are given by straight lines in the \((F,G)\)-plane and their intersections give Hopf-Hopf bifurcations. We have shown that such a codimension two bifurcation point acts as an organising centre in the original model when it destabilises the equilibrium \( x_F \) and when it occur for \( G \) close to 0.

To illustrate, in the special case of dimension \( n = 12 \) a Hopf-Hopf bifurcation lies on the line \( G = 0 \) and is in fact also the first bifurcation through which the trivial equilibrium bifurcates for \( F > 0 \) in the original model. Two Neimark-Sacker bifurcation curves emanate from the Hopf-Hopf point and bound a lobe-shaped region in the \((F,G)\)-plane in which two stable travelling waves with different wave numbers coexist, as shown in chapter 4. Since this region actually intersects the line \( G = 0 \), multistability also occurs in the original Lorenz-96 model, meaning that different stable waves coexist for the same parameter values.

In general, we find two Hopf-Hopf bifurcations near the \( F \)-axis for all dimensions \( n > 12 \), that can create two multistability lobes intersecting each other. This phenomenon leads to the coexistence of three stable waves for \( G = 0 \) and an interval of \( F \)-values. To
conclude, adding an extra parameter to the Lorenz-96 model helps to explain the dynamics that is observed in the original model.

**bifurcations and waves for \( F < 0 \)**  For \( F < 0 \), we show in chapter 3 that the bifurcation pattern of the stable attractor depends on the dimension of the model. We can distinguish three different cases. Firstly, the case of odd dimensions for which symmetry does not play a role:

1. For odd \( n \), the first bifurcation of the equilibrium \( x_F \) is a supercritical Hopf bifurcation, similar to the case of positive \( F \) — see above. The periodic attractor that appears after the Hopf bifurcation is a travelling wave, whose wave number equals \((n - 1)/2\) and the period scales as \( O(4n) \).

The other two cases are both for even dimensions, where symmetries play an important role in the dynamics. Using a theorem on bifurcations in \( \mathbb{Z}_n \)-equivariant dynamical systems, we have proven analytically the existence of a supercritical pitchfork bifurcation at \( F_{P,1} = -\frac{1}{2} \) in dimension \( n = 2 \). Consequently, by utilising the invariant manifolds, we establish that in any even dimension a pitchfork bifurcation takes place at \( F_{P,1} \) as the first bifurcation of \( x_F \). Besides, a pitchfork bifurcation gives rise to two stable \( \gamma_n/2 \)-conjugate equilibria, whose bifurcations depend again on the dimension:

2. If \( n = 4k + 2 \) for some \( k \in \mathbb{N} \), then each of these two equilibria undergoes a supercritical Hopf bifurcation.

3. If \( n = 4k \), \( k \in \mathbb{N} \), the conjugate equilibria have enough symmetry left, such that a second, successive pitchfork bifurcation occurs at \( F_{P,2} = -3 \). Again, this is analytically shown for the smallest dimension \( n = 4 \). All four stable equilibria, generated through this second pitchfork bifurcation, exhibit a supercritical Hopf bifurcation subsequently.

The occurrence of pitchfork bifurcations before the Hopf bifurcation leads again to multistability: the Hopf bifurcations that occur for all stable equilibria after one, resp. two, pitchfork bifurcations lead to the coexistence of two, resp. four, stable periodic attractors.
It is shown that these attractors represent stationary waves in the model. The role of the pitchfork bifurcation is to change the mean flow which in turn changes the propagation of the wave.

Additionally, numerical investigation showed that there are more successive pitchfork bifurcations if the dimension is divisible by larger powers of two. Let the dimension be uniquely given by $n = 2^q p$, where $q$ is a non-negative integer and $p$ is odd. We then conjecture that in any dimension $n$ the number of successive pitchfork bifurcations is exactly equal to $q$. However, to prove this result analytically is a nontrivial task, since obtaining explicit formulae for the nontrivial equilibria that arise from each pitchfork bifurcation is not feasible and, moreover, their Jacobian is no longer circulant. Besides, to prove other dynamical properties beyond the $l$-th pitchfork bifurcation becomes increasingly difficult, since the lowest dimension needed is $n = 2^l$ and thus increases exponentially with $l$. On the other hand, once we have found an equilibrium in a certain invariant subspace $\text{Fix}(G^m_n)$, the relation (3.25) guarantees that the $m-1$ conjugate equilibria have the same properties.

**patterns** Finally, in chapter 5 we have numerically investigated the dynamics of the model for dimensions up to $n = 100$ and parameter values $F > 0$ beyond the first Hopf bifurcation value. We focused in particular on the routes to chaos of the periodic attractors. For general $n$, these routes are numerous and can comprise intermittent transitions, period-doubling cascades and possibly Newhouse-Ruelle-Takens scenarios.

A pattern of attractors with a persisting period-doubling bifurcation has been observed for $n = 5k$, $k = 1, \ldots, 10$. This phenomenon can be explained by the invariant manifolds. However, the pattern is interrupted at $k = 11$, since another bifurcation takes place before the phenomena extrapolated from low dimension and thus a different attractor gains stability.

