Dynamics of phase oscillators with generalized frequency-weighted coupling

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Heterogeneous coupling patterns among interacting elements are ubiquitous in real systems ranging from physics, chemistry to biology communities, which have attracted much attention during recent years. In this paper, we extend the Kuramoto model by considering a particular heterogeneous coupling scheme in an ensemble of phase oscillators, where each oscillator pair interacts with different coupling strength that is weighted by a general function of the natural frequency. The Kuramoto theory for the transition to synchronization can be explicitly generalized, such as the expression for the critical coupling strength. Also, a self-consistency approach is developed to predict the stationary states in the thermodynamic limit. Moreover, Landau damping effects are further revealed by means of linear stability analysis and resonance poles theory below the critical threshold, which turns to be far more generic. Our theoretical analysis and numerical results are consistent with each other, which can help us understand the synchronization transition in general networks with heterogeneous couplings.

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I. INTRODUCTION

The emergence of spontaneous synchronization in a population of interacting elements is one important issue in the nonlinear dynamics and complex networks. Investigating the intrinsic microscopic mechanism of such phenomena provides insights for understanding the collective behaviors in a wide variety of fields, such as the flashing of fireflies, circadian rhythms, electrochemical and spin-toque oscillators, applause formation in a large audience, the power grids, and then again in other real systems [1–4]. Mathematically, the most successful model for studying the synchronization problem was introduced by Kuramoto [5], which stands for the classical Kuramoto model. The Kuramoto theory for the transition to synchronization can be explicitly generalized, such as the expression for the critical coupling strength. Also, a self-consistency approach is developed to predict the stationary states in the thermodynamic limit. Moreover, Landau damping effects are further revealed by means of linear stability analysis and resonance poles theory below the critical threshold, which turns to be far more generic. Our theoretical analysis and numerical results are consistent with each other.

The Kuramoto model describes the evolution of an ensemble of coupled phase oscillators by means of a set of time differential equations,

\[ \dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, N, \]  \hspace{1cm} (1)

where \( \theta_i(t) \) is instantaneous phase of the \( i \)th oscillator, \( \omega_i \) is its natural frequency usually extracted from a certain probability density function \( g(\omega) \), and \( K > 0 \) is the global coupling strength. To characterize the degree of synchrony of system (1), it is useful to define the order parameter,

\[ z(t) = r(t)e^{i\Psi(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)}, \]  \hspace{1cm} (2)

where \( z(t) \) is a complex valued vector on the complex plane, \( r(t) \) is the collective amplitude, \( \Psi(t) \) is its phase. It has been shown that with the increase of the coupling strength \( K \), the system may experience a transition from the incoherent state, in which \( r = 0 \), and the oscillators evolve almost according to their natural frequency, into the coherent state, in which \( r > 0 \), and a number of oscillators became synchronized, sharing a common effective frequency \( \Omega \). As already well-known results [8], Kuramoto showed that in the limit \( N \to \infty \), when the natural frequency distribution function \( g(\omega) \) is unimodal and symmetric with respect to its center \( \bar{\omega} \), a continuous second-order phase transition occurs at the critical coupling strength,

\[ K_c = \frac{2}{\pi g(\bar{\omega})}, \]  \hspace{1cm} (3)

and the collective amplitude \( r \) satisfies the 1/2 scale-law

\[ r \propto \sqrt{K - K_c}, \]  \hspace{1cm} (4)

close to the critical point \( K_c \). With further increasing \( K \) above \( K_c \), oscillators with natural frequencies sufficiently close to the collective frequency will become entrained by the synchronized group, forming a macroscopic oscillating cluster, and the size of the cluster becomes larger and larger. Eventually, when \( K \) is large enough, all oscillators coincide with each other and \( r \) approaches to 1, implying a complete synchrony of the system.

