Dynamics of the Lorenz-96 model
van Kekem, Dirk Leendert

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INTRODUCTION

“The relevance of mathematically defined systems cannot be too strongly emphasized; much of what we know, or believe that we know, about real systems has come from the study of models.”
— Lorenz (2006a)

In this thesis we study a dynamical system designed by Edward Lorenz, known as the Lorenz-96 model. Initially, Lorenz introduced the model to gain a better understanding of the predictability of the atmosphere as part of his life-long study of weather forecasting and atmospheric predictability. Therefore, we devote a substantial part of this introductory chapter to the work and life of Lorenz. We give a short historical sketch of Lorenz’s pioneering work on chaos and weather, climate and predictability, starting with his first works till the end of his life. His oeuvre shows the power of constructing models that are further simplifications of existing models of reality. We will discuss his three most important models, one of which the model that lies on the basis of this thesis. Thus, the Lorenz-96 model will be put in the context of his other contributions to the field and connected with the other models he developed.
1.1 PREDICTABILITY: LORENZ’S VOYAGE

1.1.1 Chaos (un)recognised

POINCARÉ The discovery of chaos in a system of ordinary differential equations (ODEs) is often attributed to Edward Lorenz. But even before Lorenz was born, Poincaré had already observed chaotic behaviour when he studied the three-body problem in AD 1880. At that time the significance of this phenomenon was not fully understood and therefore Poincaré’s observations did not lead to any significant breakthrough. Neither did his comments about the weather lead meteorology into a different direction, when he noted that (Poincaré, 1912):

[Even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that very small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible and we have the fortuitous phenomenon.

Similar statements about the irregular and unpredictable weather are made in the years after Poincaré. For example, the mathematician Norbert Wiener pointed out that the weather could not be treated as a deterministic process, since unobserved parts of the weather could be of importance for longer periods (Wiener, 1956). One year later, Philip Thompson published a paper in which he acknowledged that the uncertainty of the initial state would disturb the predictability of future states, though he considered it to be possible to eliminate the error growth almost entirely by increasing the density of observations (Thompson, 1957).

LORENZ It took more than 80 years after Poincaré’s discovery before the notion of a chaotic system was recognised by Edward Lorenz coincidentally, while studying a small model for a hydrodynamical flow. He revealed his findings in the seminal article
predictability: Lorenz’s voyage

Where others did not recognise the phenomenon that prevented them from solving their problems, Lorenz realised that he needed chaos in order to understand the irregularity of the atmosphere (Lorenz, 1993). It is his merit that a whole new field came to existence: Chaos theory. Many other discoveries and studies by others contributed to the development of this area in the following decades.

The discovery of Lorenz not only opened a whole new direction of research, but also clarified the unpredictability or unsolvability of many old problems. For example, it became more and more clear why it is so difficult to forecast the weather a few days in advance correctly. In particular, Lorenz himself made it plausible that the atmosphere is a chaotic system, based on the study of mathematical models. In the next subsection we will follow him on his journey through the study of weather forecasting and the unpredictable atmosphere. To be able to study important concepts (such as predictability), he often simplified models as much as possible leaving only some essential features. This leads to a couple of elementary, conceptual models that also advanced the field of mathematics. We will discuss this in subsection 1.1.3.

1.1.2 Unpredictable atmosphere and weather forecast

Edward Norton Lorenz1 (23 May, 1917 – 16 April, 2008) started his academic career by obtaining a Master’s degree in Mathematics. Due to the Second World War he became involved in Meteorology, which influenced the course of his academic research for the rest of his life.

Weather forecasting before Lorenz

At that time, forecasting the weather was a quite subjective task: after a meteorologist has analysed the weather maps and the specific meteorological objects present therein, the forecast follows by displacing and adjusting these entities or by removing old and adding new ones. How these adjustments should be done, was based on the experience of expert-forecasters and the known physical laws, such as Buys-Ballot’s law. While such a forecast was far from perfect, it

1 To our best knowledge, there is no fully detailed or scientific biography of Edward Lorenz today, although there exist his autobiographical note (Lorenz, 1991) and short biographical memoirs by Palmer (2009) and Emmanuel (2011). We do, however, not aim to fill this gap. An incomplete overview of important life-events of Edward Lorenz can be found in table 1.1.
Table 1.1: Important events in the life of Edward Lorenz.

