Dynamics of the Lorenz-96 model
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Recall from chapter 2 that the Lorenz-96 model (2.1) has the trivial equilibrium solution $x_F$ that exists for all $n \geq 1$ and any $F \in \mathbb{R}$. In order to detect bifurcations in the Lorenz-96 model, we take $F$ as the bifurcation parameter and vary it along the real line. In this chapter, we start the analysis of bifurcations with the equilibrium $x_F$ and the stable equilibria resulting from one or more bifurcations. Our investigation builds on the results of chapter 2 — namely the analysis of the eigenvalues and the symmetries of the Lorenz-96 model. In this way we encounter several Hopf, Hopf-Hopf and pitchfork bifurcations.

The bifurcations presented in this chapter only involve bifurcations of equilibria — bifurcations of periodic orbits and attractors are discussed in chapter 5. In particular, we are interested in the first bifurcation that destabilises the equilibrium $x_F$, because this bifurcation determines for a large part the route to chaos. The cases $F > 0$ and $F < 0$ are treated separately, as in the latter case symmetry places an important role in the bifurcation structure. For positive $F$, the first bifurcation for $n \geq 4$ is either a supercritical Hopf bifurcation or a Hopf-Hopf bifurcation. For negative $F$, the bifurcation pattern for $n \geq 4$ splits into three different cases, depending on the dimension:

1. In all odd dimensions, the first bifurcation of the equilibrium $x_F$ is a supercritical Hopf bifurcation.
2. In dimensions \( n = 4k + 2, k \in \mathbb{N} \), only one pitchfork bifurcation takes place, followed by two simultaneous Hopf bifurcations.

3. For dimensions \( n = 4k, k \in \mathbb{N} \), two subsequent pitchfork bifurcations occur, followed by four simultaneous Hopf bifurcation.

Part of this chapter is devoted to an analysis of the bifurcations of the two-parameter Lorenz-96 model (2.13), to highlight the Hopf-Hopf bifurcation as an organising centre.

This chapter is mainly based on (Van Kekem & Sterk, 2018b; Van Kekem & Sterk, 2018a; Van Kekem & Sterk, 2018c).

3.1 BIFURCATIONS FOR POSITIVE \( F \)

In section 2.1 we have computed the eigenvalues of the equilibrium \( x_F \). For convenience, we repeat the expression (2.9) of the eigenvalues here:

\[
\lambda_j(F,n) = -1 + Ff(j,n) + Fg(j,n)i, \tag{3.1}
\]

where \( f \) and \( g \) are defined as

\[
f(j,n) = \cos \frac{2\pi j}{n} - \cos \frac{4\pi j}{n},
\]

\[
g(j,n) = -\sin \frac{2\pi j}{n} - \sin \frac{4\pi j}{n}.
\]

Clearly, for \( F = 0 \) the equilibrium \( x_F \) is stable as \( \text{Re} \lambda_j = -1 \) for all \( j = 0, \ldots, n-1 \). Numerical simulations show that for \( F = 1.2 \) the dynamics of the model is periodic for all \( n \geq 4 \). This suggests that for \( 0 < F < 1.2 \) a supercritical Hopf bifurcation occurs at which the equilibrium \( x_F \) loses its stability and gives birth to a periodic attractor — see also section 4.2.

EIGENVALUE CROSSING Indeed, as equation (2.12) shows, the eigenvalues come in pairs and, moreover, each complex \( j \)-th eigenvalue pair \( \{\lambda_j, \lambda_{n-j}\} \) has a particular parameter value \( F \) for which it crosses the imaginary axis and thus causes a Hopf bifurcation. In the following we prove that for all \( n \geq 4 \) the trivial equilibrium \( x_F \) can exhibit several Hopf or Hopf-Hopf bifurcations. In case of a Hopf bifurcation, we also prove whether the bifurcation
is sub- or supercritical. In particular, the first Hopf bifurcation for both $F < 0$ and $F > 0$ is always supercritical.

Before we formulate our results on the Hopf-Hopf and Hopf bifurcations in Theorems 3.2 and 3.5, respectively, let us first state the following preliminary result, which proves that we have the desired eigenvalue crossing that is needed for both cases:

**Lemma 3.1 (eigenvalue crossing).** Let $n \geq 4$ and $l \in \mathbb{N}$ such that $0 < l < \frac{n}{2}$, $l \neq \frac{n}{3}$, then the following holds:

1. The $l$-th eigenvalue pair of the trivial equilibrium $x_F$ of system (2.1) crosses the imaginary axis transversally at the parameter value
   \[
   F_H(l,n) := \frac{1}{f(l,n)}
   \]
   and thus the equilibrium changes stability.

2. $F_H(l,n)$ lies in the domain $(F_{\min}(n), -\frac{1}{2}) \cup \left[\frac{8}{9}, F_{\max}(n)\right)$ with
   \[
   F_{\min}(n) = \begin{cases} 
   -\frac{1}{2} & \text{if } n = 4, 6, \\
   \frac{1}{f(r+1,n)} & \text{otherwise},
   \end{cases}
   \]
   \[
   F_{\max}(n) = \begin{cases} 
   \frac{1}{f(2,7)} & \text{if } n = 7, \\
   \frac{1}{f(1,n)} & \text{otherwise},
   \end{cases}
   \]
   where $r$ satisfies $r = \left\lfloor \frac{n}{3} \right\rfloor$.

**Proof.** See appendix B.1.

Due to the shape of the function $f(j, n)$, at most two eigenvalue pairs can have simultaneously vanishing real part for a particular value of $F$ — see figure 2.1. This indicates that the crossing of eigenvalue pairs, described in Lemma 3.1, can lead to Hopf bifurcations and Hopf-Hopf bifurcations only.

**Hopf-Hopf bifurcations** Let us first describe the Hopf-Hopf case: suppose that we have two distinct eigenvalue pairs with $l_1$ and $l_2$ which both cross the imaginary axis at the same parameter value $F_{HH} := F_H(l_1, n) = F_H(l_2, n)$. In that case a Hopf-Hopf bifurcation occurs:
**Theorem 3.2 (Hopf-Hopf Bifurcation).** Let $l_1$, $l_2$ and $n$ satisfy the assumptions of Lemma 3.1 with $l_1 \neq l_2$. Then the equilibrium $x_F$ exhibits a Hopf-Hopf bifurcation at $F_{HH}$ if and only if $l_1$ and $l_2$ satisfy

$$\cos \frac{2nl_1}{n} + \cos \frac{2nl_2}{n} = \frac{1}{2}. \quad (3.3)$$

**Proof.** Throughout the proof we assume that $l_1 < l_2$, without loss of generality. Suppose that at a certain parameter value $F_{HH}$ a Hopf-Hopf bifurcation occurs, for which the $l_1$-th and $l_2$-th eigenvalue pairs both have real part equal to 0. So, we need to have that $F_{HH}(l_1, n) = F_{HH}(l_2, n)$ or, equivalently,

$$f(l_1, n) = f(l_2, n), \quad (3.4)$$

where it should hold that

$$0 < l_1 < \frac{n}{2\pi} \cos^{-1} \left( \frac{1}{4} \right) < l_2 < \frac{n}{3}. \quad (3.5)$$

Here, the second and third inequality follow from the fact that if $l_1$ and $l_2$ give the same value for $f$, then for the continuous function $\tilde{f}(y) = \cos y - \cos 2y$ we need $y_1 = \frac{2nl_1}{n}$ to be left and $y_2 = \frac{2nl_2}{n}$ right of the top $y_{\text{top}} = \cos^{-1} (\frac{1}{4})$ in the domain of consideration, $(0, \pi)$. So, $y_1$ and $y_2$ have to satisfy $0 < y_1 < y_{\text{top}} < y_2 < \frac{2n}{3}$ — see figure 2.1 for the picture. This is equivalent to equation (3.5).

Since $f(l, n)$ can be written as

$$f(l, n) = -2 \cos^2 \frac{2nl}{n} + \cos \frac{2nl}{n} + 1,$$

the substitution $x = \cos \frac{2nl}{n}$ gives the function

$$h(x) = -2x^2 + x + 1.$$

Condition (3.4) then becomes

$$h \left( \cos \frac{2nl_1}{n} \right) = h \left( \cos \frac{2nl_2}{n} \right).$$

By condition (3.5) on $l_1$ and $l_2$, $\cos \frac{2nl_1}{n}$ is on the left and $\cos \frac{2nl_2}{n}$ is on the right of the maximum $x = \frac{1}{4}$ of $h$. Since the function $h(x)$ is symmetric around the maximum, $l_1$, $l_2$ and $n$ should satisfy

$$\frac{1}{2} \left( \cos \frac{2nl_1}{n} + \cos \frac{2nl_2}{n} \right) = \frac{1}{4}.$$

This provides the condition for a Hopf-Hopf bifurcation to occur.
Conversely, suppose that equation (3.3) holds for \( l_1 \) and \( l_2 \) satisfying \( 0 < l_1, l_2 < \frac{n}{2} \), \( l_1, l_2 \neq \frac{n}{3} \) and \( l_1 \neq l_2 \). The existence of a Hopf-Hopf bifurcation can be derived from Lemma 3.1 by showing that the bifurcation parameters \( F_H \) corresponding to each of these eigenvalue pairs coincide, i.e. \( F_H(l_1, n) = F_H(l_2, n) \).

