In chapter 3 we have proven that the stable equilibria of the Lorenz-96 model (2.1) eventually lose stability through either a supercritical Hopf or a Hopf-Hopf bifurcation for all dimensions $n \geq 4$. At these bifurcations a periodic attractor is born which has the physical interpretation of a travelling or stationary wave — see chapter 4.

In this chapter, we explore the dynamics of the Lorenz-96 model numerically for dimensions up to $n = 100$ and for $F > 0$ beyond the first Hopf bifurcation. Thereby, we cover the parameter values that are used most often in applications — see table 1.2.

Firstly, for a few dimensions we comment on routes to chaos and the resulting attractors using tools such as continuation, integration, Poincaré sections and Lyapunov exponents. Our emphasis is on the bifurcations through which the periodic attractor loses stability and the first parameter value of $F$ for which chaos sets in. We designate the system to be chaotic whenever we measure at least one positive Lyapunov exponent.

Moreover, a natural question is to what extent these bifurcations depend on the dimension $n$. Therefore, the second part of this chapter is devoted to the generalisation of the dynamics to higher dimensions and to identify whether patterns can be found in the bifurcations and routes to chaos.
The numerical analysis is carried out using mainly the original Lorenz-96 model (2.1). In some cases, the two-parameter model (2.13) turns out to be useful to explain features observed in the one-parameter model. Whenever the two-parameter system is used, this is stated explicitly — otherwise, $G$ is assumed to be equal to 0.$^\text{1}$

The results in this chapter are mainly contained in (Van Kekem & Sterk, 2018b).

5.1 Individual Routes to Chaos

Beyond the bifurcation value $F_{H}(l_{1}^{+}(n), n)$ for the first Hopf bifurcation we can encounter further bifurcations of the stable periodic orbit. Eventually, this leads to chaotic behaviour. In this section, we will discuss the routes to chaos and some features of the found attractors for a few dimensions and positive $F$. We have selected the dimensions $n \geq 4$ that are small, generate potentially characteristic dynamics for higher dimensions and also based on how often they are used in other studies — see table 1.2. Note that the widely used dimensions $n = 8$ and $n = 40$ are also discussed in (Orrell & Smith, 2003).

In addition, due to multistability of attractors — see section 4.4 — different attractors might be involved in the route to chaos. For negative $F$ the coexistent periodic attractors have the same properties by symmetry, so one may study just one of them. For positive $F$ — the case that is discussed here — we only track the attractor that is stable, show its bifurcations and how it evolves to chaos. For example, in dimension $n = 40$, three stable periodic orbits co-exist in some interval of parameter values, but before chaos sets in already two of them became unstable.

5.1.1 Dimension $n = 4$

In the four-dimensional Lorenz-96 model there is only one Hopf bifurcation, which takes place at $F_{H}(1, 4) = 1$. Continuing the periodic attractor originating from this bifurcation numerically in $F$
and plotting its period against $F$ gives the diagram in figure 5.1. The original periodic orbit disappears through a \textit{saddle-node bifurcation of limit cycles (LPC)} at $F_{\text{LPC}} \approx 11.8382$. In figure 5.2 we observe chaos for parameter values $F \geq 11.84$. Figure 5.3 compares the periodic attractor for $F = 11.83$ with the chaotic attractor for $F = 11.9$, while figure 5.4 shows time series of the first variable for both parameter values. Observe that the dynamics for $F = 11.9$ alternates between approximate periodic behaviour and chaotic behaviour. This is the classical \textit{type 1 intermittency} scenario as described in (Pomeau & Manneville, 1980; Eckmann, 1981). Note that for intermittency we not only need an attractor that has disappeared through a bifurcation, but we also need the global dynamics to be such that it enables recurrent visits to the location of the formerly existing attractor in state space. In our case, such a