A typical heterogeneity of the classical Kuramoto model lies in the natural frequencies of oscillators, while the dispersion of frequencies competes with the attractive coupling \( K \), in a way that a phase transition to synchronization takes place when the coupling strength is strong enough, or the frequency distribution \( g(\omega) \) is sufficiently narrow. A straightforward extension of the current model is to add a
new component of heterogeneity into the coupling strength, which is a natural attribute in the realistic systems [9–14]. For example, when the phase oscillators are set on complex networks, the network properties strongly impact the route to synchronization, moreover, this structural heterogeneity is equivalent to the coupling heterogeneity in a mean-field form under suitable approximation [15,16]. In particular, recent studies of the traveling wave state, and the π-state in an ensemble of coupled phase oscillators is essentially a special case of a heterogeneous coupling pattern, where the interacting strength among elements has only two kinds of possible choices, either $K$ or $-K$ with different proportion coefficients [17,18]. Also it should be noted that most of the previous works are concerned with the topology of the coupling scheme, instead the role of oscillators themselves on synchrony, for it has been widely known that the interaction between two individuals would be influenced by the characteristics of their own. For example, a power grid network can be described as a network of Kuramoto oscillators, where the weighted coupling coefficient between two oscillators is related to their natural frequencies [19]. In social networks, different individuals may have different strengths of response to the same influence/stimuli, e.g., rumors from friends. If we roughly describe the human dynamics by Kuramoto phase oscillators, a weighted factor in terms of the natural frequencies can be either positive or negative according to the sign of the interaction to oscillator $i$. Here $Krf(\omega_i)$ can be interpreted as an effective coupling function, in contrast to the previous studies of the frequency-weight coupling, where $f(\omega_i) = \omega_i^b$ or $|\omega_i|$ [23–28]. Equation (5) defines a heterogeneous interaction pattern underlying the system, apart from the global coupling $K > 0$, and the coupling strength is weighted by the frequency $\omega_i$ of the oscillators, which is a reasonable consideration in the role of realistic systems. The definition Eq. (2) allows us to rewrite Eq. (5) in the mean-field form, which yields

$$\dot{\theta}_i = \omega_i + Krf(\omega_i) \sin(\Psi - \theta_i).$$

Here $Krf(\omega_i)$ can be interpreted as an effective coupling strength, Eq. (6) reflects that the interaction to oscillator $i$ is equivalent to a phase $\Psi$, and weighted by the effective coupling $Krf(\omega_i)$. It should be noted that the effective coupling here can be either positive or negative according to the sign of the interaction to oscillator $i$. If $f(\omega_i) > 0$, the coupling to the mean-field is attractive, and synchronization is triggered by these oscillators, whereas $f(\omega_i) < 0$, the interaction to the mean-field turns out to be repulsive, as a result, synchronization between these oscillators is suppressed.

Since we are interested in the steady states of the system, the self-consistencies method turns out to be effective. In the long time limit, we assume Eq. (5) approaches to a stationary state, where the collective amplitude $r$ is time independent and the mean-field phase $\Psi$ rotates uniformly with a frequency, i.e., $\dot{\Psi}(t) = \Omega r + \Psi_0$, after an appropriate phase shift, we can set $\Psi_0 = 0$. By introducing the phase difference

$$\psi_i = \theta_i - \Psi,$$

Eq. (6) can be transformed into

$$\dot{\psi}_i = \omega_i - \Omega - Krf(\omega_i) \sin(\psi_i).$$

In the rotating frame, the evolution of each oscillator can be thought of as resulting from the interaction with the mean-field which is generated by the ensemble. In the thermodynamic limit $N \to \infty$, a density function $\rho(\varphi,\omega,t)$ in the $(\varphi,t)$ space, with dependence on the parameter $\omega$ is needed, where $\rho(\varphi,t,\omega)d\omega$ gives the fraction of oscillators with natural frequency $\omega$, and the phase deviation lying between $\varphi$ and $\varphi + d\varphi$ at time $t$ with the normalization condition,

$$\int_0^{2\pi} \rho(\varphi,\omega,t) d\varphi = 1,$$
and $2\pi$-periodic in $\varphi$. Consequently, Eq. (8) is equivalent to the following continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \varphi}(\rho \cdot (\omega - \Omega - K \cdot r f(\omega) \sin \varphi)) = 0. \quad (10)$$

The stationary solutions of Eq. (10) $\partial \rho/\partial t = 0$ should be discussed in two distinct cases, respectively,

$$\rho(\varphi, \omega) = \begin{cases} \delta(\varphi - \arcsin(\omega / Krf(\omega))), & |\omega - \Omega| \leq |Krf(\omega)|, \\ \sqrt{(\omega - \Omega)^2 - (Krf(\omega))^2} / 2\pi |\omega - \Omega - Krf(\omega) \sin \varphi|, & \text{otherwise}. \end{cases} \quad (11)$$

The first expression in Eq. (11) corresponds to the phase-locked oscillators, and is therefore a fixed point of Eq. (8), which stands for the time independent asymptotic phase deviation of an entrained phase oscillator of natural frequency $\omega$ and with respect to the synchronization frequency $\Omega$.