<table>
<thead>
<tr>
<th>YEAR</th>
<th>EVENT</th>
<th>COMMENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1917</td>
<td>Born on 23 May as Edward Norton Lorenz</td>
<td>In West Hartford, Connecticut, USA</td>
</tr>
<tr>
<td>1938</td>
<td>Bachelor’s degree Mathematics</td>
<td>Dartmouth College in Hanover, New Hampshire</td>
</tr>
<tr>
<td>1940</td>
<td>Master’s degree Mathematics (A.M.)</td>
<td>Harvard University</td>
</tr>
<tr>
<td>1943</td>
<td>Second Master’s degree in Meteorology (S.M.)</td>
<td>Massachusetts Institute of Technology (MIT)</td>
</tr>
<tr>
<td>1946</td>
<td>Employed as (assistant) meteorologist</td>
<td>MIT, till 1954</td>
</tr>
<tr>
<td>1948</td>
<td>Received his doctorate in Meteorology (Sc.D.)</td>
<td>MIT</td>
</tr>
<tr>
<td>1948</td>
<td>Married Jane Loban</td>
<td>MIT</td>
</tr>
<tr>
<td>1954</td>
<td>Appointed assistant professor in Meteorology</td>
<td>MIT</td>
</tr>
<tr>
<td>1961</td>
<td>Elected fellow of American Academy of Arts and Sciences</td>
<td>MIT</td>
</tr>
<tr>
<td>1962</td>
<td>Appointed full Professor in Meteorology</td>
<td>American Meteorological Society</td>
</tr>
<tr>
<td>1969</td>
<td>Awarded Clarence Leroy Meisinger Award and Carl Gustaf Rossby Research Medal Award</td>
<td>Lorenz (1972)</td>
</tr>
<tr>
<td>1972</td>
<td>Coined the term <em>Butterfly effect</em></td>
<td>Royal Meteorological Society</td>
</tr>
<tr>
<td>1973</td>
<td>Awarded Symons Memorial Gold Medal</td>
<td>United States</td>
</tr>
<tr>
<td>1975</td>
<td>Elected fellow of National Academy of Sciences</td>
<td>MIT, till 1981</td>
</tr>
<tr>
<td>1977</td>
<td>Head of Department</td>
<td>MIT, till 1981</td>
</tr>
<tr>
<td>1981</td>
<td>Elected member of Indian Academy of Sciences and Norwegian Academy of Science and Letters</td>
<td>Royal Swedish Academy of Sciences</td>
</tr>
<tr>
<td>1983</td>
<td>Awarded Crafoord Prize</td>
<td>Royal Swedish Academy of Sciences</td>
</tr>
<tr>
<td>1984</td>
<td>Elected honorary member of Royal Meteorological Society</td>
<td>MIT</td>
</tr>
<tr>
<td>1987</td>
<td>Retirement</td>
<td>MIT</td>
</tr>
<tr>
<td>1989</td>
<td>Awarded Elliott Cresson Medal</td>
<td>Franklin Institute</td>
</tr>
<tr>
<td>1991</td>
<td>Awarded Kyoto Prize</td>
<td>Inamori Foundation; Japanese equivalent of Nobel prize</td>
</tr>
<tr>
<td>1991</td>
<td>Awarded Roger Revelle Medal</td>
<td>American Geophysical Union</td>
</tr>
<tr>
<td>1991</td>
<td>Awarded Louis J. Battan Author’s Award</td>
<td>American Meteorological Society</td>
</tr>
<tr>
<td>2000</td>
<td>International Meteorological Organization Prize</td>
<td>World Meteorological Organization</td>
</tr>
<tr>
<td>2004</td>
<td>Awarded Buys Ballot medal</td>
<td>Royal Netherlands Academy of Arts and Sciences</td>
</tr>
<tr>
<td>2004</td>
<td>Awarded Lomonosov Gold Medal</td>
<td>Russian Academy of Sciences</td>
</tr>
<tr>
<td>2008</td>
<td>Awarded Premio Felice Pietro Chiesi e Caterina Tomassoni</td>
<td>Sapienza University of Rome</td>
</tr>
<tr>
<td>2008</td>
<td>Died on 16 April at the age of 90</td>
<td>In Cambridge, Massachusetts, USA</td>
</tr>
</tbody>
</table>
was thought generally that an accurate forecasting of the weather is possible, although very difficult due to the intricacy of the atmosphere.

Even though weather forecasting at that time was done mostly by hand, there existed dynamic equations for the atmosphere already. A failed attempt to calculate the weather was made in 1922 by the meteorologist Richardson using a modified model of Bjerknes. The breakthrough came with the introduction of the computer. In 1950, Charney developed the first model that produced moderately good weather forecasts by filtering the small scale features (Charney, et al., 1950; Lorenz, 1993; Lorenz, 1996).

**Lorenz’s contribution** At that time, Lorenz (1950) obtained models for the circulation in the weather atmosphere via a generalised vorticity equation. He realised that for a better understanding of atmospheric phenomena it is legitimate to simplify the existing models to the extent that they are realistic enough to clarify and describe qualitatively some of the important physical phenomena in the atmosphere (Lorenz, 1960). Such a procedure is used, for example, in (Lorenz, 1980) in order to gain more insight in the attractor that plays a role in numerical weather forecasting.

Besides, it also led to the construction of his famous Lorenz-63 model, a three-dimensional model based on a seven-dimensional model by Saltzman (Lorenz, 1963) — see also section 1.1.3. This system as a mathematical abstraction is one of the simplest to indicate the chaotic nature of the atmosphere. He discovered using this model that taking two slightly different initial conditions will result after some time in completely different states.\(^2\) He realised that this *sensitive dependence* also implies that the predictability of the future state of the system is limited (Lorenz, 1963):

If, then, there is any error whatever in observing the present state — and in any real system such errors seem inevitable — an acceptable prediction of an instantaneous state in the distant future may well be impossible.\(^3\)

Lorenz has investigated the irregularity of the atmosphere and its implications on the predictability of the weather in numerous following studies, e.g. (Lorenz, 1969a; Lorenz, 1975; Lorenz, 1982),

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\(^2\) Lorenz tells the story how he found this phenomenon by accident in (Lorenz, 1993).

\(^3\) Note the similarity to the statement of Poincaré, quoted earlier.
Also the construction and use of the Lorenz-84 model (Lorenz, 1984a) — discussed further in section 1.1.3 — is part of this research.

Now the weather turned out to be chaotic, the question arose to what extent the atmosphere — and hence the weather — is actually predictable in terms of time range and accuracy. Besides the chaotic nature of the atmosphere, there are two other limitations to this predictability (Lorenz, 1969b): Firstly, observations of the current state of the weather will always be imperfect and incomplete. Secondly, we are not able to formulate the real governing laws of the weather and any model of them is inevitably an approximation.