Let us denote \( y_1 = \frac{2\pi l_1}{n} \) and \( y_2 = \frac{2\pi l_2}{n} \), then equation (3.3) becomes:

\[
\cos y_1 + \cos y_2 = \frac{1}{2}.
\]

From this equation, we obtain \( \cos y_2 = \frac{1}{2} - \cos y_1 \) and, with a little trigonometry, its double angle reads

\[
\begin{align*}
\cos 2y_2 &= 2\cos^2 y_2 - 1 = -\frac{1}{2} - 2\cos y_1 + 2\cos^2 y_1 \\
&= \frac{1}{2} - 2\cos y_1 + \cos 2y_1.
\end{align*}
\]

Now, observe that the following holds:

\[
\hat{f}(y_2) = \cos y_2 - \cos 2y_2 = \left( \frac{1}{2} - \cos y_1 \right) - \left( \frac{1}{2} - 2\cos y_1 + \cos 2y_1 \right) = \cos y_1 - \cos 2y_1 = \hat{f}(y_1).
\]

Hence, it holds that \( f(l_1, n) = \hat{f}(y_1) = \hat{f}(y_2) = f(l_2, n) \) and therefore \( F_H(l_1, n) = F_H(l_2, n) \) as desired.

\[Q.E.D.\]

From equation (3.3) we can deduce two infinite sequences of dimensions for which a Hopf-Hopf bifurcation takes place:

**Corollary 3.3.** Let \( m \in \mathbb{N} \), then a Hopf-Hopf bifurcation occurs if we select \( l_1, l_2 \) and \( n \) according to one of the following criteria:

1. \( n = 10m \) and \( l_1 = m, l_2 = 3m \), which corresponds to \( F_{HH} = 2 \);

2. \( n = 12m \) and \( l_1 = 2m, l_2 = 3m \), which corresponds to \( F_{HH} = 1 \).

**Proof.** From equation (3.3) we can determine explicit combinations of \( l_1, l_2 \) and \( n \) for which a Hopf-Hopf bifurcation will occur. To begin with the easiest one, criterion 2: choose \( l_2/n \) such that \( \cos \frac{2\pi l_2}{n} = 0 \), i.e. \( l_2/n = 1/4 \). This implies that \( l_1/n \) has to be equal
to 1/6 to satisfy equation \((3.3)\). Since all numbers have to be integers, we should take \(n = 12m\) with \(m \in \mathbb{N}\) and hence, \(l_1 = 2m\) and \(l_2 = 3m\).

Criterion 1 is obtained by observing that

\[
\cos \frac{n\pi}{5} = \frac{1}{4}(1 + \sqrt{5}), \quad \text{and} \quad \cos \frac{3n\pi}{5} = \frac{1}{4}(1 - \sqrt{5}),
\]

so that we have the relations \(2l_1/n = 1/5\) and \(2l_2/n = 3/5\). These relations are satisfied by taking multiples of \(m \in \mathbb{N}\) as follows: \(n = 10m\) and \(l_1 = m, l_2 = 3m\).

\[\text{Q.E.D.}\]

Remark 3.4. Equation \((3.3)\) gives a necessary and sufficient condition for the occurrence of a Hopf-Hopf bifurcation. However, the explicit values of \(l_1, l_2\) and \(n\) given in Corollary 3.3 possibly do not provide all occasions in the Lorenz-96 model where a Hopf-Hopf bifurcation occurs.

**Hopf Bifurcations** Conversely, if equation \((3.3)\) is not satisfied then we have only one eigenvalue pair crossing the imaginary axis, which implies a Hopf bifurcation:

**Theorem 3.5 (Hopf Bifurcation).** Let \(l\) and \(n\) be as in Lemma 3.1. If the \(l\)-th eigenpair is the only one crossing the imaginary axis at the corresponding parameter value \(F_H(l, n)\), then the equilibrium \(x_F\) exhibits a Hopf bifurcation at \(F_H\).

The first Lyapunov coefficient for this bifurcation is given by

\[
\ell_1(l, n) = \frac{4}{n} \tan \left( \frac{\pi l}{n} \right) \sin^2 \left( \frac{3\pi l}{n} \right) \cdot \frac{5 \cos \left( \frac{2\pi l}{n} \right) + 8 \cos \left( \frac{4\pi l}{n} \right) - 2 \cos \left( \frac{6\pi l}{n} \right) - 8}{4 \cos \left( \frac{2\pi l}{n} \right) - 4 \cos \left( \frac{4\pi l}{n} \right) + 9}. \tag{3.6}
\]

Fix \(y_0 \in (0, \pi)\) such that

\[
5 \cos y_0 + 8 \cos 2y_0 - 2 \cos 3y_0 - 8 = 0,
\]

then \(\ell_1(l, n)\) is

- positive if \(l\) and \(n\) satisfy \(0 < \frac{l}{n} < \frac{y_0}{2\pi} \approx 0.0883\), which corresponds to a subcritical bifurcation;
negative if $\frac{1}{n} \in \left(\frac{w_0}{2\pi}, \frac{1}{2}\right) \setminus \{\frac{1}{3}\}$ holds, which corresponds to a super-critical bifurcation.

Proof. See appendix B.2.

Remark 3.6. Our analytical results on the parameter value $F_H$ and the value of the first Lyapunov coefficient coincide with the numerical estimates by MatCont, the continuation toolbox for MATLAB. In general, MatCont predicts the Hopf and Hopf-Hopf bifurcations for exactly the same bifurcation value. In addition to the well-predicted value $F_H$, the computation of the first Lyapunov coefficient by MatCont is also very accurate. Note that MatCont scales the Lyapunov coefficient (B.5) by the (positive) factor $\omega_0$ (Dhooge et al., 2011):

$$\tilde{\ell}_1(l, n) = \omega_0(l, n)\ell_1(F_H(l, n)). \quad (3.7)$$

Table 3.1 shows a comparison of the analytical values of the Lyapunov coefficient computed by formula (3.6) and those of formula (3.7) with the output of MatCont for several combinations of $l$ and $n$. The scaled versions of the coefficients almost coincide. Also, observe for which values we obtain a positive Lyapunov coefficient — in accordance with Theorem 3.5 — which is confirmed by figure B.2.

The number of possible Hopf bifurcations for a given dimension $n$ is exactly equal to the number of conjugate eigenvalue pairs which satisfy Lemma 3.1. Using equation (2.12), we can count the number of such eigenvalue pairs by the number of eigenvalues with $0 < j < \frac{n}{2}$, which gives the number $\lceil n/2 - 1 \rceil$ (we need the ceiling function here if $n$ is odd). However, as described in section 2.1, if $n$ is a multiple of 3, then the eigenvalue pair with $j = \frac{n}{3}$ is not complex, so in this case the number of such eigenvalue pairs equals $\lceil n/2 - 2 \rceil$. For the actual number of Hopf bifurcations in the Lorenz-96 model, we should decrease these numbers by the number of Hopf-Hopf bifurcations.

First bifurcation for positive $F$. Let us now restrict our attention to the parameter range $F \geq 0$. We are interested in the smallest value of $F > 0$ at which the equilibrium bifurcates and
Table 3.1: Comparison of values of the Lyapunov coefficient computed via equation (3.6) ($\ell_1(l,n)$), the same value multiplied by $\omega_0(l,n)$ (as in equation (3.7)) and those computed by MatCont ($\tilde{\ell}_1(l,n)$). The numbers $l$ and $n$ are chosen according to the conditions in Lemma 3.1, while avoiding Hopf-Hopf bifurcations.

<table>
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<tr>
<th>$n$</th>
<th>$l$</th>
<th>$F_H$</th>
<th>$\ell_1$</th>
<th>$\omega_0 \cdot \ell_1$</th>
<th>$\tilde{\ell}_1$ (MatCont)</th>
</tr>
</thead>
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<tr>
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<td>0.8944272</td>
<td>$-0.6153846$</td>
<td>$-0.6153846$</td>
<td>$-0.6153846$</td>
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<tr>
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</tr>
<tr>
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<tr>
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<td>$-0.3076923$</td>
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<tr>
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</tr>
<tr>
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<td>$1.752696 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

becomes unstable. From the previous results we know that this must be either a Hopf or a Hopf-Hopf bifurcation.

**Proposition 3.7.** Let $n \geq 4$ be fixed. For $F > 0$ the first Hopf or Hopf-Hopf bifurcation occurs for the eigenvalue pair(s) with index

$$l_1^+(n) = \arg \max_{0 \leq l < n/3} f(l,n),$$

which satisfies the bounds

$$\frac{n}{6} \leq l_1^+(n) \leq \frac{n}{4},$$

except for $n = 7$, in which case we have to take $l_1^+ = 1$.

In particular, if the first bifurcation is a Hopf bifurcation, then this bifurcation is supercritical. Its bifurcation value satisfies $F_H(l_1^+(n), n) \in \left[ \frac{8}{9}, 1.19 \right]$ and converges to $\frac{8}{9}$ as $n \to \infty$.

**Proof.** Lemma 3.1 implies that the trivial equilibrium undergoes a Hopf bifurcation at the parameter value $F_H(l,n) = 1/f(l,n)$. The first Hopf bifurcation for $F > 0$ takes place for the integer $l_1^+(n) \in$
(0, \frac{n}{3}) that minimises the value of \( F_H(l, n) \), which is equivalent to maximising \( f(l, n) \).

For all \( n \geq 4 \) except \( n = 7 \) there exists at least one integer \( l \in \left[ \frac{n}{6}, \frac{n}{4} \right] \). Indeed, for \( n = 4, 5 \) and 6 this follows by simply taking \( l_1^+(n) = 1 \), and for \( n = 8, 9, 10 \) and 11 it follows by taking \( l_1^+(n) = 2 \). For \( n \geq 12 \) the width of the interval is larger than 1. We now claim that this implies that

\[
l_1^+(n) = \arg \max_{0 < l < n/3} f(l, n) \in \left[ \frac{n}{6}, \frac{n}{4} \right], \quad n \neq 7,
\]
as well. To that end we use \( \tilde{f}(y) := \cos y - \cos 2y \). Note that \( y \in \left[ \frac{n}{6}, \frac{n}{4} \right] \) implies that \( \tilde{f}(y) \geq 1 \) and \( y \in (0, \frac{n}{3}) \cup (\frac{n}{2}, \frac{2n}{3}) \) implies that \( 0 < \tilde{f}(y) < 1 \). Moreover, \( l \in \left[ \frac{n}{6}, \frac{n}{4} \right] \) implies that \( \frac{2\pi l}{n} \in \left[ \frac{\pi}{3}, \frac{\pi}{2} \right] \).

Therefore, \( f(l, n) \) is maximised for some integer \( l \in \left[ \frac{n}{6}, \frac{n}{4} \right] \).

In case \( n = 7 \), we can easily compute the smallest value \( F_H(l, n) \) for which a Hopf bifurcation occurs. We have shown in the proof of Lemma 3.1 that this is the case for \( l = 1 \) — see appendix B.1.