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_1}
\caption{Continuation of the periodic orbit for $n = 4$, originating from the first Hopf bifurcation at $F_H$. For parameter values where the cycle is stable, the curve is coloured blue; where it is unstable, it is coloured red. The periodic attractor remains stable until $F \approx 5.0584$ where it exchanges stability with another periodic attractor. However, at $F \approx 8.9432$ the original periodic attractor gains stability again. Also, from $F \approx 8.5405$ additional limit cycles are created through saddle-node bifurcations of limit cycles (LPCs). Finally, at $F_{\text{LPC}} \approx 11.8382$, it disappears through an LPC.}
\end{figure}
Figure 5.2: Bifurcation diagrams of attractors in the Lorenz-96 model for $n = 4$. The three largest Lyapunov exponents are plotted as a function of the parameter $F$. At $F_{\text{LPC}} \approx 11.8382$ a periodic attractor disappears through an LPC and a chaotic attractor is detected — see the magnification in the right panel. Compare with figure 5.1.

A global mechanism might be provided by a heteroclinic structure, as we will show below.

At $F \approx 8.5405$ an additional limit cycle appears through an LPC, which is stable for only a short interval. This bifurcation is followed by more saddle-node bifurcations, which accumulate for $F$ between 11.73 and 11.77, as can be seen from figure 5.1. This phenomenon suggests a homoclinic or heteroclinic structure (Kuznetsov, 2004). Similar behaviour has been observed in other atmospheric models (Van Veen, 2003). Analysis of the system for this parameter value indicates a heteroclinic structure. At $F \approx 8.8990$, namely, four pairs of two equilibria appear through fold bifurcations. By numerical continuation we found at $F \approx 12.0812$ — the importance of this value will become clear in a moment — the following coordinates for these equilibria:

$$x^4 \approx (-1.1822, -0.2331, 11.5431, 1.1263),$$

$$y^4 \approx (-2.6682, -1.1663, 6.8133, 1.8484),$$

while the other six equilibria can be obtained by applying the cyclic shift $\gamma_4$ repeatedly, as explained in section 2.3.1.² Both types of equilibria are hyperbolic saddles with three, resp. two, stable eigenvalues. However, only at $F \approx 12.0812$ (which is in the chaotic

² Note that our notation resembles the form of the equilibrium (2.17).
Figure 5.3: Plot of the attractors for $n = 4$ and $F = 11.83$ (red) and $F = 11.9$ (grey). At $F = 11.83$ we have a stable periodic orbit, whereas $F = 11.9$ gives a chaotic attractor which partly resembles the stable periodic orbit. See also figure 5.4.

Figure 5.4: Time series of the first coordinate for the attractors from figure 5.3 with $n = 4$ and $F = 11.83$ (red, periodic) and $F = 11.9$ (black, chaotic). The black curve shows alternating dynamics between approximate periodic and chaotic behaviour which is typical for intermittency.
region) we have numerically detected a \textit{heteroclinic cycle} between the equilibria \(x_j^4, 0 \leq j \leq 3\), using \textsc{MatCont}. A continuation of these connections in the \((F,G)\)-plane for the two-parameter system does not yield any other value \(F\) for which a heteroclinic cycle exist at \(G = 0\). The heteroclinic cycle for \((F,G) \approx (12.0812,0)\) is shown in figure 5.5. Notice the similarity between the right panel and the periodic attractor in figure 5.3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Heteroclinic cycle with four orbits connecting the equilibrium \((5.1)\) and its three \(\gamma_4\)-conjugates for \(n = 4\) and \(F \approx 12.0812\) in three coordinates (left panel). The right panel is a projection on the \((x_1, x_2)\)-plane and shows also the location of the equilibria. Notice the resemblance to the periodic attractor in figure 5.3.}
\end{figure}

\section{Dimension \(n = 5\)}

For \(n = 5\), the first bifurcation after the Hopf bifurcation at \(F_H(1,5) \approx 0.8944\) is a \textit{period-doubling bifurcation} (\textsc{pd}) which occurs at \(F_{\text{PD},1} \approx 3.9379\). This is followed by more \textsc{pds}: the next three \textsc{pds} occur for the parameter values \(F_{\text{PD},2} \approx 4.9819, F_{\text{PD},3} \approx 6.3715, F_{\text{PD},4} \approx 6.6410\), consecutively.