The predictability of a system can be determined by the rate at which an error in the system typically grows or decays by increasing time range (Lorenz, 2006a). Here, ‘error’ is either the difference between the assumed initial state of the system and its actual state, or the difference between the future state predicted by the model and the actual future state and refers to either one of the two limitations above. The time range at which this happens gives an indication how far in advance the model can predict without being meaningless.

The Lorenz-96 model was designed by Lorenz (2006a) particularly to study the error growth with increasing range of prediction for the atmosphere. He started with an initial error and looked at its growth in time, while assuming that the model itself is correct. The long-term average factor of the error growth coincides with the largest Lyapunov number of the system (except for the early stages). Furthermore, he introduced a multiscale model by which he showed that the small-scale features of the atmosphere act like small random forcing.

The Lorenz-96 model should not be considered as a model attempting to describe the real atmosphere, but rather as a conceptual model resembling its behaviour only to a small extent. The simplicity of the model allows to gain more insight in problems that are difficult or time-consuming to investigate with large and more realistic models, as is shown in (Lorenz & Emanuel, 1998).

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4 Although the model was presented in 1996 at an ECMWF workshop on predictability, the corresponding paper was not published until 2006, when it was included in a book on predictability in weather and climate by experts in that field (Palmer & Hagedorn, 2006).
A more detailed description of the Lorenz-96 model will follow in section 1.2.

**Current weather forecasting** Without doubt, we can say that the work and models of Lorenz has changed the fields of both meteorology and mathematics. In meteorology, he demonstrated that the forecast of the weather will always be accompanied with uncertainty even within the limits of predictability and exactness in measurements can never be attained. Therefore, modern weather forecasting models, like that of the European Centre for Medium-Range Weather Forecasts (ECMWF), are based on a probabilistic approach: they use the method of ensemble forecasting in which several predictions are made with slightly different initial conditions and slightly perturbed models (Buizza, 2006). See figure 1.2 for an example of such computations. From the set of predictions one can derive their uncertainty and the probability that a certain type of weather will occur. Such an approach was already suggested by Eady (1951) and Lorenz (1965).

Concerning mathematics, Lorenz introduced a few models that are interesting to mathematicians as well. In the following section we give a short discussion of these models to illustrate the influence of Lorenz on mathematics.

![Figure 1.2](image-url)

**Figure 1.2:** Forecast of the temperature for the Groningen (Eelde) weather station using ensemble forecasting. The computation is done by the Royal Netherlands Meteorological Institute (KNMI) at 14th of February 2018, using the weather model Ensemble Prediction System from the ECMWF with 52 runs. Source: KNMI.
1.1.3 Lorenz’s contribution to mathematics

In the study of the real world, one has to deal with representative models to gain insight in the present phenomena. Besides more natural models, one can also design artificial systems that only describe the key features (or at least those that are subject of study) to study particular problems of the reality. The preceding historical discussion of Lorenz’ work shows that simplifying models as much as possible can be very successful in studying and explaining basic concepts. The resulting models of Lorenz are also mathematically interesting, not only because their simplicity eases the mathematical investigation, but also because they have further advanced the field of nonlinear dynamical systems.

**FORCED DISSIPATIVE SYSTEMS** The fundamental models designed by Lorenz all belong to the same type of systems, namely the class of *forced dissipative systems* with quadratic nonlinear terms. This class of systems — introduced in (Lorenz, 1963) — can be described by the general \( n \)-dimensional system

\[
\dot{x}_j = \sum_{k,l=1}^{n} a_{jkl} x_k x_l - \sum_{k=1}^{n} b_{jk} x_k + c_j, \quad j = 1, \ldots, n, \tag{1.1}
\]

where the constant coefficients are chosen such that \( \sum a_{jkl} x_k x_l \) vanishes\(^5\) and \( \sum b_{jk} x_j x_k \) is positive definite (Lorenz, 1963; Lorenz, 1980; Lorenz, 1984b). The class (1.1) is constructed in such a way that the quadratic part does not affect the total energy of the system, \( E = \frac{1}{2} \sum_{j=1}^{n} x_j^2 \). This property can be used to show the existence of a trapping region in these systems. We will prove this fact later in this thesis, together with special versions for the Lorenz-96 model — see section 2.1.

A subclass of the class of systems (1.1) is defined by imposing the extra condition that \( a_{jkl} = 0 \) whenever \( k = j \) or \( l = j \), which implies that the divergence is equal to \( -\sum_j b_{jj} \). It can be shown that the attractor of systems of this subclass has zero volume (Lorenz, 1980; Lorenz, 1984b). This subclass of systems includes the famous Lorenz-63 model, as well as his well-known models from 1984 and 1996.

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5 This condition may be omitted, but then the trapping region is not a global one — see (Lorenz, 1980).
1.1 Predictability: Lorenz’s Voyage

Lorenz-63 Model The Lorenz-63 model is a three-dimensional model that simulates the convective motion of a fluid between two parallel infinite horizontal plates in two dimensions, where the lower plated is heated and the upper one is cooled. The model is derived from a system of partial differential equations (PDEs) describing Rayleigh-Bénard convection (Hilborn, 2000; Broer & Takens, 2011). Its equations are given by

\[
\begin{align*}
\dot{x} &= -\sigma x + \sigma y, \\
\dot{y} &= -xz + rx - y, \\
\dot{z} &= xy - bz,
\end{align*}
\]

where \(x\) is proportional to the intensity of the convective motion, \(y\) to the temperature difference between the rising and falling parts of the fluid and \(z\) to the deviation of the temperature profile from its equilibrium, which increases linearly with height (Lorenz, 1963).