Finally, assume that the first bifurcation is a Hopf bifurcation, i.e. only one eigenvalue pair crosses the imaginary axis. Since \( l_1^+(n)/n \in \left[ \frac{1}{6}, \frac{1}{4} \right] \), it follows immediately from Theorem 3.5 that the first Lyapunov coefficient is negative, which means that the bifurcation is supercritical. This bifurcation happens for parameter values \( \frac{8}{9} \leq F \leq F_H(1, 7) \approx 1.1820 \), by the proof of Lemma 3.1. In the limit \( n \to \infty \) the bifurcation value of the first Hopf bifurcation satisfies

\[
\lim_{n \to \infty} F_H(l_1^+(n), n) = \lim_{n \to \infty} \frac{1}{f(l_1^+(n), n)} = \frac{8}{9},
\]

where we use the fact that \( \tilde{f}(y) \) attains its maximum \( \frac{9}{8} \) at \( y_{\text{top}} = \cos^{-1} \left( \frac{1}{4} \right) \) — cf. the limiting value for the fraction \( \frac{n}{l_1^+(n)} \) in equation (4.6).

Q.E.D.

In section 4.2 we will show that the periodic orbit that is born at the first Hopf bifurcation can be interpreted as a travelling wave.

Remark 3.8. The smallest dimension for which the first bifurcation is a Hopf-Hopf bifurcation instead of a Hopf bifurcation, is
\[ n = 12. \] In this case we have \( I_1^+(n) = \{2, 3\} \) as shown in Corollary 3.3. A more detailed exposition of the dynamics in this particular case can be found in section 3.2.2.

### 3.2 Unfolding: Two-Parameter Model

Corollary 3.3 shows that for \( n = 12 \) the trivial equilibrium \( x_F \) loses stability through a Hopf-Hopf bifurcation at \( F_{\text{HH}} = 1 \). Note that the Hopf-Hopf bifurcation is a codimension two bifurcation which means that generically two parameters must be varied in order for the bifurcation to occur (Kuznetsov, 2004). However, symmetries such as those in the Lorenz-96 model can reduce the codimension of a bifurcation. In order to study this codimension two bifurcation we exploit the two-parameter unfolding (2.13) of the original Lorenz-96 model, introduced in section 2.2. We take the codimension two Hopf-Hopf bifurcation as an organising centre for this family of systems. This clarifies the role of the Hopf-Hopf bifurcation and, in the meanwhile, it sheds more light on the original model. A similar approach is taken in (Broer et al., 2002; Broer, et al., 2005a; Broer, et al., 2007; Blackbeard, et al., 2014), showing the existence and influence of several codimension three or two points that act as an organising centre.

First, we present here an analysis of the bifurcations of system (2.13) locally around the Hopf-Hopf point. Thereafter, we describe the exemplary case \( n = 12 \) in more detail and show the role of the Hopf-Hopf bifurcation as organising centre. In section 5.1.6, we will show by numerical computations how it dominates the dynamics in its neighbourhood and even influences the phase space for larger parts of the parameter space.

### 3.2.1 General dimensions

**Hopf Bifurcations** In section 2.2 we showed that the eigenvalues \( \kappa_j \) of the two-parameter system (2.13) are equal to

\[ \kappa_j(F, G, n) = -1 - 2G \left( 1 - \cos \frac{2\pi j}{n} \right) + Ff(j, n) + Fg(j, n)i, \quad (3.9) \]
for any \( j = 0, \ldots, n - 1 \) and with \( f \) and \( g \) as in formula (3.1). The following lemma demonstrates that the two-parameter system can exhibit as many different Hopf bifurcations as the original one-parameter system (2.1).

**Lemma 3.9 (Hopf bifurcation curves).** Let \( n \geq 4 \) and \( l \in \mathbb{N} \) such that \( 0 < l < \frac{n}{2}, l \neq \frac{n}{3} \), then the equilibrium \( x_F \) of system (2.13) exhibits a Hopf bifurcation on the linear bifurcation curves

\[
G = H_l(F, n) = \frac{F f(l, n) - 1}{2(1 - \cos \frac{2\pi j}{n})},
\]

where \( F \in \mathbb{R} \setminus \{0\} \).

**Proof.** Let \( n \) and \( l \) be as given. We choose \( F \) as the bifurcation parameter. In order to have a Hopf bifurcation for the \( l \)-th eigenvalue pair \( \{\kappa_l, \kappa_{n-l}\}(F, G, n) \), the real part \( \mu_l := \text{Re} \kappa_l \) has to vanish. This occurs if \( F \) equals

\[
F_H(G, l, n) = \frac{1}{f(l, n)} \left( 1 + 2G(1 - \cos \frac{2\pi j}{n}) \right).
\]

For these parameter values we have a purely imaginary eigenvalue pair with the absolute value of the imaginary part given by

\[
\omega_0(G, l, n) = -F_H(G, l, n)g(l, n) = \left| 1 + 2G \left( 1 - \cos \frac{2\pi j}{n} \right) \right| \cos \frac{2\pi l}{n} \sin \frac{\pi l}{n}.
\]

Note that \( \omega_0 = 0 \) if \( l = \frac{n}{2} \) or if \( G = -\frac{1}{2} (1 - \cos \frac{2\pi l}{n})^{-1} \) for some \( l \). The last condition is equivalent to \( F_H = 0 \) by formula (3.11) and implies even that \( \kappa_l = 0 \). Therefore, this parameter value needs to be excluded.

Furthermore, the eigenvalue pair crosses the imaginary axis with nonzero speed, since, by the restriction on \( l \),

\[
\mu'_l(F, G, n) = f(l, n) \neq 0,
\]

where the derivative is with respect to the bifurcation parameter \( F \).

Thus, equation (3.11) gives us for general \( n \) and for each allowed \( l \) a whole line of Hopf bifurcations, which is linear in \( G \). Rewritten in terms of \( F \) provides the linear curves (3.10).

Q.E.D.

---

3 If we would have taken \( G \) as bifurcation parameter, then \( \mu'_l \) will be nonzero as well.
The Hopf bifurcation points are now turned into straight lines in the \((F,G)\)-plane. Along these curves (3.10) it is possible to determine the type of the bifurcation by computing the first Lyapunov coefficient explicitly in a similar manner as done in the proof of Theorem 3.5, but we will not repeat the procedure here — see appendix B.2.

**Hopf-Hopf bifurcations** The lines (2.13) have a different slope for all \(0 < l < \frac{n}{\pi} \) and \(l \neq \frac{n}{3}\) and, hence, they mutually intersect each other. It is obvious that these intersections of Hopf-lines cause Hopf-Hopf bifurcations. One can find all Hopf-Hopf bifurcation points of the trivial equilibrium by equating two Hopf-lines from formula (3.10) with different \(l\). The following result gives the maximum number of such points.

**Proposition 3.10.** The maximum number of Hopf-Hopf bifurcations of the trivial equilibrium \(x_F\) in the two-parameter Lorenz-96 system (2.13) is given by

\[
N_{HH}(n) = \begin{cases} 
\frac{1}{2} (\lceil \frac{n}{2} \rceil - 1)(\lceil \frac{n}{2} \rceil - 2) & \text{if } n \neq 3m, \\
\frac{1}{2} (\lceil \frac{n}{2} \rceil - 2)(\lceil \frac{n}{2} \rceil - 3) & \text{if } n = 3m,
\end{cases}
\]  

where \(m\) is some positive integer.

**Proof.** Since a Hopf-Hopf bifurcation occurs if two eigenvalue pairs cross the imaginary axis simultaneously, we can locate and count all occurrences by the intersections of Hopf bifurcation curves. In our case, the Hopf curves for the trivial equilibrium are given by equation (3.10), which are all straight lines. We can therefore determine on forehand how many intersections there can be, given the dimension.

Lemma 3.9 provides the conditions on \(l\) to give a Hopf bifurcation line. Define \(N_H(n)\) as the number of Hopf lines in dimension \(n\), then \(N_H\) is given by

\[
N_H(n) = \begin{cases} 
\lceil \frac{n}{2} \rceil - 1 & \text{if } n \neq 3m, \\
\lceil \frac{n}{2} \rceil - 2 & \text{if } n = 3m,
\end{cases}
\]
with \( m \in \mathbb{N} \). We can show that all these lines \( H_l \) have different slopes and therefore eventually intersect each other. The derivative of \( H_l(F) \) with respect to \( F \) is equal to

\[
H'_l = \frac{f(l,n)}{2(1 - \cos(\frac{2\pi l}{n}))} = \frac{1}{2} + \cos\left(\frac{2\pi l}{n}\right).
\]

Since this is an injective function with respect to \( l \), it follows that the slope \( H'_l \) is different for each \( l \).

Now we have derived that every Hopf-line intersects with all other Hopf lines, it is time to count intersections. The number of intersections should be equal to the triangular number of \( N_H - 1 \) (one less than \( N_H \), because we do not have self-intersections), i.e. \( N_{HH} = \frac{1}{2}N_H(N_H - 1) \). Inserting the values of \( N_H \) from equation (3.13) gives the maximal number of Hopf-Hopf bifurcation points for the trivial equilibrium, as in formula (3.12).

Q.E.D.

Remark 3.11. The preceding result shows that the maximum number of Hopf-Hopf bifurcation points grows quadratically with \( n \). Note that this is indeed an upper bound for the number of Hopf-Hopf points, since we did not take into account the possibility of three Hopf-lines intersecting each other at the same value of \( F \), although this is not a generic situation. Furthermore, since \( F = 0 \) does not yield any Hopf bifurcation, we should not count intersections which take place for \( F = 0 \). However, a quick look at equation (3.10) shows that this never happens.

Note also that there might be other Hopf-Hopf points which are not caused by these intersections. However, another equilibrium should then be involved.