Considering a weak perturbation away from the fixed point of Eq. (8) given by $\varphi_i = (\omega_i - \Omega) / Krf(\omega_i)$, we obtain the linear perturbation equation about $\delta \varphi_i$, i.e.,

$$\delta \varphi_i = -Krf(\omega_i) \cos \varphi_i \delta \varphi_i - Krf(\omega_i) \sin \varphi_i \sum_{j=1}^{N} \frac{\partial r}{\partial \varphi_j} \delta \varphi_j. \quad (12)$$

According to the definition above, the amplitude of the order parameter $z$ is

$$r = \frac{1}{N} \sum_{k=1}^{N} e^{i\varphi_k} - \frac{1}{N} \sum_{i=1}^{N} e^{i\varphi_i}, \quad (13)$$

and

$$\frac{\partial r}{\partial \varphi_j} = \frac{1}{N} \sum_{k \neq j}^{N} e^{i\varphi_k} \delta_{jk} - \frac{i e^{i\varphi_j}}{N}. \quad (14)$$

In particular, we choose a specific perturbation direction for every trajectory $\varphi_i$, i.e., we set $\delta \varphi_i = 0$ for $i = 1, 2, \ldots, N$ and $j \neq i$. In this case, the second term in Eq. (12) tends to 0 in the limit $N \rightarrow \infty$. Since $K > 0$ and the $r > 0$, the stability of the fixed point of Eq. (8) is determined by the sign of $f(\omega) \cos \varphi_i$. As a consequence, the value of $\cos \varphi_i$ of the stable fixed point of Eq. (8) should take the following form:

$$\cos \varphi_i = \text{sgn}(f(\omega_i)) \sqrt{1 - \left(\frac{\omega_i - \Omega}{Krf(\omega_i)}\right)^2}, \quad (15)$$

where $\text{sgn}(x)$ is the sign function, $\text{sgn}(x) = 1$ when $x \geq 0$ and $\text{sgn}(x) = -1$ when $x < 0$. The second term of Eq. (11) represents the drifting oscillators, where these oscillators could not be entrained by the mean-field Eq. (8). Equation (11) relates the stationary density function with the natural frequency $\omega$ and $f(\omega)$, accordingly, the order parameter defined in Eq. (2) takes the integral form:

$$r = \int_{-\infty}^{\infty} \int_{0}^{2\pi} g(\omega) \rho(\varphi, \omega) e^{i\varphi} d\varphi d\omega. \quad (16)$$

It is easy to obtain that for the drifting oscillators, the integral

$$\int_{0}^{2\pi} \cos \varphi \sqrt{\frac{(\omega - \Omega)^2 - (Krf(\omega))^2}{2\pi |\omega - \Omega - Krf(\omega) \sin \varphi|}} d\varphi = 0 \quad (17)$$

and

$$\int_{0}^{2\pi} \sin \varphi \sqrt{\frac{(\omega - \Omega)^2 - (Krf(\omega))^2}{2\pi |\omega - \Omega - Krf(\omega) \sin \varphi|}} d\varphi \not= 0. \quad (18)$$

Taking into account the contribution of both the phase-locked and the drifting oscillators to the order parameter. By inserting Eq. (11) into Eq. (16) and separating the real and imaginary parts, one can obtain the self-consistency equations,

$$r = \int_{-\infty}^{\infty} d\omega g(\omega) \text{sgn}(f(\omega)) \sqrt{1 - \left(\frac{\omega - \Omega}{Krf(\omega)}\right)^2} \times \Theta\left[1 - \frac{\omega - \Omega}{Krf(\omega)}\right]. \quad (19)$$

for the collective amplitude $r$, and

$$0 = \int_{-\infty}^{\infty} d\omega g(\omega) \frac{\omega - \Omega}{Krf(\omega)} - \int_{-\infty}^{\infty} d\omega g(\omega) \frac{\omega - \Omega}{Krf(\omega)} \times \sqrt{1 - \left(\frac{Krf(\omega)}{\omega - \Omega}\right)^2} \Theta\left[1 - \frac{\omega - \Omega}{Krf(\omega)}\right] - 1. \quad (20)$$

for the mean-field frequency $\Omega$, and here, $\Theta(\chi)$ is the Heaviside function.

Equation (19) together with Eq. (20) provide closed equations for the dependence of the magnitude $r$ and the frequency $\Omega$ of the mean-field on $K$, in terms of the distribution $g(\omega)$ and weighted function $f(\omega)$. Theoretically, an explicit expression for $r(K)$ and $\Omega(K)$ can be solved analytically or numerically for given $g(\omega)$ and $f(\omega)$. However, a full analysis of the solution for an arbitrary form of $g(\omega)$, and any possible dependence on the weighted function $f(\omega)$ is very difficult. Correspondingly, we focus on our attentions to the representative and generic properties of the self-consistency equations, and we look for the critical point with the onset of a non-vanishing mean-field. Hence, in the limit case $r \rightarrow 0$, by taking into account the Taylor expansion of Eqs. (19) and (20), one obtains the critical coupling strength $K_c$ as

$$K_c = \text{sgn}(f(\Omega_c)) \frac{2}{\pi g(\Omega_c) f(\Omega_c)}, \quad (21)$$

and the critical mean-field frequency $\Omega_c$ satisfies the balance equation

$$P \cdot \int_{-\infty}^{\infty} \frac{g(\omega) f(\omega)}{\omega - \Omega_c} d\omega = 0. \quad (22)$$

The symbol $P$ means the principal-value integral along the real $\omega$.