The bifurcation diagrams in figure 5.6 suggest that a cascade of period-doubling bifurcations takes place. After the cascade, a chaotic attractor is detected at \(F = 6.72\) — see figure 5.7. The Poincaré section of this attractor appears to have the structure of a fattened curve. This suggests that the attractor is of \textit{Hénon-like type}, which means that it is the closure of an unstable manifold of an
unstable periodic point of the Poincaré map. We have numerically detected an unstable periodic orbit at \( F = 6.72 \), which corresponds to an unstable period-3 point for the Poincaré return map to the section \( \Sigma = \{ x_1 = 0 \} \). The unstable manifold of this period-3 point was computed with standard numerical techniques which are described in (Simó, 1990). Figure 5.8 shows a magnification of the unstable manifold along with the attractor of the Poincaré map. The two plots are in very good agreement with each other. Therefore, we conjecture the attractor in figure 5.7 to be the closure of the unstable manifold of an unstable periodic orbit.

The bifurcation scenario of \( n = 5 \) turns out to be typical for higher dimensions that are multiples of 5. We will discuss this observation later on in section 5.2.2.

### 5.1.3 Dimension \( n = 6 \)

For \( n = 6 \), the first bifurcation after the Hopf bifurcation at \( F_H(1,6) = 1 \) is a Neimark-Sacker bifurcation (ns), which occurs at \( F_{NS} \approx 5.4567 \). At this bifurcation the periodic attractor loses stability and gives birth to a quasi-periodic attractor in the form of a two-dimensional torus — see figure 5.9. The attractor becomes chaotic for \( F > 6.31 \).
Figure 5.7: A chaotic attractor (left panel) for \((n, F) = (5, 6.72)\), which is after the pd-cascade, and a corresponding Poincaré section defined by \(\Sigma = \{x_1 = 5\}\) (right panel). The latter appears to have the structure of a fattened curve. See also figure 5.8.

Figure 5.8: Magnification of the Poincaré section in the right panel of figure 5.7 (left panel) and the unstable manifold of the period-3 point of the Poincaré return map at the same parameter values (right panel). The plots agree very well with each other which suggests that the attractor in figure 5.7 is the closure of the unstable manifold of the unstable period-3 point.
5.1 Individual Routes to Chaos

Figure 5.9: A 2-torus attractor (left panel) for \((n,F,G) = (6,5.6,0)\) after the \(nS\) bifurcation and the corresponding invariant circle of the Poincaré return map defined by the section \(\Sigma = \{x_1 = 0\}\) (right panel).

Figure 5.10: The three largest Lyapunov exponents of the Lorenz-96 model as a function of the parameter \(F\) for \((n,G) = (6,0)\) (left panel) and a Lyapunov diagram in the parameters \((U,V)\) defined by the affine transformation \((F,G) = (U + 6V + 1, 0.35V - 0.25)\) (right panel). The colour coding for the right panel is almost the same as in table 5.1, except that blue indicates a periodic attractor for wave number \(l = 1\). The Arnol’d tongues emanating from the \(nS\)-curve are clearly visible.

The Lyapunov diagram in figure 5.10 (left panel) clearly shows alternating intervals of periodic behaviour and quasi-periodic behaviour. This phenomenon can be clarified by the two-parameter system (2.13). In the \((F,G)\)-plane this alternation organises itself in the form of the well-known Arnol’d resonance tongues, which emanate from the \(nS\)-curve (Kuznetsov, 2004). For a better visualisation of these tongues the affine transformation \((F,G) = (U + 6V +\)
1, 0.35 V \ - \ 0.25) has been used to obtain the panel on the right of figure 5.10. The original Lorenz-96 model is then parametrised by the line \( V = \frac{5}{7} \), with the ns-point \( F_{NS} \) at \( (U, V) \approx (0.1709, \frac{5}{7}) \).