Under the assumption that all nondegeneracy conditions are satisfied, the truncated normal form for a Hopf-Hopf bifurcation reads (Kuznetsov, 2004)

\[
\begin{align*}
\dot{\xi}_2 &= \xi_2(\mu_2 - \sigma \xi_2 - 8 \xi_3 + \Theta \xi_2^2), \\
\dot{\xi}_3 &= \xi_3(\mu_3 - \sigma \delta \xi_2 - \xi_3 + \Delta \xi_2^2), \\
\dot{\phi}_2 &= \omega_2, \\
\dot{\phi}_3 &= \omega_3,
\end{align*}
\]

(3.14)
after a suitable choice of phase variables. Here, \( \mu_j \) is defined as \( \mu_j := \text{Re} \kappa_j \), \( \sigma = \pm 1 \) and \( \vartheta, \delta, \Theta, \Delta \) are other normal form coefficients. The sign \( \sigma \) and the values of the coefficients \( \vartheta \) and \( \delta \) mainly determine the unfolding of the Hopf-Hopf bifurcation \( \text{(Kuznetsov, 2004)} \). In total, there are eleven different bifurcation scenarios to consider. In any case, two Neimark-Sacker bifurcation (ns) curves emanate from the Hopf-Hopf point. The directions of these ns-curves depend on \( \vartheta \) and \( \delta \) and can be computed up to first order via the real part of the eigenvalues at the Hopf-Hopf point \( \text{(Kuznetsov, 2004)} \). Note that the type of dynamics we have around the Hopf-Hopf bifurcation point does not depend on the choice of unfolding \( \text{(2.13)} \), since the normal form coefficients should be evaluated at the bifurcation point.

3.2.2 Unfolding for \( n = 12 \)

The case of dimension \( n = 12 \) is particularly interesting, since it is the smallest dimension for which the first bifurcation of \( x_F \) is a Hopf-Hopf bifurcation instead of a Hopf bifurcation. In this section we describe this situation more explicitly, using the results we obtained for general dimensions. Therefore, consider system \( \text{(2.13)} \) and let \( n = 12 \). The eigenvalues of the Jacobian belonging to the trivial equilibrium \( x_F \) are given by equation \( \text{(3.9)} \).

A Hopf bifurcation for the \( l \)-th eigenvalue pair (with \( 0 < l < 6, \ l \neq 4 \)) occurs along the Hopf-lines in equation \( \text{(3.10)} \). So, we obtain explicitly the following Hopf bifurcation curves as a function of \( F \in \mathbb{R} \setminus \{0\} \):

\[
H_1(F, 12) = \frac{2 + (1 - \sqrt{3})F}{2\sqrt{3} - 4}; \\
H_2(F, 12) = F - 1; \\
H_3(F, 12) = \frac{1}{2}(F - 1); \\
H_5(F, 12) = -\frac{2 + (1 + \sqrt{3})F}{2\sqrt{3} + 4}.
\]

The curves for \( l = 2, 3 \) intersect each other at \( (F, G) = (1, 0) \), which is the Hopf-Hopf bifurcation point we discovered in the original system \( \text{(2.1a)} \). It is easy to see that for \( G = 0 \) this is the first bifur-
cation of $x_F$ one encounters by increasing the parameter $F$, since the only Hopf bifurcation with positive $F_H$-value has $F_H(1,12) \approx 2.7321 > F_{HH} = 1$. Observe that there is no resonance present at the Hopf-Hopf point, since $\omega_2 = \text{Im}(\kappa_2(1,H_2(1,12),12)) = \sqrt{3}$ and $\omega_3 = \text{Im}(\kappa_3(1,H_3(1,12),12)) = 1$. Furthermore, numerical computation of the normal form coefficients using MatCont (Dhooge et al., 2011) yields the following values

$$(\sigma, \delta, \delta, \Theta, \Delta) = (1,1.4135,1.2584,-0.2001,0.6779),$$

showing that the bifurcation is indeed nondegenerate and that the normal form (3.14) is valid.

From the values of $\sigma$, $\delta$ and $\delta$ it follows that the dynamics of system (2.13) with $n = 12$ is of “type I in the simple case” as described by (Kuznetsov, 2004). This means that the two ns-curves are the only bifurcation curves that emanate from the Hopf-Hopf point and between these curves there exists a region in the $(F,G)$-plane where two stable periodic orbits coexist with an unstable $2$-torus. These ns-curves correspond to the limit cycles with $l = 2$ and $l = 3$ and are approximated by, respectively,

$$\mu_3 = \delta \mu_2 + O(\mu_2^2), \quad \mu_2 > 0,$n
$$\mu_2 = \delta \mu_3 + O(\mu_3^2), \quad \mu_3 > 0.$$n

Removing higher order terms and solving for $G$ gives the following linear curves in $F$:

$$G = NS_2(F) = \frac{1-\delta}{2-\delta}(F-1),$$

$$G = NS_3(F) = \frac{1-\delta}{1-2\delta}(F-1).$$

These lines are tangent to the real ns-curve at the Hopf-Hopf point.

The preceding results show how the Hopf-Hopf bifurcation acts as an organising centre in this particular dimension. Figure 3.1 displays the local bifurcation diagram with the Hopf-lines and approximated ns-curves $NS_l$ with $l = 2,3$ together with the phase portraits for each region. Note that in region 4 there are two stable periodic attractors that coexist for the same parameter values. We
Figure 3.1: Local bifurcation diagram for the two-parameter system (2.13) with $n = 12$ near the Hopf-Hopf bifurcation point located at $(F, G) = (1, 0)$ (top panel). The lines $H_2$ and $H_3$ are the exact Hopf bifurcation curves from (3.15), whereas the lines $NS_2$ and $NS_3$ are the linear approximations of the Neimark-Sacker curves (3.16). The phase portraits (lower panels) show the dynamics in each of the domains in the top figure. In region 1 there is only one stable equilibrium. In all other domains at least one stable periodic orbit exists. Here, a blue orbit corresponds to wave number 2, a red orbit corresponds to wave number 3, while a dashed line means that the orbit is unstable. Moreover, besides the two stable periodic attractors in region 4 an unstable 2-torus is present, which is not shown in the corresponding phase portrait. Note also that the phase portrait of region 5, respectively 6, is similar to that of region 3, respectively 2, though the stable attractor has wave number 3 instead of 2. Compare with the numerical results presented in figures 5.14 and 5.17.
will investigate this phenomenon of multistability in section 4.4 in more detail. In section 5.1.6, we verify these results numerically.

3.3 BIFURCATIONS FOR NEGATIVE $F$

Contrary to the case of positive forcing, in the bifurcation structure for negative forcing a lot of symmetry is involved. Of particular interest is when the dimension $n$ is even, in which case we can obtain a $\mathbb{Z}_2$-symmetry by $\gamma_{n/2}^{n/2}$. In this section we investigate the codimension 1 bifurcations that can be found in the Lorenz-96 model for $F < 0$, using the results on symmetries and invariant manifolds from section 2.3.

By means of equivariant bifurcation theory we show that for even dimensions the equilibrium $x_F$ exhibits a pitchfork bifurcation. The emerging stable equilibria both exhibit again a pitchfork bifurcation if the dimension equals $n = 4k$, $k \in \mathbb{N}$. Both cases will be proven for the smallest possible dimension using a theorem from Kuznetsov (2004) on bifurcations for systems with $\mathbb{Z}_2$-symmetry. A generalisation to all dimensions $n = 2k$, resp. $n = 4k$, is then provided by Proposition 2.6.

Furthermore, for $n \geq 4$ a supercritical Hopf bifurcation destabilises all present stable equilibria after at most two pitchfork bifurcation. Of particular interest is the observation that for specific dimensions there can be more than two subsequent pitchfork bifurcations, which however occur after the equilibria undergo the Hopf bifurcations. We formulate a conjecture about the number of subsequent pitchfork bifurcations that occur in a given dimension $n$.

3.3.1 Preliminaries

Before we start our analysis for $F < 0$, we state some preliminary theory about bifurcations in symmetric systems. Our proofs on pitchfork bifurcations in the Lorenz-96 model rely on these results.
Let us first introduce some notation. Let $R_n$ be an $n \times n$ matrix that defines a symmetry transformation $x \mapsto R_n x$. Furthermore, we decompose $\mathbb{R}^n$ into a direct sum $\mathbb{R}^n = X^+_n \oplus X^-_n$, where

$X^+_n := \{ x \in \mathbb{R}^n : R_n x = x \}$,

$X^-_n := \{ x \in \mathbb{R}^n : R_n x = -x \}$.

The following theorem demonstrates the existence of either a fold or a pitchfork bifurcation in $\mathbb{Z}_2$-equivariant systems:

**Theorem 3.12 (Kuznetsov (2004), Theorem 7.7).** Suppose that a $\mathbb{Z}_2$-equivariant system

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^1,$$

with smooth $f$, $R_n f(x, \alpha) = f(R_n x, \alpha)$ and $R_n^2 = \text{Id}_n$, has at $\alpha = 0$ the fixed equilibrium $x_0 = 0$ with simple zero eigenvalue $\lambda_1 = 0$, and let $v \in \mathbb{R}^n$ be the corresponding eigenvector.

Then the system has a one-dimensional $R_n$-invariant center manifold $W^c_\alpha$ and one of the following alternatives generically takes place:

- **(Fold)** If $v \in X^+_n$, then $W^c_\alpha \subset X^+_n$ for all sufficiently small $|\alpha|$ and the restriction of the system to $W^c_\alpha$ is locally topologically equivalent near the origin to the normal form

$$\dot{\xi} = \beta \pm \xi^2;$$

- **(Pitchfork)** If $v \in X^-_n$, then $W^c_\alpha \cap X^+_n = x_0$ for all sufficiently small $|\alpha|$ and the restriction of the system to $W^c_\alpha$ is locally topologically equivalent near the origin to the normal form

$$\dot{\xi} = \beta \xi \pm \xi^3.$$

**Remark 3.13.** At the pitchfork bifurcation the equilibrium that satisfies $R_n x_0 = x_0$ changes stability, while two $R_n$-conjugate equilibria appear (Kuznetsov, 2004). In terms of the fixed-point subspaces, this means that the resulting $R_n$-conjugate equilibria are contained in a larger subspace than the original. In section 3.3.5 we will elaborate further on this.
The proofs of the first and the second pitchfork bifurcations — see sections 3.3.2 and 3.3.3 — are based on the lowest possible dimensions, i.e. \( m = 2 \) and \( m = 4 \). In both cases we start with equilibria in \( \text{Fix}(G_m^{m/2}) \) and \( \mathbb{Z}_2 \)-symmetry is realised by \( \gamma_m^{m/2} \). Consequently, we set

\[
R_m := \gamma_m^{m/2},
\]

and the pitchfork bifurcation will result in two extra \( \gamma_m^{m/2} \)-conjugate equilibria in \( \text{Fix}(G_m^{m/2}) \). Likewise, we have that \( X_n^+ = \text{Fix}(G_m^{m/2}) \) and \( X_n^- = \text{Fix}(G_m^{m/2})^\perp \). Compare this with example 2.5 that treats this particular case.