After the Lorenz-63 model became known to mathematicians in the seventies it gained a huge interest and — besides the new insights in chaotic dynamics — inspired new research in dynamical systems and other fields. Soon, there appeared studies into the dynamics of the model (Sparrow, 1982), its attractor and a geometric version of the Lorenz model (Afraimovich, et al., 1977; Guckenheimer & Williams, 1979; Williams, 1979). Meanwhile, Hénon (1976) constructed a simple two-dimensional mapping that has the same essential properties as the Lorenz-63 model and contains a strange attractor as well.

The proof that the attractor of the Lorenz-63 model actually exists lasted for quite some time and the final proof gave rise to new advanced techniques (Tucker, 1999; Viana, 2000), such as the use of a computer to establish certain features of the geometry of the solutions. Likewise, for the computation of the two-dimensional stable manifold of the origin of system (1.2) algorithms to compute global manifolds in vector fields had to be developed (Krauskopf & Osinga, 2003; Krauskopf, et al., 2005), as well as methods to visualise its intriguing geometry (Osinga & Krauskopf, 2002).
LORENZ-84 MODEL  The Lorenz-84 model is another simple three-dimensional model describing the long-term atmospheric circulation at midlatitude. It is given by the following equations:

\[
\begin{align*}
\dot{x} &= -y^2 - z^2 - ax + aF, \\
\dot{y} &= xy - bxz - y + G, \\
\dot{z} &= bxy + xz - z,
\end{align*}
\]

where \( x \) denotes the intensity of the symmetric globe-encircling westerly wind current and \( y \) and \( z \) denote the cosine and sine phases of a chain of superposed waves transporting heat poleward (Lorenz, 1984a). A derivation of the model as a reduction of a Galerkin approximation of a quasi-geostrophic two-layer PDE model is given by Van Veen (2003).

The Lorenz-84 model also attracted the attention of mathematicians, who studied its dynamics thoroughly (Masoller, et al., 1995; Sicardi Schifino & Masoller, 1996; Shilnikov, et al., 1995; Van Veen, 2003). The model also inspired further research: for example, in (Broer, et al., 2002; Broer, et al., 2005b) a periodic forcing was added to the model (1.3), which led to the discovery of quasi-periodic Hénon-like attractors, a new class of strange attractors (Broer, et al., 2008a; Broer, et al., 2010). The model has been used for various other applications as well, such as the study of the interaction of the atmosphere with the ocean by combining the Lorenz-84 model with a box model into a slow-fast system (Van Veen, et al., 2001).

LORENZ-96 MODEL  This thesis concentrates on another model by Lorenz, namely, his 1996 model, which also belongs to the class of forced dissipative systems with quadratic nonlinear terms and can be considered as one of the simplest of them. Already in 1984 he studied a four-dimensional version of the model in his search for the simplest nontrivial system (1.1) that still contains the basic properties of the subclass and capable of exhibiting chaotic behaviour (Lorenz, 1984b). By imposing symmetry conditions on the equations, he came up with the monoscale version of the \( n \)-dimensional Lorenz-96 model. In the next section we will define
this model and discuss it in more detail, together with applications in other fields.

1.2 LORENZ-96 MODEL

Actually, Lorenz designed two variants of the Lorenz-96 model: a monoscale version, with only one time scale, and a multiscale version, in which two different time scales are obtained by coupling two suitably scaled versions of the monoscale model. This thesis is devoted solely to the monoscale version. For the multiscale model — which will not be discussed in this thesis — the reader is referred to (Lorenz, 2006a) or to appendix A.1.

1.2.1 The monoscale Lorenz-96 model

The equations of the monoscale Lorenz-96 model are equivariant with respect to a cyclic permutation of the variables. Therefore, the system with dimension \( n \in \mathbb{N} \) is completely determined by the equation for the \( j \)-th variable, which is given by

\[
\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) - x_j + F, \quad j = 1, \ldots, n, \tag{1.4a}
\]

where we take the indices modulo \( n \) by the following ‘boundary condition’

\[
x_{j-n} = x_{j+n} = x_j, \tag{1.4b}
\]

resulting in a model with circulant symmetry. Note that both the dimension \( n \in \mathbb{N} \) and the forcing parameter \( F \in \mathbb{R} \) are free parameters.

Here, the variables \( x_j \) can be interpreted as values of some atmospheric quantity (e.g., temperature, pressure or vorticity) measured along a circle of constant latitude of the earth (Lorenz, 2006a). The latitude circle is divided into \( n \) equal sectors, with a distinct variable \( x_j \) for each sector such that the index \( j = 1, \ldots, n \) indicates the longitude — see figure 1.3. In this way, the model (1.4) can be interpreted as a model that describes waves in the atmosphere. Lorenz observed that for \( F > 0 \) sufficiently large the waves in the model slowly propagate “westward”, i.e. in the direction of de-
Figure 1.3: Example of a latitude circle of the earth, divided into \( n \) equal sized sectors.

creasing \( j \) (Lorenz, 2006a). Figure 1.4 illustrates travelling waves for dimension \( n = 24 \) and parameter values \( F \) in the periodic and the chaotic regime.

However, the Lorenz 96 model is not designed to be realistic. Indeed, as we described in section 1.1.2, the aim for Lorenz to introduce his so-called Lorenz-96 model was to study fundamental issues regarding the predictability of the atmosphere and weather forecasting (Lorenz, 2006a). For this reason, he did not aim to design a complicated and physically realistic model of the atmosphere, but just a simple test model that is easy to use in numerical experiments. Or, as Lorenz & Emanuel (1998) wrote:

We know of no way that the model can be produced by truncating a more comprehensive set of meteorological equations. We have merely formulated it as one of the simplest possible systems that treats all variables alike and shares certain properties with many atmospheric models.