5.1.4 Dimension \( n = 7 \)

Figure 5.11 shows the bifurcation diagram of the Lorenz-96 model for dimension \( n = 7 \). The equilibrium \( x_F \) becomes unstable at \( F \approx 1.1820 \) through a supercritical Hopf bifurcation. The periodic attractor remains stable until \( F \approx 2.7171 \) where it bifurcates through a Neimark-Sacker bifurcation. The resulting 2-torus attractor remains stable until \( F \approx 4.2720 \) where it disappears through a quasi-periodic saddle-node bifurcation (Broer, et al., 1990; Broer & Takens, 2011). Figure 5.12 shows a Poincaré section of the quasiperiodic attractor before the bifurcation and the chaotic attractor just after the bifurcation. The trace of the formerly existing 2-torus attractor is clearly visible. The dynamics is characterized by alternations between quasi-periodic and chaotic dynamics. This is a form of intermittency but of a different nature than type 2 intermittency described by Pomeau & Manneville (1980) since the latter scenario involves the disappearance of a stable periodic orbit instead of a 2-torus attractor.

![Bifurcation diagram](image)

**Figure 5.11:** Bifurcation diagram of attractors in the Lorenz-96 model for \( n = 7 \). The three largest Lyapunov exponents are plotted as a function of the parameter \( F \). In this case a 2-torus attractor disappears through a quasi-periodic saddle-node bifurcation which leads to a chaotic attractor.
5.1.5 Dimension $n = 8$

The case $n = 8$ shows an interesting example of inheritance of (part of the) dynamics due to symmetry — see also section 4.2. Here, for specific parameter values two attractors coexist, of which one is symmetric and one is non-symmetric (Orrell & Smith, 2003). The symmetric attractor is born as periodic attractor at the first Hopf bifurcation (at $F_H(2, 8) = 1$) and is contained in $\text{Fix}(G^4_8)$, so that — by Proposition 2.6 — it inherits its dynamics from the attractor of dimension $n = 4$. Indeed, the symmetric attractor undergoes exactly the same bifurcations (except for the $\text{PD}$) and the corresponding blue curve in figure 5.13 is similar to the curve in figure 5.1 (up to at least the third $\text{LPC}$) with again an accumulation of $\text{LPCs}$.

It turns out that the non-symmetric attractor emanates from the symmetric attractor via a $\text{PD}$ at $F_{PD} \approx 2.7747$ (Orrell & Smith, 2003) — see also figure 5.13. Chaos is observed for $F > 3.76$. Note that after the $\text{PD}$ (and even after the point where chaos sets in) the dynamics can still converge to the symmetric attractor provided that the initial conditions are chosen inside $\text{Fix}(G^4_8)$; otherwise, the orbit is attracted by the non-symmetric attractor.
Figure 5.13: Continuation of the two attractors — a symmetric (blue line) and a non-symmetric one (black line) — for $n = 8$. The dotted line is to guide the eye. The non-symmetric attractor, created at the PD, exhibits two NSs before chaos sets in for $F > 3.76$. The bifurcation sequence of the symmetric attractor is similar to the one of the attractor for $n = 4$ — compare with figure 5.1. From $F \approx 8.5406$ additional limit cycles are created through saddle-node bifurcations of limit cycles (LPCs). Finally, at $F_{\text{LPC}} \approx 11.8382$, it disappears through a saddle-node bifurcation.

5.1.6 Dimension $n = 12$

Part of the dynamics for this dimension is already explained in section 3.2.2 and 4.4. Here, we present the results of our numerical exploration which support the analytical results very well.