Extending these results to dimensions \( n = km \) according to Proposition 2.6 yields that the equilibria after the first pitchfork bifurcation (for which \( m = 2 \)) are \( \gamma_{km}^{m/2} = \gamma_n^1 \)-conjugate and contained in \( \text{Fix}(G_n^2) \). Similarly, for the second pitchfork bifurcation we have \( m = 4 \), so here the resulting equilibria are pairwise \( \gamma_{km}^{m/2} = \gamma_n^2 \)-conjugate and contained in \( \text{Fix}(G_n^4) \).

3.3.2 First pitchfork bifurcation

As observed in section 2.1, the equilibrium (2.7) has a real eigenvalue when the dimension \( n \) is even. In that case, the eigenvalue for \( j = \frac{n}{2} \) equals \( \lambda_{n/2} = -1 - 2F \), which is the only eigenvalue that is purely real and depends on the parameter \( F \). Observe that at \( F = -\frac{1}{2} \) we have \( \lambda_{n/2} = 0 \). This gives rise to the following result:

**Theorem 3.14 (Pitchfork Bifurcation).** Let \( n \in \mathbb{N} \) be even. Then the trivial equilibrium \( x_F \) exhibits a supercritical pitchfork bifurcation at the parameter value \( F_{P,1} := -\frac{1}{2} \).

Note that the index of \( F_P \) anticipates the possibility of more pitchfork bifurcations, of which this is the first one in line for decreasing \( F \).

By Proposition 2.6, the proof of Theorem 3.14 reduces to proving the case \( n = 2 \) — see section 2.3.2. In order to prove the existence of a pitchfork bifurcation in the two-dimensional Lorenz-96 model, all we have to do is to show that it satisfies the second
case of Theorem 3.12. This is in brief how the following lemma is proven:

**Lemma 3.15.** Let \( n = 2 \), then the equilibrium \( x_F \) of the Lorenz-96 model exhibits a pitchfork bifurcation at the parameter value \( F_{P,1} = -\frac{1}{2} \).

**Proof.** For the two-dimensional model the eigenvalues of \( x_F \) are given by equation (3.1), so that

\[
\lambda_0 = -1, \quad \lambda_1 = -1 - 2F.
\]

Therefore, when \( F = F_{P,1} \) we have that \( \lambda_1 = 0 \) and a bifurcation takes place. An eigenvector at \( F_{P,1} \) corresponding to \( \lambda_1 \) is given by

\[
v_1 = (-1, 1).
\]

In order to show that system (2.1) with \( n = 2 \) has a \( \mathbb{Z}_2 \)-equivariance we use the symmetry transformation

\[
R_2 := \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

as defined by formula (3.17). It follows easily from section 2.3 that this matrix has the following properties:

- \( R_2^2 = \text{Id}_2 \);
- \( R_2 f_2(x,F) = f_2(R_2 x, F) \), by Proposition 2.4;
- \( R_2 \) defines a symmetry transformation on \( \mathbb{R}^2 = X_2^+ \oplus X_2^- \) with

\[
X_2^+ = \text{Fix}(G_2^0) = V^0,
\]

\[
X_2^- = \text{Fix}(G_2^1)^\perp = \{ x \in \mathbb{R}^2 : x_0 = -x_1 \}.
\]

With these preliminaries the conditions of Theorem 3.12 are satisfied (up to a transformation to the origin). Additionally, it is easy to see that we are in the pitchfork-case, since we have

\[
R_2 v_1 = -v_1,
\]

i.e. the eigenvector with respect to \( \lambda_1(F_{P,1}) \) lies in \( X_2^- \). Hence, the two-dimensional Lorenz-96 model has a one-dimensional \( R_2 \)-
invariant center manifold $W_c^F$ with $W_c^F \cap X_2^+ = x_F$ for all $F$ sufficiently close to $F_{P,1}$ and the restriction of the system to $W_c^F$ is locally topologically equivalent near $x_F$ to the normal form of a pitchfork bifurcation.

Q.E.D.

Proof of Theorem 3.14. The result of Lemma 3.15 extends to all dimensions $n = 2k, k \in \mathbb{N}$ by Proposition 2.6.

Q.E.D.

Remark 3.16. Theorem 3.14 can also be proven via a center manifold reduction (Guckenheimer & Holmes, 1983; Kuznetsov, 2004; Wiggins, 2003); this proof can be found in appendix B.3.1. The result yields the normal form of a supercritical pitchfork bifurcation, meaning that the equilibrium $x_F$ is stable for $F > F_{P,1}$ and loses stability at $F = F_{P,1}$, while two other stable equilibria exist for $F < F_{P,1}$. Notice that this proof is much longer, but it has the advantage that it also shows that the pitchfork bifurcation is supercritical for any even dimension.

At the pitchfork bifurcation the equilibrium $x_F \in V^0$ loses stability and gives rise to two stable equilibria $\xi^1_j \in V^1, j = 0, 1$, that exist for $F < -\frac{1}{2}$. This implies that the bifurcation is supercritical, as is confirmed by the proof via a center manifold reduction in appendix B.3. These new equilibria are given by

$$\xi^1_0(F) = (a_+, a_-, \ldots, a_+, a_-), \quad a_\pm = -\frac{1}{2} \pm \frac{1}{2} \sqrt{-1 - 2F},$$

(3.19)

while $\xi^1_1$ is obtained by swapping the indices $+$ and $-$. So, each $\xi^1_j$ has a structure like $x^m$ in formula (2.17) with $m = 2$ and has precisely half the symmetry of $x_F$ since it has $G^2_n = \langle \gamma^2_n \rangle$ as isotropy subgroup (whose order is half the order of the full group). Moreover, the relation between these two equilibria is given by $\xi^1_1 = \gamma_n \xi^1_0$, which means that these equilibria are $\gamma_n$-conjugate as predicted by Remark 3.13. In other words: applying the matrix $\gamma_n$ means geometrically a switch from one branch of equilibria to the other.
3.3.3 Second pitchfork bifurcation

The pitchfork bifurcation described in the previous section is followed by a second subsequent pitchfork bifurcation for $F < F_{P,1}$ when the dimension is a multiple of 4. This time, there are two bifurcations, each of which takes place at a different branch of the equilibria emanating from the first pitchfork bifurcation and described by equation (3.19).

**Theorem 3.17 (Second pitchfork bifurcation).** Let $n = 4k$ with $k \in \mathbb{N}$. Then both equilibria $\xi^1_{0,1}(F)$ emanating from the pitchfork bifurcation at $F_{P,1} = -\frac{1}{2}$ exhibit a supercritical pitchfork bifurcation at the parameter value $F_{P,2} := -3$.

The proof goes in exactly the same way as the proof for the first pitchfork bifurcation. Again, we first prove a lemma that describes the occurrence of a second pitchfork bifurcation in the lowest possible dimension, $n = 4$. Thereafter, we generalise to higher dimensions $n = 4k$ using Proposition 2.6.

**Lemma 3.18.** Let $n = 4$, then the equilibria $\xi^1_{0,1}(F)$ emanating from the pitchfork bifurcation at $F_{P,1} = -\frac{1}{2}$ both exhibit a pitchfork bifurcation at the parameter value $F_{P,2} := -3$.

**Proof.** The four eigenvalues of both equilibria $\xi^1_{0,1}$ are given by:

\[
\begin{align*}
\lambda^1_{0,1} &= \frac{1}{2}(-1 \pm \sqrt{9 + 16F}), \\
\lambda^1_{2,3} &= \frac{1}{2}(-3 \pm \sqrt{-3 - 4F}).
\end{align*}
\tag{3.20}
\]

Since $\lambda^1_2 = 0$ when $F = F_{P,2}$, a bifurcation takes place. An eigenvector corresponding to $\lambda^1_2(F_{P,2})$ is given by

\[
v^1_2 = (2 + \sqrt{5}, -1, -2 - \sqrt{5}, 1).
\]

Recall from formula (3.17) that the symmetry transformation

\[
R_4 := \gamma^2_4 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\tag{3.21}
\]
makes the four-dimensional system (2.1) $Z_2$-equivariant. Again, this matrix satisfies the requirements of Theorem 3.12, since

- $R_4^2 = \text{Id}_4$;
- $R_4 f_4(x, F) = f_4(R_4 x, F)$, by Proposition 2.4;
- $R_4$ defines a symmetry transformation on $\mathbb{R}^4 = X_4^+ \oplus X_4^-$, where
  \[
  X_4^+ = \text{Fix}(G_4^2) = V^1,
  X_4^- = \text{Fix}(G_4^2)^\perp = \{x \in \mathbb{R}^4 : x_0 = -x_2, x_1 = -x_3\}.
  \]

Note that the group $\{\text{Id}_4, R_4\}$ has $V^1$ as its fixed-point subspace, so it contains all the symmetries of $\xi^1 \in V^1$ (i.e. $R_4 \xi^1_j = \xi^1_j$). In contrast, it holds that

\[ R_4 v_2^1 = -v_2^1, \]

i.e. the eigenvector with respect to $\lambda_2^{1}(F_{P,2})$ lies in $X_4^-$. By Theorem 3.12 this implies that a pitchfork bifurcation takes place and the four-dimensional Lorenz-96 model has a one-dimensional $R_4$-invariant center manifold $W^c_F$ with $W^c_F \cap X_4^+ = \xi^1_j$ for all $F$ sufficiently close to $F_{P,2}$.

Q.E.D.