Indeed, the following physical mechanisms are present in system (1.4):
1.2 Lorenz-96 model

Figure 1.4: Hovmöller diagrams of a periodic attractor (left, $F = 2.75$) and a chaotic attractor (right, $F = 3.85$) in the Lorenz-96 model for $n = 24$. The value of $x_j(t)$ is plotted as a function of $t$ and $j$. For visualisation purposes linear interpolation between $x_j$ and $x_{j+1}$ has been applied in order to make the diagram continuous in the variable $j$.

1. Advection, that conserves the total energy, simulated by the quadratic terms;

2. Damping, through which the energy decreases, is represented by the linear terms;

3. External forcing keeps the total energy away from zero and is described by the constant terms.

The traditional Lorenz-63 model (1.2) — which does have a clear physical interpretation — has two disadvantages. Firstly, it consists of only three ordinary differential equations. Secondly, for the classical parameter values the model has Lyapunov spectrum $(0.91, 0, -14.57)$, which makes the model very dissipative. Such properties are not typical for atmospheric models. In contrast with the Lorenz-63 model, the dimension of the Lorenz-96 model can be chosen arbitrarily large and for suitable values of the parameters it has multiple positive Lyapunov exponents, which is similar to models obtained from discretising PDEs.

The value of the Lorenz-96 model lies primarily in the fact that it has a very simple implementation in numerical codes while at the same time it can exhibit very complex dynamics for suitable choices of the parameters $n$ and $F$. The properties of the Lorenz-96 model make the model attractive and very useful for various
other applications. It is sometimes even called “a hallmark representative of nonlinear dynamical behavior” (Frank, et al., 2014).

1.2.2 Applications

The applications of the Lorenz-96 model are broad and range from data assimilation and predictability (Ott, et al., 2004; Trevisan & Palatella, 2011; De Leeuw, et al., 2017) to studies in spatiotemporal chaos (Pazó, et al., 2008). It is often used to test new ideas in various fields. For example, Lorenz himself used his own model to study the atmosphere and related problems, (Lorenz & Emanuel, 1998; Lorenz, 2006a; Lorenz, 2006b).

In (Sterk & Van Kekem, 2017), we use the Lorenz-96 model to study the distribution of finite-time growth rates of errors in initial conditions along the attractor of the system. To illustrate a method for quantifying the predictability of a certain specified event, we study the predictability of extreme amplitudes of travelling waves in the Lorenz-96 model. It turns out that the predictability of extremes depends on the dynamical regime of the model.

Table 1.2 gives an overview of recent papers in which the Lorenz-96 model has been used together with the values of the parameters that were used. In most studies the dimension $n$ is chosen ad hoc, but $n = 36$ and $n = 40$ appear to be popular choices. Many applications are related to geophysical problems, but the model has also attracted the attention of mathematicians working in the area of dynamical systems for phenomenological studies in high-dimensional chaos.