Recall that the first bifurcation of the trivial equilibrium for $G = 0$ is a Hopf-Hopf bifurcation, which is rather exceptional for the original Lorenz-96 model. This codimension two point acts as an organising centre, as explained in section 3.2. Two codimension one NS-curves originate from this bifurcation point, each corresponding to one of the wave numbers $l = 2$ or $l = 3$. The local bifurcation diagram obtained using MatCont is presented in figure 5.14 and should be compared with the analytically computed bifurcation diagram in figure 3.1.
In the region enclosed by both \textit{ns}-curves multistability occurs, due to the coexistence of the periodic attractors for both \( l = 2 \) and \( l = 3 \). Both attractors are plotted for the same parameter values \((F, G) = (1.5, 0)\) in figure 5.15. Together with their Hovmöller diagrams in figure 5.16, this shows that both waves are of a different nature. Multistability is also reflected by the Lyapunov diagrams in figure 5.17. The left (resp. right) panel is obtained by fixing the parameter \( F \) and increasing (resp. decreasing) the parameter \( G \). Along each vertical line in the parameter plane we have used the last point on the attractor detected in the previous step as an initial condition for the next one. In both diagrams we have used a grid of size 1000 by 1000. The colouring for each region is explained in table 5.1. Figure 5.17 clearly shows that there is a region in the parameter plane where two different periodic attractors coexist. Also note that the bifurcation curves of figure 5.14 are clearly visible in these diagrams. Lastly, figure 5.17 shows the role of the Hopf-Hopf bifurcation as organising centre, that influences a large portion of the parameter space as well as the phase space.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{bifurcation_diagram.png}
\caption{Local bifurcation diagram for \( n = 12 \) around the Hopf-Hopf bifurcation point obtained by numerical continuation. The blue and red lines are the Hopf bifurcation curves (3.10) for \( l = 2 \) and \( l = 3 \), respectively. The light-blue and orange curves are \textit{ns}-curves for the periodic orbit originating from the Hopf bifurcation with \( l = 2 \) and \( l = 3 \), respectively. The \textit{ns}-curve for \( l = 3 \) ends at the corresponding Hopf line. The points on the \textit{ns}-curves denote other codimension two bifurcations. Also compare with figure 3.1 and 5.17, both indicating the (global) dynamics in each region.}
\end{figure}
Figure 5.15: Projections onto the \((x_1,x_2)\)-plane of coexisting periodic attractors with wave numbers \(l = 2\) (blue) and \(l = 3\) (red) for \((n,F,G) = (12,1.5,0)\), which is in the region enclosed by the two ns-curves where multistability occurs — see figure 5.14.

Figure 5.16: Hovmöller diagrams of the periodic attractors from figure 5.15 with wave numbers \(l = 2\) (left panel) and \(l = 3\) (right panel) for \((n,F,G) = (12,1.5,0)\). The value of \(x_j(t)\) is plotted as a function of \(t\) and \(j\). For visualisation purposes linear interpolation between \(x_j\) and \(x_{j+1}\) has been applied in order to make the diagram continuous in the variable \(j\). Note that the difference in both the period and the wave number is clearly visible.
Figure 5.17: Lyapunov diagrams for $n = 12$ and domain $(F, G) \in [0, 3] \times [-0.25, 0.25]$, computed from bottom to top (left panel) and from top to bottom (right panel). See table 5.1 for the colour coding. Note that the bifurcation curves shown in figure 5.14 are clearly visible.

Table 5.1: Colour coding for the Lyapunov diagram in figure 5.17.