Furthermore, in dimension $n = 4$ a periodic attractor disappears through a saddle-node bifurcation of limit cycles. After this bifurcation intermittency is detected, which is possibly explained by a nearby heteroclinic cycle between four equilibria. A similar
bifurcation scenario is found for $n = 8$ for a symmetric attractor, which coexists with a non-symmetric attractor. In general, due to the fact that symmetric attractors are contained in a nontrivial invariant subspace $\text{Fix}(G^n)$, they inherit their properties from the attractor in dimension $m$. Thus, they cause repeating dynamics.

6.2 MAJOR CONCLUSIONS

The results in this thesis provide a coherent overview of the bifurcation patterns and the spatiotemporal properties of the resulting waves in the Lorenz-96 model for $n \geq 4$ and $F \in \mathbb{R}$. Our contribution enhances the understanding of the dynamics of the Lorenz-96 model. Based on these findings we can draw five major conclusions:

1. Firstly, our results provide important insights into the symmetrical structure of the Lorenz-96 model. This helps to understand the bigger dynamical structure and the wave structures, described in this work and other dynamical studies — see table 1.3. In particular, the invariant manifolds allow us to reduce the dimension and therefore to extrapolate results that are proven for a certain dimension $n$ to its multiples.

2. Secondly, we have shown the persistence of Hopf and Hopf-Hopf bifurcations for positive forcing and the persistence of up to two pitchfork bifurcations followed by a Hopf bifurcation for negative forcing in any dimension $n \geq 4$. Consequently, for $n \geq 4$ the stable equilibrium eventually bifurcates into one or more periodic attractors. Contrary to this structure, no clear pattern on bifurcations of the periodic attractors — besides symmetric ones — is found, but the routes to chaos seem to depend very much on $n$.

3. Another important point that is illustrated in this thesis is the following: both qualitative and quantitative aspects of the dynamics of the Lorenz-96 model depend on the parity of $n$. Also, the dimension determines the possible symmetries of the model. The dependence of the dynamics on $n$ shows the importance
of choosing appropriate values of the parameters. Since the Lorenz-96 model is often used as a model for testing purposes, our results can be used to select the most appropriate values of $n$ and $F$ for a particular application, such as those listed in table 1.2. Meanwhile, the periodic attractors representing travelling or stationary waves can bifurcate into chaotic attractors representing irregular versions of these waves and their spatiotemporal properties are inherited from the periodic attractor. This means that our results on the spatiotemporal properties of waves apply to broader ranges of the parameter $F$ than just in a small neighbourhood of the Hopf bifurcation.

4. Multistability occurs in the Lorenz-96 model in (at least) two ways:
   - Organising centres in the form of Hopf-Hopf bifurcations generate multiple stable periodic attractors that can coexist for some $F > 0$.
   - Simultaneous Hopf bifurcations subsequent to one or two pitchfork bifurcations generate two or four stable and conjugate periodic orbits that coexist for some $F < 0$ below the corresponding Hopf bifurcation value.

5. Lastly, the observation that the wave number of the travelling waves for both positive and negative $F$ increases linearly with $n$ shows that the Lorenz-96 model is not the discretisation of a PDE model.

6.3 Further research

Despite the lack of a clear bifurcation pattern for all dimensions $n$, the Lorenz-96 remains an interesting model to study for its rich dynamics. There are several open problems that arise from our research, which have not been addressed in this work:

1. Further investigation is needed in order to unravel the bifurcations and routes to chaos of the attractors for negative $F$ below the Hopf bifurcation value. How does the symmetry influence the bifurcation patterns for parameter values $F < 0$ beyond the
Hopf bifurcations? In particular, note that, up to and including the Hopf bifurcations, these patterns can be divided into three different cases, which might have consequences for the number of different routes to chaos.

2. Next, Conjecture 3.22 has been verified numerically up to $q = 9$. Show, either numerically or analytically, that the statement holds for all $q \in \mathbb{N}$, so that our conjecture covers all dimensions $n \in \mathbb{N}$. What is the influence of the $2^{q+1} - 1$ conjugate equilibria resulting from the $q$ pitchfork bifurcations on the dynamics of the model?

3. An interesting question regarding symmetries is: what are the consequences of the existence of the invariant manifolds on the routes to chaos for coexisting symmetric and non-symmetric stable attractors?

4. We have reported on a conjecture for periodic orbits $x_P(t)$ whose spatial wave number has a divisor in common with the dimension, $\gcd(l, n) = g > 1$ — see sections 4.2 and 5.2.1. Prove that this implies that then $x_P(t) \in \text{Fix}(G^n_g)$, i.e. such a periodic orbit has some symmetry, which might be inherited as well by the chaotic attractor that arise from it.

5. We have observed that for $n = 36$ a 3-torus exists in a small interval of $F$-values before chaotic attractors are observed — see section 5.1.7. Further studies need to be carried out to unravel the bifurcations of these 3-tori and the associated routes to chaos in the Lorenz-96 model. One possibility might be that it exists due to the coexistence of three stable periodic attractors.

6. In some dimensions, we observed chaotic attractors with more than one positive Lyapunov exponent — for example, $n = 9, 10$ and 24. It would be interesting to investigate how such chaotic attractors can arise.