1.2.3 Setting of the problem

The Lorenz-96 model (1.4) is one of the simplest systems of the class of systems (1.1) showing chaotic behaviour and has been studied by many researchers. In contrast to its importance, only a few studies have investigated the dynamics of this model. Table 1.3 lists a selection of papers that investigate part of the dynamics of the Lorenz-96 model. The papers by Lorenz present a few basic
Table 1.2: Recent papers with applications of the monoscale ($^a$) Lorenz-96 model (1.4) or the multiscale ($^b$) Lorenz-96 model (A.1) and the main values of $n$ and $F$ that were used. Almost all values are chosen in the chaotic domain ($F = 8$) of dimension $n = 36$ or 40.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Application</th>
<th>$n$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basnarkov &amp; Kocarev (2012)</td>
<td>Forecast improvement</td>
<td>960$^a$</td>
<td>15</td>
</tr>
<tr>
<td>Boffetta, et al. (2002)</td>
<td>Predictability</td>
<td>36$^b$</td>
<td>10</td>
</tr>
<tr>
<td>Crommelin &amp; Vanden-Eijnden (2008)</td>
<td>Subgrid scale parameterisation</td>
<td>18$^b$</td>
<td>10</td>
</tr>
<tr>
<td>Danforth &amp; Kalnay (2008)</td>
<td>State-dependent model errors</td>
<td>8$^b$</td>
<td>8, 14, 18</td>
</tr>
<tr>
<td>Danforth &amp; Yorke (2006)</td>
<td>Making forecasts</td>
<td>40$^a$</td>
<td>8</td>
</tr>
<tr>
<td>Dieci, et al. (2011)</td>
<td>Approximating Lyapunov exponents</td>
<td>40$^a$</td>
<td>8</td>
</tr>
<tr>
<td>Gallavotti &amp; Lucarini (2014)</td>
<td>Non-equilibrium ensembles</td>
<td>32$^a$</td>
<td>≥ 8</td>
</tr>
<tr>
<td>Hallerberg, et al. (2010)</td>
<td>Bred vectors</td>
<td>1024$^a$</td>
<td>8 (6, … , 20)</td>
</tr>
<tr>
<td>Hansen &amp; Smith (2000)</td>
<td>Operational constraints</td>
<td>40$^a$</td>
<td>8</td>
</tr>
<tr>
<td>Haven, et al. (2005)</td>
<td>Predictability</td>
<td>40$^a$</td>
<td>8</td>
</tr>
<tr>
<td>De Leeuw et al. (2017)</td>
<td>Data assimilation</td>
<td>36$^a$</td>
<td>8</td>
</tr>
<tr>
<td>Lieb-Lappen &amp; Danforth (2012)</td>
<td>Shadowing time</td>
<td>4, 5, 6$^b$</td>
<td>14</td>
</tr>
<tr>
<td>Lorenz (1984a)</td>
<td>Chaotic attractor</td>
<td>4$^a$</td>
<td>−100</td>
</tr>
<tr>
<td>Lorenz &amp; Emanuel (1998)</td>
<td>Optimal sites</td>
<td>40$^a$</td>
<td>8</td>
</tr>
<tr>
<td>Lorenz (2005)</td>
<td>Designing chaotic models</td>
<td>30$^a$</td>
<td>10 (2.5, … , 40)</td>
</tr>
<tr>
<td>Lorenz (2006a)</td>
<td>Predictability</td>
<td>36$^a$</td>
<td>8 (15, 18)</td>
</tr>
<tr>
<td>Lorenz (2006b)</td>
<td>Regimes in simple systems</td>
<td>21$^a$</td>
<td>5.1</td>
</tr>
<tr>
<td>Lucarini &amp; Sarno (2011)</td>
<td>Ruelle linear response theory</td>
<td>40$^a$</td>
<td>8</td>
</tr>
<tr>
<td>Orrell, et al. (2001)</td>
<td>Model error</td>
<td>8$^{a,b}$</td>
<td>10</td>
</tr>
<tr>
<td>Orrell (2002)</td>
<td>Metric in forecast error growth</td>
<td>8$^{a,b}$</td>
<td>10</td>
</tr>
<tr>
<td>Orrell (2003)</td>
<td>Model error and predictability</td>
<td>8$^b$</td>
<td>10</td>
</tr>
<tr>
<td>Ott et al. (2004)</td>
<td>Data assimilation</td>
<td>40, 80, 120$^a$</td>
<td>8</td>
</tr>
<tr>
<td>Roulston &amp; Smith (2003)</td>
<td>Combining ensembles</td>
<td>8$^b$</td>
<td>8</td>
</tr>
<tr>
<td>Stappers &amp; Barkmeijer (2012)</td>
<td>Adjoint modelling</td>
<td>40$^a$</td>
<td>8</td>
</tr>
<tr>
<td>Sterk, et al. (2012)</td>
<td>Predictability of extremes</td>
<td>36$^a$</td>
<td>8</td>
</tr>
<tr>
<td>Sterk &amp; van Kekem (2017)</td>
<td>Predictability of extremes</td>
<td>4, 7, 24$^a$</td>
<td>11.85, 4.4, 3.85</td>
</tr>
<tr>
<td>Trevisan &amp; Palatella (2011)</td>
<td>Data assimilation</td>
<td>40, 60, 80$^a$</td>
<td>8</td>
</tr>
<tr>
<td>Vannitsem &amp; Toth (2002)</td>
<td>Model errors</td>
<td>36$^b$</td>
<td>10</td>
</tr>
<tr>
<td>Verkley &amp; Severijns (2014)</td>
<td>Maximum entropy principle</td>
<td>36$^a$</td>
<td>2.5, 5, 10, 20</td>
</tr>
<tr>
<td>Wilks (2005)</td>
<td>Stochastic parameterisation</td>
<td>8$^b$</td>
<td>18, 20</td>
</tr>
</tbody>
</table>
dynamical properties of the system (Lorenz, 1984b; Lorenz, 2005). Bifurcation diagrams in low dimensions of the Lorenz-96 model have been studied in (Orrell & Smith, 2003), although the emphasis of their work was on methods to visualise bifurcations by means of spectral analysis, rather than exploring the dynamics itself. Karimi & Paul (2010) explored the high-dimensional chaotic dynamics by means of the fractal dimension. A recent study on patterns of order and chaos in the multiscale model by Frank et al. (2014) has reported the existence of regions with standing waves.

Table 1.3: Overview of the research into the dynamics of the monoscale Lorenz-96 model (1.4) and the main values of the parameter $F$ that were used. In most cases, only the range for positive $F$ has been analysed.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Subject</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lorenz (1984a)</td>
<td>Chaotic attractor</td>
<td>−100</td>
</tr>
<tr>
<td>Orrell &amp; Smith (2003)</td>
<td>Spectral bifurcation diagram</td>
<td>[0, 17]</td>
</tr>
<tr>
<td>Lorenz (2005)</td>
<td>Designing chaotic models</td>
<td>2.5, 5, 10, 20, 40</td>
</tr>
<tr>
<td>Pazó et al. (2008)</td>
<td>Lyapunov vectors</td>
<td>8</td>
</tr>
<tr>
<td>Van Kekem &amp; Sterk (2018a)</td>
<td>Symmetries &amp; bifurcations</td>
<td>(−9, 0)</td>
</tr>
<tr>
<td>Van Kekem &amp; Sterk (2018b)</td>
<td>Travelling waves &amp; bifurcations</td>
<td>[0, 13]</td>
</tr>
<tr>
<td>Van Kekem &amp; Sterk (2018c)</td>
<td>Wave propagation</td>
<td>(−4, 4)</td>
</tr>
</tbody>
</table>

The works above already revealed an extraordinarily rich structure of the dynamical behaviour of the Lorenz-96 model for specific values of $n$. However, there has been no systematic study of its dynamics yet. We aim to fill this gap in this thesis. The main question that will be studied is the following:

**Research question.** *How do the quantitative and qualitative features of the dynamics of the Lorenz-96 model (1.4) depend on $n \in \mathbb{N}$?*

System (1.4) is in fact a family of dynamical systems parameterised by the discrete parameter $n \in \mathbb{N}$ that gives the dimension of its state space. A coherent overview of the dependence of spatio-temporal properties on the parameters $n$ and $F$ is useful to assess
1.3 Overview of this Thesis

The research question of this thesis will be answered by studying the dynamical nature of the Lorenz-96 model in greater detail, both analytically and numerically. Particular attention will be paid to properties that stabilise in the limit \( n \to \infty \). In the following, we present an overview of the main results of this work.
1.3.1 Main results

An important equilibrium solution of the Lorenz-96 model is the equilibrium \( x_F = (F, \ldots, F) \), which exists for all \( n \geq 1 \) and \( F \in \mathbb{R} \). This equilibrium plays a key role in the dynamics of the Lorenz-96 model and therefore we will study it extensively. In the analysis we take advantage of the symmetry of the system.