Remark 3.19. Lemma 3.18 can be proven by a center manifold reduction, like in the alternative proof of Theorem 3.14 — see Remark 3.16. For $n = 4$ this is sketched in appendix B.3.2 and shows that both pitchfork bifurcations at $F_{P,2}$ are supercritical.

Proof of Theorem 3.17. The result of Lemma 3.18 extends to all dimensions $n = 4k, k \in \mathbb{N}$ by Proposition 2.6.

Q.E.D.

Remark 3.20. A generalisation to all $n = 4k - 2$ is not provided by Proposition 2.6. Indeed, the second pair of eigenvalues $\lambda_2^{1,3}$ of (3.19) occurs only in the form of equation (3.20) when the dimension has the form $n = 4k$. If instead the dimension equals $n = 2$ then there are no more eigenvalues that can cross the imaginary axis, whereas for $n = 4k - 2, k \geq 2$, the eigenvalue pairs are different from the case $n = 4k$, as numerical computations show
— see section 3.3.4. Therefore, in dimensions $n = 4k - 2$, $k \geq 2$, there will not be an additional pitchfork bifurcation. The next bifurcation after the first pitchfork bifurcation is a Hopf bifurcation, as will be shown in section 3.3.4.

At the second pitchfork bifurcation the equilibria $\xi_j^1 \in V^1$, with $j = 0, 1$, lose stability and four stable equilibria $\xi_j^2 \in V^2$, $0 \leq j \leq 3$ appear that exist for $F < -3$. This again shows that the bifurcation is supercritical. In contrast with $\xi_j^1$ it is not feasible to derive analytic expressions for the equilibria $\xi_j^2$. In the four-dimensional case, these new equilibria are — by Remark 3.13 — pairwise $R_4$-conjugate in the following way: $\xi_j^2 = R_4 \xi_{j+2}^2$ (with the index modulo 4), that is, the equilibria with index $j$ and $j + 2$ emanate from the same equilibrium $\xi_j^1$ for $j = 0, 1$. By equivariance, the conjugate solutions $\gamma_4 \xi_j^2$ should be equilibria as well for all $0 \leq j \leq 3$. In fact, we observe numerically that this gives precisely the solutions from the other $R_4$-conjugate pair of solutions — see section 3.3.5. This means that we can switch between all four equilibria by subsequently applying $\gamma_4$, to be precise $\xi_j^2 = \gamma_4 \xi_0^2$.

For general dimensions $n = 4k$ similar statements hold: the new equilibria satisfy $\xi_j^2 = \gamma_n \xi_j^2$ (with the index modulo $n$) and they are of the form (2.17) with $k = 4$. This gives an extra argument why a symmetry breaking by a pitchfork bifurcation is not possible in dimensions $n = 4k - 2$: since $n$ is not divisible by four, we cannot ‘fill’ the $n$ coordinates of an equilibrium in $\mathbb{R}^n$ completely by blocks of four entries and the invariant subspace $V^2$ does not exist.

### 3.3.4 Hopf bifurcations

Recall that for $n = 2$ only one pitchfork bifurcation is possible and no further bifurcation can happen. Moreover, in dimensions $n = 1$ and 3 all eigenvalues are equal to $-1$, so that no bifurcation is possible at all. Apart from that, we show below that in any dimension $n \geq 4$ the stable equilibria for negative parameter values $F$ eventually lose stability through a supercritical Hopf bifurcation and one or more stable periodic orbits appear. Because the Hopf
bifurcation is preceded by at most two pitchfork bifurcations — depending on the dimension — we consider three different cases according to the number of occurring pitchfork bifurcations: zero, one or two. In section 4.3, we discuss the properties and implications for the resulting periodic orbits.

**Case 1: No Pitchfork Bifurcations**  
For odd dimensions, no pitchfork bifurcation will occur, since the equilibrium (2.7) has only complex eigenvalue pairs (2.9) that can cross the imaginary axis and no single real eigenvalue that depends on $F$ — see section 2.1. Since formula (3.3) cannot be satisfied for $F < 0$, the bifurcations of $x_F$ are completely covered by Theorem 3.5, which implies the occurrence of at least one Hopf bifurcation in each odd dimension. There are as many Hopf bifurcations for $F < 0$ as there are integers $l \in \left( \frac{n}{3}, \frac{n}{2} \right)$ and corresponding eigenvalue pairs.

In particular, for the first Hopf bifurcation for decreasing negative $F$ the following holds:

**Proposition 3.21.** Let $n \geq 4$ be fixed. For $F < 0$ the first Hopf bifurcation of $x_F$ occurs for the eigenvalue pair with index

$$l_1^{-}(n) := \frac{n - 1}{2}.$$  

In particular, this first bifurcation is supercritical and its bifurcation value $F_H(l_1^+, n)$ converges to $-\frac{1}{2}$ as $n \to \infty$.

**Proof.** Given odd $n$, the first Hopf bifurcation for $F < 0$ occurs when formula (3.2) is maximised, which is equivalent to minimising $f(l, n)$. We obtain that the first bifurcating eigenvalue pair $\{\lambda_{l_1^-}, \lambda_{n-l_1^-}\}$ has index

$$l_1^{-}(n) := \frac{n - 1}{2}.$$  

Hence, the corresponding bifurcation value for general $n$ is bounded as

$$F_H(l_1^{-}(5), 5) \approx -0.8944 \leq F_H(l_1^{-}(n), n) < -0.5,$$

since $n \geq 5$ and the function

$$\cos \frac{2n l_1^{-}(n)}{n} - \cos \frac{4n l_1^{-}(n)}{n} = -\cos \frac{n}{n} - \cos \frac{2n}{n}$$
is negative and strictly decreasing as \( n \) increases. In particular, if \( n \) goes to infinity then
\[
\lim_{n \to \infty} \frac{l_1^-(n)}{n} = \lim_{n \to \infty} \frac{n - 1}{2n} = \frac{1}{2},
\]
and, hence, the limiting bifurcation value satisfies
\[
\lim_{n \to \infty} F_{H}(l_1^-(n), n) = \frac{1}{f(1, 2)} = -\frac{1}{2}.
\]
Since the index of the bifurcating eigenvalue pairs for \( F < 0 \) satisfy \( l \in (\frac{n}{3}, \frac{n}{2}) \), Theorem 3.5 directly implies that all Hopf bifurcations of the equilibrium \( x_F \) for negative \( F \) are supercritical.

Q.E.D.

The fact that this first Hopf bifurcation for negative values is supercritical implies that the stable equilibrium \( x_F \) loses stability and a stable periodic orbit appears after the bifurcation. Figure 3.2 displays the described situation of odd \( n > 3 \) schematically.

\[\text{Figure 3.2: Schematic representation of the attractors for negative } F \text{ in an } n\text{-dimensional Lorenz-96 model with odd } n > 3, \text{ so without any pitchfork bifurcation. The label } H \text{ stands for a (supercritical) Hopf bifurcation and occurs for } -0.8944 \leq F_H < -\frac{1}{2}. \text{ The only equilibrium is given by } x_F \in \text{Fix}(G^1_n). \text{ A solid line represents a stable attractor; a dashed line represents an unstable one.}\]

**Case 2: One Pitchfork Bifurcation** Let \( n \) be even with \( n = 4k + 2, k \in \mathbb{N} \). By Remark 3.20 only one pitchfork bifurcation occurs in this case. At \( F_{P,1} = -\frac{1}{2} \) the trivial equilibrium (2.7) loses stability and the two stable, \( \gamma_n \)-conjugate equilibria (3.19) appear, according to Theorem 3.14. The now unstable equilibrium \( x_F \) still
exhibits as many supercritical Hopf bifurcations for $F < F_{p,1}$ as there are integers $l \in (\mathbb{N}, \mathbb{N})$ and corresponding eigenvalue pairs, by Theorem 3.5. However, the emerging periodic attractors will be unstable at first.

We now demonstrate numerically that both equilibria exhibit a supercritical Hopf bifurcation simultaneously, by studying dimension $n = 6$ in greater detail. Each of the equilibria $\xi_{0,1}$ emerging from the first pitchfork bifurcation may bifurcate again as $F$ decreases. We first consider the equilibrium $\xi_{1}^{1}$ for which the Jacobian is given by

$$
J = \begin{pmatrix}
-1 & a_{-} & 0 & 0 & -a_{-} & a_{-} - a_{+} \\
-1 & a_{+} & 0 & 0 & a_{+} & -a_{-} \\
-a_{-} & a_{-} - a_{+} & -1 & a_{-} & 0 & 0 \\
0 & -a_{+} & a_{+} - a_{-} & -1 & a_{+} & 0 \\
0 & 0 & a_{-} & a_{-} - a_{+} & -1 & a_{-} \\
a_{+} & 0 & 0 & -a_{+} & a_{+} - a_{-} & -1
\end{pmatrix},
$$

(3.23)

where $a_{\pm}$ are given by formula (3.19). Note that $J$ is no longer circulant: in addition to shifting each row in a cyclic manner, the values of $a_{+}$ and $a_{-}$ also need to be interchanged. In particular, this means that the eigenvalues are no longer determined by means of equation (3.1). Symbolic manipulations with the computer algebra package Mathematica (Wolfram Research, 2016) show that an eigenvalue crossing occurs for $F = -\frac{7}{2}$ in which case $a_{\pm} = \frac{1}{2}(-1 \pm \sqrt{5})$, so that the characteristic polynomial of $J$ is given by

$$
\det(J - \lambda I) = 468 + 219\lambda + 246\lambda^{2} + 91\lambda^{3} + 33\lambda^{4} + 6\lambda^{5} + \lambda^{6} = (3 + \lambda^{2})(12 + \lambda + \lambda^{2})(13 + 5\lambda + \lambda^{2}).
$$

This expression shows that $J$ has two purely imaginary eigenvalues $\pm i\sqrt{3}$ and the remaining four complex eigenvalues have negative real part. Therefore the equilibrium $\xi_{1}^{1}$ undergoes a supercritical Hopf bifurcation at $F'_{H}(n = 6) := -\frac{7}{2}$. The computations for
the equilibrium $\xi_1^1$ are similar and show that another Hopf bifurcation takes place at $F = -\frac{7}{2}$. This means that for $F < F'_H(6)$ there exist two stable periodic orbits, born at the Hopf bifurcations of $\xi_0^1$ and $\xi_1^1$, that coexist.