<table>
<thead>
<tr>
<th>COLOUR</th>
<th>TYPE OF ATTRACTOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>Stable equilibrium</td>
</tr>
<tr>
<td>Blue</td>
<td>Periodic attractor for $l = 2$</td>
</tr>
<tr>
<td>Green</td>
<td>Periodic attractor for $l = 3$</td>
</tr>
<tr>
<td>Grey</td>
<td>Quasi-periodic attractor</td>
</tr>
<tr>
<td>Black</td>
<td>Chaotic attractor</td>
</tr>
</tbody>
</table>

5.1.7 Dimension $n = 36$

Dimension $n = 36$ provides another illustration of the phenomenon that two or more stable attractors can coexist when a Hopf-Hopf bifurcation occurs for a small value of $G$ and close to the first Hopf bifurcation. We observe again coexistence of attractors, like in the case $n = 12$ — see section 4.4. The Hopf-Hopf bifurcation that induces this phenomenon occurs at the intersection of the Hopf-lines for wave numbers $l = 7$ and $l = 8$ where $(F, G) \approx (0.9196, 0.0144)$, i.e. close to the $F$-axis. Note that these wave numbers correspond to the first two Hopf bifurcations of the trivial equilibrium for $F > 0$ and $G = 0$. From the normal form coefficients, we deduce that the Hopf-Hopf bifurcation is of the
same type as for \( n = 12 \), meaning that only two \( \text{ns} \)-curves arise from the codimension two point — see section 3.2.2. The local bifurcation diagram in figure 5.18 shows these two curves together with their corresponding Hopf-lines. The blue \( \text{ns} \)-curve (corresponding to \( l = 7 \)) intersects the line \( G = 0 \) at \( F \approx 0.9093 \), so we can observe multistability in the one-parameter model (2.1) for \( F \) somewhat larger than this value. Again, the Hopf-Hopf bifurcation point acts as an organising centre.

In figures 5.19 and 5.20 the Lyapunov diagrams are shown for \( l = 7 \) and \( l = 8 \), respectively, with \( G = 0 \) fixed. For wave number \( l = 7 \), the first bifurcation after the Hopf bifurcation at \( F_H(7,36) \approx 0.9025 \) is the mentioned \( \text{ns} \) at \( F \approx 0.9093 \), which is followed by another \( \text{ns} \) at \( F \approx 4.3891 \). The resulting quasi-periodic attractor then bifurcates to a \( 3 \)-torus — see below. For \( l = 8 \), a stable periodic attractor originates from the supercritical Hopf bifurcation at \( F_H(8,36) \approx 0.8982 \). This attractor exhibits a \( \mathbb{R}^3 \) at

---

**Figure 5.18:** Local bifurcation diagram obtained by numerical continuation for \( n = 36 \) around the Hopf-Hopf bifurcation point at \((F,G) \approx (0.9196, 0.0144)\). The blue and red lines are the Hopf bifurcation curves for \( l = 7 \) and \( l = 8 \), respectively. The light-blue and orange curves are \( \text{ns} \)-curves for the periodic orbit originating from the Hopf bifurcation with \( l = 7 \) and \( l = 8 \), respectively. The box magnifies the region around the Hopf-Hopf point and the line \( G = 0 \).
F \approx 3.1555 and becomes unstable via a subcritical ns at \( F \approx 3.1626 \), which can be seen from the right panel of figure 5.20. The only stable attractor for \( F > 3.1626 \) is the one with wave number \( l = 7 \). This is reflected in the Lyapunov diagrams of figure 5.20, where the Lyapunov exponents take up the values for \( l = 7 \) right after

**Figure 5.19:** The three largest Lyapunov exponents of the Lorenz-96 model as a function of the parameter \( F \) for \( n = 36 \) and wave number \( l = 7 \) (left panel). The right panel is a magnification of the right part of the left panel, which displays the appearance of a 3-torus for \( F \in [4.45, 4.48] \). In both panels \( G = 0 \).