**Symmetries and Invariant Manifolds** In section 1.1.3 it is described that the Lorenz-96 model is constructed by imposing a symmetry condition on the equations. We will show that for any \( n \in \mathbb{N} \) the \( n \)-dimensional model is equivariant with respect to a cyclic left shift — i.e. the model possesses \( \mathbb{Z}_n \)-symmetry. An important question is then how the symmetry influences the dynamics of the model.

First of all, from the theory of equivariant dynamical systems it is known that equivariance gives rise to invariant linear subspaces (Golubitsky, et al., 1988). These invariant subspaces turn out to be particularly useful in our research, since they enable us to generalise results that are proven for a low dimension to all multiples of that dimension. We investigate the properties of these invariant manifolds and show how they can be utilised.

Secondly, due to the symmetry of the model we are able to prove its dynamical properties to quite some extent using analytical methods only. This includes proofs of the bifurcations that destabilise the only stable equilibrium \( x_F \) around \( F = 0 \) for all dimensions \( n \). It turns out that the bifurcation structure is different for positive and negative values of \( F \).

**Dynamics for \( F > 0 \)** For positive \( F \), we prove that the equilibrium \( x_F \) exhibits several Hopf or Hopf-Hopf bifurcations for all \( n \geq 4 \). In case of a Hopf bifurcation, it is also possible to show whether the bifurcation is sub- or supercritical. In particular, the first Hopf bifurcation is always supercritical, which implies the birth of a stable periodic attractor.

The Hopf and Hopf-Hopf bifurcation are not induced by symmetry. However, it turns out that the generated periodic orbits
are symmetric if their wave number has a common divisor with the dimension. Here, the wave number should be interpreted as the spatial frequency of the wave, which measures the number of ‘highs’ or ‘lows’ on the latitude circle — see for example chapter 4 and (Lorenz & Emanuel, 1998).

It turns out that these periodic orbits have the physical interpretation of a travelling wave. To illustrate, a travelling wave in dimension $n = 6$ is shown in the right panel of figure 1.5 by means of a so-called Hovmöller diagram (Hovmöller, 1949). In such diagrams the value of the variables $x_j(t)$ is plotted as a function of time $t$ and “longitude” $j$. In this thesis we also study how the spatiotemporal properties of waves in system (1.4) — such as their period, wave number and symmetry — depend on the dimension $n$ and whether these properties tend to a finite limit as $n \to \infty$. For instance, figure 1.6 shows that the wave number of the periodic attractor increases linearly with the dimension $n$ and is, consequently, unbounded.

![Figure 1.5](image)

**Figure 1.5:** As figure 1.4, but with a stationary wave (left, $F = -3.6$) and a travelling wave (right, $F = 1.2$) for $n = 6$.

**Hopf-Hopf Bifurcation: Organising Centre** To unfold the codimension two Hopf-Hopf bifurcation we add an extra parameter $G$ to the original model (1.4) via a Laplace-like diffusion term in such a way that the original model is easily retrieved by setting $G = 0$. The thus obtained two-parameter system clarifies the role of the Hopf-Hopf bifurcation as organising centre and so it sheds more light on the original model. Especially for $n = 12$,
Figure 1.6: The wave number of the periodic attractor of the Lorenz-96 model after the first Hopf bifurcation as a function of the dimension $n$. Note that the wave number increases linearly with $n$.

the Hopf-Hopf bifurcation is the first bifurcation for $F > 0$ in the original model and gives rise to two coexisting stable periodic orbits by the two subcritical Neimark-Sacker curves that emanate from the Hopf-Hopf point — see figure 1.7.

The amount of Hopf-Hopf bifurcations in the two-parameter model scales quadratically with the dimension. For larger values of $n$ these bifurcations are closer to the $F$-axis in the $(F,G)$-plane, which means that these points are likely to affect the dynamics of the original Lorenz-96 model for $G = 0$. Actually, two or more nearby Hopf-Hopf points cause bifurcation scenarios by which two or more stable waves with different spatiotemporal properties coexist for the same values of the parameters $n$ and $F$. We will demonstrate that such a phenomenon is indeed typical for the Lorenz-96 model.

**Dynamics for $F < 0$** For negative $F$, the dynamics is very much influenced by the symmetry of the model. As a result, the bifurcation structure depends on the dimension $n$. In all *odd* dimensions, the first bifurcation of the equilibrium $x_F$ is a supercritical Hopf bifurcation. The periodic orbit that results can be interpreted again as a *travelling wave*. 
Figure 1.7: Local bifurcation diagram for a Hopf-Hopf bifurcation unfolded by the two parameters \(F\) and \(G\). The Hopf-Hopf bifurcation point occurs due to the intersection of the two Hopf-lines \(H_2\) and \(H_3\) and act as an organising centre for the dynamics. From this codimension two point two subcritical Neimark-Sacker bifurcation curves \(NS_2\) and \(NS_3\) emanate. In the region between these Neimark-Sacker curves two stable periodic attractors with different wave numbers can coexist. Compare with figures 3.1 and 5.14 for the case \(n = 12\).