For general $n = 4k + 2$, with $k \in \mathbb{N}$, numerical continuation using the software package Auto-07p (Doedel & Oldeman, 2007), shows that each of the two equilibria (3.19) undergoes a supercritical Hopf bifurcation, which leads to the coexistence of two stable waves. Figure 3.3 suggests that the Hopf bifurcation value $F'_H(n)$ is not constant, but satisfies $F'_H(6) = -3.5 \leq F'_H(n) \leq -3$ and seem to converge to $-3$ as $n \to \infty$. Hence, for parameter values $F < F'_H(n)$ the two equilibria $\xi_j^1, j = 0, 1$, are unstable and two stable periodic orbits coexist. The schematic bifurcation diagram for all dimensions $n = 4k + 2$ is sketched in figure 3.4.

Figure 3.3: Bifurcation values of the first Hopf bifurcation for $F < 0$ and even values of the dimension $n$. For clarity the bifurcation values $F'_H$ (corresponding to the case $n = 4k + 2$) and $F''_H$ (corresponding to $n = 4k$) have been marked with different symbols in order to emphasize the differences between the two cases.

Case 3: Two Pitchfork Bifurcations  
If the dimension $n$ is of the form $n = 4k$, $k \in \mathbb{N}$, then Theorems 3.14 and 3.17 guarantee the occurrence of two subsequent pitchfork bifurcations: one at $F_{P,1} = -\frac{1}{2}$ and two simultaneous bifurcations at $F_{P,2} = -3$. So,
Figure 3.4: Schematic bifurcation diagram for negative \( F \) of a \( 2^1 p \)-dimensional Lorenz-96 model with \( p > 1 \) odd. The label \( PF_1 \) denotes the only (supercritical) pitchfork bifurcation with bifurcation value \( F_{P,1} = -\frac{1}{2} \); \( H \) stands for a (supercritical) Hopf bifurcation with bifurcation value \(-3.5 \leq F_H' \leq -3\), depending on \( n \). The equilibria are \( \xi^0 \equiv x_F \in V^0 \) and \( \xi^1_j \in V^1, \ j = 0, 1 \), given by equation (3.19). A solid line represents a stable equilibrium; a dashed line represents an unstable one.

for \( F < -3 \) there are four stable equilibria \( \xi^2_j, \ 0 \leq j \leq 3 \). Again, for \( F < F_{P,1} \) the equilibrium \( x_F \) is unstable and exhibits as many supercritical Hopf bifurcations as there are integers \( \ell \in \left( \frac{n}{3}, \frac{n}{2} \right) \).

The analysis of the lowest possible dimension — i.e. \( n = 4 \) — turns out to be more complicated than the case \( n = 6 \) and, surprisingly, not analytically tractable. Numerical continuation using the software package \textsc{Auto-07p} (Doedel & Oldeman, 2007) of the four branches after the two subsequent pitchfork bifurcations — while monitoring their stability — indicates that at \( F \approx -3.8531 \) in total four supercritical Hopf bifurcations occur simultaneously: one at each branch.

For general \( n = 4k \), numerical continuation shows that each of the four stable equilibria \( \xi^2_j \) undergoes a supercritical Hopf bifurcation, each at exactly the same bifurcation value \( F''_{H}(n) \). Fig-
Figure 3.5: Schematic bifurcation diagram for negative $F$ of a $2^p$-dimensional Lorenz-96 model with $p \in \mathbb{N}$. The label $PF_1$, resp. $PF_2$, denotes the first, resp. second, (supercritical) pitchfork bifurcation with bifurcation value $F_{P,1} = -\frac{1}{2}$, resp. $F_{P,2} = -3$; $H$ stands for a (supercritical) Hopf bifurcation with bifurcation value $-3.9 < F''_H < -3.5$, depending on $n$. The equilibria are $\xi^0 = x_F \in V^0$ and $\xi^1_j \in V^1$, $j = 0, 1$, given by equation (3.19), while $\xi^2_j \in V^2$, $j = 0, \ldots, 3$. A solid line represents a stable equilibrium; a dashed line represents an unstable one.

Figure 3.3 suggests that $F''_H(n)$ is not constant in $n$, but satisfies $-3.9 < F''_H(n) < -3.6$ and tends to $-3.64$ as $n \to \infty$. Thus, in this case four stable periodic orbits coexist in at least a small region below the bifurcation value $F''_H(n)$. See figure 3.5 for a schematic view of the bifurcation diagram, which is illustrative for all dimensions $n = 4k$ up to $F = F''_H$. 
Concluding, we see that a supercritical Hopf bifurcation destabilises the present stable equilibria after at most two pitchfork bifurcations. Therefore, the dynamical structure for \( n \geq 4 \) and \( F < 0 \) can be divided into three classes, depending on the dimension:

1. In all odd dimensions, the first bifurcation of the equilibrium \( x_F \) is a supercritical Hopf bifurcation. This yields one stable periodic orbit.

2. In dimensions \( n = 4k + 2, k \in \mathbb{N} \), only one pitchfork bifurcation takes place, followed by two simultaneous Hopf bifurcations. This leads to two coexisting stable periodic orbits.

3. For all dimensions \( n = 4k, k \in \mathbb{N} \), all four stable equilibria generated by the second pitchfork bifurcation exhibit a Hopf bifurcation simultaneously, resulting in four coexisting periodic orbits.

We will study the spatiotemporal properties of these periodic orbits in section 4.3.

3.3.5 *Multiple pitchfork bifurcations*

In section 3.3.3 we have proven that it is possible to have two pitchfork bifurcation after each other, namely, when \( n \) is a multiple of 4. Numerically, we observe that there can be even more subsequent pitchfork bifurcations after these two bifurcations. Even though these additional bifurcations happen after the Hopf bifurcations of case 3 in the previous subsection — and therefore for unstable equilibria, generating unstable equilibria — they entail arbitrarily large groups of symmetries, an exponentially increasing number of equilibria and they show a beautiful structure.

Since this possibly influences the dynamical structure for smaller \( F \), we present in this section a more detailed overview on what can be found regarding symmetries and the series of pitchfork bifurcations. We discuss and explain the possibility of more than two subsequent pitchfork bifurcations for specific dimensions and show how many of them can be expected in each dimension. Most of these results follow from numerical observations. We interpret
these observations by means of the theoretical description in section 2.3 without aiming to be complete. Especially, proving facts after many pitchfork bifurcations will become increasingly difficult, since the lowest dimension that is needed increases exponentially.

In the following exposition we write the dimension uniquely as \( n = 2^q p \), with \( q \in \mathbb{N} \) arbitrary and \( p \) odd. One should bear in mind that the cases \( q = 1 \) and \( q = 2 \) are completely covered by the results proven in section 2.3. Consequently, we assume \( q \geq 3 \) in the following, which enables the occurrence of more than two subsequent pitchfork bifurcations. Let us start with some notation that anticipates the results later on.

**Notation** First of all, we call the pitchfork bifurcation which is the \( l \)-th in the row the \( l \)-th pitchfork bifurcation and denote its bifurcation value as \( F_{P,l} \). Clearly, \( F_{P,l} < F_{P,l-1} \) and by definition the \( l \)-th pitchfork bifurcation occurs for equilibria generated through the \((l - 1)\)-th bifurcation. In the previous sections we have already used this nomenclature for the cases \( l = 1, 2 \).

Furthermore, inspired by Remark 3.13 and equation (3.19) we define

\[
\xi^l_j \in V^l, \quad 0 \leq j \leq 2^l - 1, \tag{3.24}
\]

to be the equilibria generated by the \( l \)-th pitchfork bifurcation, which have the same symmetry as equilibria of the form \( x^{2^l} \). Below we will indeed observe such equilibria \( \xi^l_j \) that come along with multiple pitchfork bifurcations.

Similarly, let \( \xi^l \) be the collection of all equilibria \( \xi^l_j \),

\[
\xi^l := \{ \xi^l_j, \ 0 \leq j \leq 2^l - 1 \} \subset V^l,
\]

which turns out to contain all equilibria that share the same properties. Accordingly, for \( l = 1 \) we have the equilibria as defined in equation (3.19): \( \xi^1 = \{ \xi^1_0, \xi^1_1 \} \subset V^1 \). Likewise, we can also define \( \xi^0 \equiv x_F \in V^0 \), for convenience (even though a ‘0-th pitchfork bifurcation’ does not exist).
Table 3.2: The number of successive pitchfork bifurcations and the corresponding total number of equilibria after the last pitchfork bifurcation as observed in selected even dimensions.

<table>
<thead>
<tr>
<th>$n$</th>
<th>#PF's</th>
<th>#Equilibria</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>31</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>24</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>32</td>
<td>5</td>
<td>63</td>
</tr>
<tr>
<td>36</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>64</td>
<td>6</td>
<td>127</td>
</tr>
<tr>
<td>128</td>
<td>7</td>
<td>255</td>
</tr>
</tbody>
</table>

**Numerical Observations**  In table 3.2 we list the numbers of successive pitchfork bifurcations (middle column) that are observed for specific even dimensions as well as the total number of equilibria generated through these bifurcations including the trivial equilibrium $x_F$ (right column). The number of pitchfork bifurcations for a specific dimension $n = 2^q p$, as above, turns out to be precisely the exponent $q$. Accordingly, we assume in the following that $0 \leq l \leq q$, which coincides with the restriction for $V^l$ in equation (2.20).