**Figure 5.20:** The three largest Lyapunov exponents of the Lorenz-96 model as a function of the parameter \( F \) for \( n = 36 \) and wave number \( l = 8 \) (left panel). The right panel shows a magnification of the left panel around \( F = 3.15 \), showing the disappearance of the stable attractor for \( l = 8 \) at \( F \approx 3.1626 \). For larger \( F \) the Lyapunov exponents take up the values of the stable attractor with wavenumber \( l = 7 \) — see figure 5.19. In both panels \( G = 0 \).
the subcritical ns at $F \approx 3.1626$ — compare with figure 5.19. These observations show that the region of multistability is bounded for $G = 0$.

The Lyapunov diagram in the right panel of figure 5.19 suggests that for $G = 0$ a 3-torus exists in a small interval of $F$-values before chaotic attractors are observed. Figure 5.21 shows a 3-torus attractor for $(n, F, G) = (36, 4.45, 0)$ together with a corresponding 2-torus attractor from a Poincaré section defined by $\Sigma = \{x_1 = 2\}$. The occurrence of an attractor in the form of a 3-torus has also been observed for $n = 24$ (not shown). Newhouse, Ruelle and Takens (Newhouse, et al., 1978) proved that small perturbations of a quasi-periodic flow on the 3-torus can lead to strange Axiom A attractors. Concrete routes of the nrt-scenario were reported in (Broer et al., 2008a; Broer, et al., 2008b) in the setting of a model map for the Hopf-saddle-node bifurcation in diffeomorphisms. Some techniques to study bifurcations of 3-tori in continuous-time dynamical systems are described in (Kamiyama, et al., 2015). Unravelling the bifurcations of 3-tori and the associated routes to chaos in the Lorenz-96 model is left for future research.

Figure 5.21: A 3-torus attractor (left panel) for $(n, F, G) = (36, 4.45, 0)$ and the corresponding 2-torus attractor of the Poincaré return map defined by the section $\Sigma = \{x_1 = 2\}$ (right panel).
5.2 Patterns

5.2.1 General dimensions

We now want to compare the routes to chaos that are observed in several dimensions to reveal possible general patterns. The diagram in figure 5.22 shows the bifurcations for various dimensions $n$ and $F > 0$. To obtain this diagram, we followed only the stable attractor (starting with the one generated through the first Hopf bifurcation) numerically, until chaos sets in for the first time. The parameter values where chaos sets in are estimated by means of the Lyapunov diagrams — such as figures 5.19 and 5.20 — and are also indicated in figure 5.22. For all dimensions shown chaos sets in for $F \in (3, 7)$, except for $n = 4$ where we observe chaos for $F \geq 11.84$.

As can be seen from the diagram, there are various routes to chaos, but a clear pattern for all $n$ cannot be discerned. Nonetheless, a pattern is observed for dimensions $n \leq 100$ where $n$ is a multiple of 5, which will be discussed in the next section. Furthermore, we point out that the bifurcation scenarios, as well as the dynamical behaviour — as described in section 5.1 — of a certain dimension $m$ might be extrapolated to all dimensions $km, k \in \mathbb{N}$, by Proposition 2.6. This provides the bifurcation scenarios for attractors in which symmetry is involved in the form of symmetric periodic orbits and symmetric attractors in $n = km$, namely, attractors whose spatial wave number satisfies $\gcd(l, n) = k$ — see
Figure 5.22: Diagram showing the bifurcations of the stable attractor for $F > 0$ (until chaos sets in) and for various values of $n$. Each symbol denotes a bifurcation or onset of chaos at the corresponding value of $F$. The type of bifurcation is shown by the legend at the right. Note that we only show bifurcations of the stable orbits which lead eventually to chaos. Also, we do not include bifurcations of other stable branches.
5.2 Patterns

section 4.2. However, as pointed out earlier in this thesis, this does not provide the full bifurcation scenario, but only the invariant subspace $\text{Fix}(G_n^{m})$ — i.e. for the symmetric attractor, as we have seen in the case $n = 8$, for instance. It may happen that — apart from this symmetric attractor — there exists another attractor without any symmetry, that is contained in $\mathbb{R} \setminus \text{Fix}(G_n^{m})$. Such a coexistence of multiple stable attractors can occur due to the presence of a Hopf-Hopf bifurcation, as discussed in section 4.4. Attractors with different wave numbers and sometimes also different symmetries thus appear. Using arbitrary initial conditions, one most likely encounters the non-symmetric attractor — an orbit is only attracted to a symmetric attractor if the initial conditions are chosen with the same symmetry. In general, these attractors do not have the same bifurcation scenarios.