In even dimensions, \(\mathbb{Z}_2\)-symmetry causes the occurrence of a pitchfork bifurcation for the equilibrium \(x_F\). The resulting stable equilibria can exhibit a pitchfork bifurcation again. However, a supercritical Hopf bifurcation occurs after at most two pitchfork bifurcations and destabilises all present stable equilibria, resulting in two or four coexisting periodic orbits. In contrast with the previous cases, the periodic orbits in the even dimensions have the physical interpretation of a stationary wave — see the left panel of figure 1.5 for an example of a stationary wave for \(n = 6\). In a recent paper by Frank et al. (2014) stationary waves have also been discovered in specific regions of the multiscale Lorenz-96 model. Their paper uses dynamical indicators such as the Lyapunov dimension to identify the parameter regimes with stationary waves.

There can be even more pitchfork bifurcations, which however occur after the equilibria undergo the Hopf bifurcations. We will formulate a conjecture about the number of subsequent pitchfork bifurcations that occur in a given dimension \(n\). An example of a bifurcation structure with three pitchfork bifurcations is given by the schematic bifurcation diagram in figure 1.8.
Figure 1.8: Schematic bifurcation diagram of an $n$-dimensional Lorenz-96 model for negative $F$ with 3 subsequent pitchfork bifurcations. Each time only one branch out of the pitchfork bifurcation is followed — for a full picture, see figure 3.6. The label $PF_l$, $1 \leq l \leq q$, denotes the $l$-th (supercritical) pitchfork bifurcation with corresponding bifurcation value $F_{P,l}$; $H$ stands for a (supercritical) Hopf bifurcation with bifurcation value $F_{P,3} < F''_{H} < F_{P,2}$. A solid line represents a stable equilibrium; a dashed line represents an unstable one. For full diagrams for all possible cases, we refer to section 3.3.4.

**Numerical results** The analytical study is complemented by numerical explorations focusing on the dynamics beyond the first bifurcation. We use numerical tools, such as the continuation packages AUTO (Doedel & Oldeman, 2012) and MatCont (Dhooge, et al., 2011). Along various routes the periodic attractors can bifurcate into chaotic attractors representing irregular waves which ‘inherit’ their spatiotemporal properties from the periodic attractor. For example, the wave shown in the left panel of figure 1.4 bifurcates into a 3-torus attractor which breaks down and gives rise to the wave in the right panel. Note that both waves have the same wave number. Figure 1.9 shows power spectra of these waves, and clearly their dominant peaks are located at roughly the same period. Inheritance of spatiotemporal properties is also observed by Sterk, et al. (2010) in a shallow water model in which
a Hopf bifurcation (related to baroclinic instability) explains the observed time scales of atmospheric low-frequency variability.

Figure 1.10 displays the bifurcations of the stable orbit for increasing $F$ and small dimensions $n$, starting with the equilibrium $x_F$ at $F = 0$. Although the Hopf or Hopf-Hopf bifurcation persists for all $n \geq 4$, the subsequent bifurcation patterns vary with the dimension $n$. Nevertheless, patterns can be observed due to symmetry, which induces attractors with similar spatiotemporal properties in higher dimensions. A clear example is observed for dimensions $n = 5m$ with $m \leq 20$.

![Figure 1.9: Power spectra of the attractors of figure 1.4. Note that the maximum spectral power (indicated by a dot) is attained at nearly the same period.](image)

1.3.2 Outline

The remainder of this thesis is structured as follows: We will start our research of the Lorenz-96 model in chapter 2 with the description of its basic properties, its symmetries and the corresponding invariant manifolds. The analytical part of the bifurcation analysis for both $F > 0$ and $F < 0$ is presented in chapter 3. Here, the emphasis is put on the bifurcation sequence that lead to the existence of one or more stable periodic attractors. In chapter 4 we study the waves — which represent these periodic attractors — and their spatiotemporal properties. By numerical analysis,
we show in chapter 5 how these periodic attractors further develop into chaotic attractors through various bifurcation scenarios, though without clear pattern for increasing dimension. Finally, chapter 6 concludes with an academic summary and the main contributions of this thesis and discusses some open problems for future research.

For readability purposes, the long proofs are gathered in appendix B. Appendix A contains additional matter regarding the multiscale Lorenz-96 model and some concepts of the theory of equivariant dynamical systems that could be helpful for reading this thesis.

By the present investigation we will better understand the dynamics of the Lorenz-96 model. The results above already show
that the spatiotemporal properties and the bifurcation patterns of
the Lorenz-96 model strongly depend on \( n \) and are influenced by
the symmetries of the model. This again shows the importance of
selecting an appropriate value of \( n \) in specific applications. Not-
ably, despite the Lorenz-96 model is sometimes interpreted as a
discretised pde (Basnarkov & Kocarev, 2012; Reich & Cotter, 2015),
the linear increase of the wave number with \( n \) indicates that the
Lorenz-96 model cannot be interpreted as such.

Publications The results presented in this thesis are to a
large extent based on the following publications:

Kekem, D.L. van & Sterk, A.E. (2018a), ‘Symmetries in the Lorenz-
96 model’, International Journal of Bifurcations and Chaos (accep-

bifurcations in the Lorenz-96 model’, Physica D 367, pp. 38–60,

the Lorenz-96 model’, Nonlinear Processes in Geophysics 25 (2),

Sterk, A.E. & Kekem, D.L. van (2017), ‘Predictability of
extreme waves in the Lorenz-96 model near intermit-
tency and quasi-periodicity’, Complexity, pp. 9419024:1–14,
ANALYSIS OF THE LORENZ-96 MODEL