By numerical continuation using the software packages Auto07p (Doedel & Oldeman, 2012) and MatCont (Dhooge et al., 2011) we observe that the bifurcation values $F_{P,l}$ are independent of $n$ for all $l$. These fixed values $F_{P,l}$ are listed in table 3.3 for $l \leq 9$ and obtained by analysing the dimensions $n = 2^l$. In addition, the $l$-th pitchfork bifurcation occurs for all equilibria $x_j^{l-1}(F)$ with $0 \leq j \leq 2^{l-1} - 1$ at exactly the same bifurcation value $F_{P,l}$. So, when
Table 3.3: List of bifurcation values $F_{P,l}$ for the $l$-th pitchfork bifurcation, that are independent of the dimension $n$ and known up to $l = 9$. The two right columns give the distances between the successive pitchfork bifurcations and their ratios $r_l = (F_{P,l-1} - F_{P,l-2})/(F_{P,l} - F_{P,l-1})$.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$F_{P,l}$</th>
<th>Distance to $F_{P,l-1}$</th>
<th>$r_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>−0.5</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>2</td>
<td>−3</td>
<td>2.5</td>
<td>−</td>
</tr>
<tr>
<td>3</td>
<td>−6.6</td>
<td>3.6</td>
<td>0.694444</td>
</tr>
<tr>
<td>4</td>
<td>−8.0107123</td>
<td>1.41071</td>
<td>2.55190</td>
</tr>
<tr>
<td>5</td>
<td>−8.4360408</td>
<td>0.425329</td>
<td>3.31676</td>
</tr>
<tr>
<td>6</td>
<td>−8.5275625</td>
<td>0.0915217</td>
<td>4.64730</td>
</tr>
<tr>
<td>7</td>
<td>−8.5474569</td>
<td>0.0198944</td>
<td>4.60037</td>
</tr>
<tr>
<td>8</td>
<td>−8.5517234</td>
<td>$4.2665 \times 10^{-3}$</td>
<td>4.66289</td>
</tr>
<tr>
<td>9</td>
<td>−8.5526377</td>
<td>$9.143 \times 10^{-4}$</td>
<td>4.66681</td>
</tr>
</tbody>
</table>

we speak about 'the $l$-th pitchfork bifurcation' there are actually $2^{l-1}$ simultaneous pitchfork bifurcations of conjugate equilibria, generating $2^l$ new equilibria.

Even more, we observe that all these new equilibria have the same entries in the same order but shifted, which justifies our notation of the equilibria (3.24). Therefore, the equilibria $\xi_j^l \in V^l$ can be obtained from one another by applying $\gamma_n$ repeatedly:

$$
\gamma^n \xi_j^l = \xi_{j+k}^l \quad \text{for all } 0 \leq j, k \leq 2^l - 1,
$$

(3.25)

where the index of $\xi$ should be taken modulo $2^l$. As described in section 2.3.2, all these conjugate solutions have the same properties and therefore it suffices to study only one copy of them, say $\xi_0^l$. We will often just refer to the set $\xi^l$ (so, without index) when we describe their common properties.

Table 3.3 also shows that the distance between successive pitchfork bifurcations decreases as $l$ increases. The values of their ratios $r_l$ suggest that

$$
\lim_{l \to \infty} r_l = \lim_{l \to \infty} \frac{F_{P,l-1} - F_{P,l-2}}{F_{P,l} - F_{P,l-1}} = \delta,
$$
where \( \delta \approx 4.66920 \) is Feigenbaum’s constant. Therefore, the \( q \)-th and last pitchfork bifurcation of a specific dimension will be expected for the bifurcation value \( F_{P,q} \geq F_{P,\infty} \approx -8.55289 \).

**Visualisation of Structure** The structure of pitchfork bifurcations and equilibria that we observed by numerical analysis is summarised in figure 3.6, which we will now explain. The figure presents a schematic view for the case \( n = 2^d p \) with \( q = 4 \) and gives an indication for the bifurcation structure for general \( q \geq 3 \).

First of all, the horizontal line in the middle represents the trivial equilibrium \( \xi^0 = x_F \) that exists for all \( F \) and is stable for \( F > F_{P,1} \). At the point \( PF_1 \) we see that two stable equilibria \( \xi^1_{0,1} \) emerge, while \( \xi^0 \) becomes unstable: the first supercritical pitchfork bifurcation. The resulting equilibria are \( \gamma_n \)-conjugate, so \( \xi^1_1 = \gamma_n \xi^1_0 \).

Secondly, both equilibria \( \xi^1_{0,1} \) exhibit a pitchfork bifurcation \( PF_2 \) at \( F_{P,2} \). In both cases a pair of stable, \( \gamma_n^2 \)-conjugate equilibria appear, i.e. \( \xi^2_2 = \gamma_n^2 \xi^2_0 \) and \( \xi^2_3 = \gamma_n^2 \xi^2_1 \). Moreover, by formula (3.25) these pairs are also \( \gamma_n \)-conjugate to each other, which means that we can switch between the branches originating from \( \xi^1_0 \) and those from \( \xi^1_1 \) by applying \( \gamma_n \).

Next, all four equilibria from \( \xi^2 \) exhibit a supercritical Hopf bifurcation, as explained in case 3 of section 3.3.4. As a result, \( \xi^2 \) and all successive equilibria \( \xi^l, 2 < l \leq q \) are unstable for \( F < F''_H \). Thereafter, the third pitchfork bifurcation \( PF_3 \) occurs at \( F_{P,3} \) and generates \( 2^3 \) unstable and pairwise \( \gamma_n^4 \)-conjugate equilibria \( \xi^3 \). Finally, the fourth pitchfork bifurcation generates the equilibria \( \xi^4 \subset V^4 \). This completes the full structure in figure 3.6 with \( 2^5 - 1 \) unstable equilibria.

**Explanation by Symmetry** The preceding phenomena can be explained using the concepts introduced in section 2.3. In general, at a pitchfork bifurcation there is a breaking of the symmetry: before the bifurcation there exists an equilibrium \( x_0 \) satisfying \( Rx_0 = x_0 \), where \( R \) represents \( \mathbb{Z}_2 \)-symmetry; after the bifurcation two additional equilibria \( x_{1,2} \) appear that satisfy \( Rx_1 = x_2 \) (Kuznetsov, 2004). So, the new equilibria \( x_{1,2} \) after the bifurcation have a lower order of symmetry than the bifurcating equilibrium
x_0$, as explained by Remark 3.13. In terms of the invariant subspaces, this means that the smallest invariant subspace containing x_{1,2} should be larger than the one containing x_0. More explicitly: if x_0 \in \text{Fix}(G_n^m), with m \leq \frac{n}{2} minimised, then the two resulting equilibria x_{1,2} are in Fix(G^m_n') with m' = 2m (due to $\mathbb{Z}_2$-symmetry).

In section 3.3.2 we have demonstrated that the equilibrium $\xi^0 = x_F \in V^0$ exhibits the first pitchfork bifurcation and that two stable equilibria $\xi^1 \subset V^1$ appear. The second pitchfork bifurcation occurs for both equilibria $\xi^1$ simultaneously and generates stable equilibria $\xi^2 \subset \text{Fix}(G^4_n) = V^2$, as shown in section 3.3.3. In general, assuming that $l \leq q$ and that the equilibria $\xi^{l-1} \subset V^{l-1}$ generated through the $(l-1)$-th pitchfork bifurcation again exhibit a pitchfork bifurcation, then the $l$-th pitchfork bifurcation generates $2^l$ new branches of equilibria $\xi_j^l(F) \in V^l$, $0 \leq j \leq 2^l - 1$, where $F < F_{P,l}$. Thus, the total number of equilibria for dimension $n$ generated by the $q$ pitchfork bifurcations (including the trivial equilibrium) is equal to $2^{q+1} - 1$, which is confirmed by the right column of table 3.2.

The observation that the $l$-th pitchfork bifurcation consists of $2^{l-1}$ simultaneous pitchfork bifurcations of conjugate equilibria can be explained by noting that all equilibria $\xi^{l-1}$ satisfy the relation (3.25) and therefore share the same properties and, in particular, the same eigenvalues. The fact that the bifurcation values $F_{P,l}$ are the same for all dimensions is a direct consequence of Proposition 2.6.

In particular, the $q$-th pitchfork bifurcation generates equilibria $\xi^q \subset V^q = \text{Fix}(G_n^{2^q})$. Consequently, there cannot be more than $q$ subsequent pitchfork bifurcations, because this requires the resulting equilibria to be in Fix($G_n^{2^{q+1}}$), which does not exist. Hence, for any dimension $n = 2^q p$ there can be at most $q$ pitchfork bifurcations.

Based on our numerical observations and their interpretation in terms of symmetry, the following conjecture seems plausible:

**Conjecture 3.22.** The number of subsequent pitchfork bifurcations in the Lorenz-96 model of dimension $n = 2^q p$, where $q \in \mathbb{N} \cup \{0\}$ and $p$ odd, is exactly equal to $q$. 


Figure 3.6: Schematic bifurcation diagram of a $2^q p$-dimensional Lorenz-96 model for negative $F$ with $q = 4$ subsequent pitchfork bifurcations. The label $PF_l$, $1 \leq l \leq q$, denotes the $l$-th (supercritical) pitchfork bifurcation with bifurcation value $F_{P,l}$ as in table 3.3; $H$ stands for a (supercritical) Hopf bifurcation with bifurcation value $-3.9 < F_{H}^{''} < -3.5$. Each branch of equilibria is labelled with $\xi_l^j$ according to equation (3.24), where $l$ indicates that the branch is generated by the $l$-th pitchfork bifurcation and contained in $V^l$ and $j$ denotes how often we have to apply $\gamma_n$ to $\xi_l^0$ to obtain this branch, as in equation (3.25). A solid line represents a stable equilibrium; a dashed line represents an unstable one. The arrows in gray indicate the relation between the mutual branches. Similar diagrams can be obtained for any $q \geq 3$. 
In summary, the results in this section show that in each dimension $n = 2^q p$ there are exactly $q$ pitchfork bifurcations for $F < 0$ and the phenomenon fits well into the theoretical description given in section 2.3. Although the periodic orbit that emerges from the supercritical Hopf bifurcation is the stable attractor for $F$ close to the bifurcation value — see section 3.3.4 — the cascade-like pitchfork bifurcations can have a big influence on the dynamics via the large number of generated equilibria. Such an influence has been observed in dimension $n = 4$ for positive $F$, where four unstable and $\gamma_4$-conjugate equilibria give rise to a heteroclinic structure that causes the dynamics on the chaotic attractor to return to nearly periodic behaviour repeatedly, i.e. the classical type 1 intermittency scenario — see section 5.1.1.