For negative $F$, we observed that there are only three different bifurcation patterns for all $n \in \mathbb{N}$ that lead to periodic orbits — see section 4.3. This might also have consequences for the number of routes to chaos. Further research is needed to unravel the bifurcation patterns and routes to chaos for $F < 0$.

5.2.2 Dimensions $n = 5k$

In section 5.1.2 it was shown that a period-doubling cascade occurs for dimension $n = 5$. It turns out by numerical continuation that the period-doubling bifurcations ($\text{pd}$s) persist in all dimensions that are multiples of 5, up to $n = 100$. Figure 5.23 shows the bifurcation scenarios for these dimensions. For each $n = 5k$, with $k = 1, \ldots, 10$, the bifurcation values of the first Hopf bifurcation and the first period-doubling are exactly the same as in the case of $n = 5$. From $n = 55$ on the pattern deviates, because the parameter value of the first Hopf bifurcation changes and, hence, the periodic orbit does not inherit its properties from the case $n = 5$ anymore — see below. Indeed, a Neimark-Sacker bifurcation ($\text{ns}$) is now the first bifurcation after the Hopf bifurcation, but the torus originating from this bifurcation disappears for slightly larger $F$ and we seem to have again the $\text{pd}$-pattern with a $\text{pd}$ at $F_{\text{PD}} \approx 3.9379$ as before.
Figure 5.23: As figure 5.22, but for $n = 5k, k = 1, \ldots, 20$. For $n \geq 55$, the Hopf bifurcation value changes and an $Ns$ appears before the usual PD, that persists up to at least dimension $n = 100$. 
This phenomenon can be explained by the wave number of the periodic attractor after the first Hopf bifurcation — see section 4.2. It turns out that for \( n = 5k \), with \( k = 1, \ldots, 10 \), the wave number \( l_1^+(n) \) of this attractor is exactly equal to \( k \). Hence, we have

\[
\frac{n}{l_1^+(n)} = \frac{n}{k} = 5 \in \mathbb{N}
\]

and therefore the periodic attractor has repeating coordinates with \( x_{j+5} = x_j \) for all \( 1 \leq j \leq n \) and indices modulo \( n \). This implies that the travelling wave is contained in the invariant subspace \( \text{Fix}(G_n^5) \subset \mathbb{R}^n \). For any \( n = 5k \), the dynamics restricted to \( \text{Fix}(G_n^5) \) is governed by the Lorenz-96 model for \( n = 5 \) by Proposition 2.6.

For \( n \geq 55 \) this phenomenon breaks down, since the wave number \( l_1^+ \) of the periodic attractor no longer satisfies the relation \( \gcd(n, l_1^+) > 1 \). Nevertheless, it appears that this periodic attractor becomes unstable and that again the symmetric periodic attractor with wave number \( n/5 \) takes up stability — see section 4.2. However, this is not guaranteed to happen in general, especially for higher dimensions, since the quotient \( n/l_1^+(n) \) for the first periodic attractor converges to a non-integer number for \( n \to \infty \) as shown in Proposition 4.1. Therefore, for increasing \( n \) an increasing number of stable and unstable periodic attractors is generated — via Hopf bifurcations with \( F_H(l, n) > F_H(l_1^+, n) \) — whose wave numbers \( l \) satisfy \( n/l_1^+(n) < n/l < 5 \) and so it may become more rare to find a stable periodic attractor which inherits its dynamics from the case \( n = 5 \).