The concise expression Eq. (21) is significant, which can be regarded as a straightforward extension of the classical Kuramoto model Eq. (3), together with the critical frequency $\Omega_c$ determined by the balance equation Eq. (22). For the
previous studies of the Kuramoto model with heterogeneous coupling, the weighted coupling is uncorrelated with the natural frequency, and therefore the second integral in Eq. 20 vanishes, consequently, the mean-field frequency \( \Omega \) is always equal to the center \( \hat{\omega} \), provided that \( g(\omega) \) is one-humped and \( g(\hat{\omega} + x) = g(\hat{\omega} - x) \). For the current model, however, there is no guarantee that \( \Omega \) is necessarily equal to \( \hat{\omega} \) due to the rotational symmetry-breaking of the system [29,30]. Actually, \( \Omega \) plays a crucial role in determining the coupling strength, since \( K_\epsilon \) is inversely proportional to \( g(\Omega_c) \) and \( f(\Omega_c) \).

III. RELAXATION DYNAMICS OF THE INCOHERENT STATE

A. Linear stability analysis

The above analysis reveals that all the stationary states for the system as well as the stationary collective dynamical properties of the synchronization can be characterized in the frame of mean-field theory, the collective amplitude \( r \) and the group velocity \( \Omega \) can be formally solved for general weighted function through the self-consistency equations. However, a thorough stability of all the possible steady states is still elusive. In the following, we pay our particular attention to the stability of this steady state, we can consider the evolution of a weak perturbation away from the incoherent state. By excluding second- and higher-order terms of \( \delta z_n \), we obtain a set of linear equations for \( \delta z_n \) as

\[
\frac{d\delta z_1}{dt} = (i\omega + K/2 f(\omega))\delta z_1 = \hat{T} \cdot \delta z_1
\]

(29)

\[
\frac{d\delta z_n}{dt} = n\omega \delta z_n, \quad n > 1.
\]

Here, \( \hat{T} \) is a linear operator defined as

\[
\hat{T} g(\omega) = \int_{-\infty}^{\infty} g(\omega) g(\omega) d\omega = (g(\omega), P_0),
\]

(30)

where \( g(\omega) \) is a function in the weighted-Lebesgue space, \( P_0 \equiv 1 \), and \(( , )\) is the concise inner product notation for the integral over \( \omega \) with respect to the weighted factor \( g(\omega) \). Since higher Fourier harmonic components have no contribution to the order parameter \( z(t) \) Eq. (28), we only focus on the evolution of \( \delta z_1(t, \omega) \). Let \( \lambda \) be the eigenvalue of linear operator \( \hat{T} \), we have

\[
\frac{d\delta z_1}{dt} = \hat{T} \cdot \delta z_1 = \lambda \delta z_1.
\]

(31)

Substituting the expression of \( \hat{T} \) into Eq. (31), and multiplying both sides by the inverse operator \((\lambda - i\omega)^{-1}\), we obtain

\[
\delta z_1 = (\lambda - i\omega)^{-1} K/2 f(\omega)(\delta z_1, P_0).
\]

(32)

Taking the inner product with \( P_0 \) for both sides, then the self-consistent eigenvalue equation for the linear operator \( \hat{T} \) takes the form

\[
\int_{-\infty}^{\infty} f(\omega) g(\omega) d\omega = \frac{2}{K}, \quad \lambda \in C \setminus i\omega,
\]

(33)

where \( \lambda \) is on the complex plane except for those points \( i\omega \). Notice that Eq. (33) relates implicitly the global coupling strength \( K \) with the eigenvalue \( \lambda \). To simplify the discussion, we rewrite Eq. (33) into two equations by setting \( \lambda = x + iy \), i.e.,

\[
\int_{-\infty}^{\infty} \frac{x}{x^2 + (\omega - y)^2} f(\omega) g(\omega) d\omega = \frac{2}{K}
\]

(34)

and

\[
\int_{-\infty}^{\infty} \frac{\omega - y}{x^2 + (\omega - y)^2} f(\omega) g(\omega) d\omega = 0.
\]

(35)

The sign of \( x \) determines the stability of the incoherent state, furthermore, it has been proven that [32,33], if the global coupling strength \( K > 0 \) and is sufficiently small, the eigenvalue \( \lambda \) of the linear operator \( \hat{T} \) actually does not exist, provided that the product \( f(\omega) g(\omega) \) is analytic and has no singularity on the real axis \( \omega \). Accordingly, the incoherent state is only neutrally stable to perturbations in the regime \( K < K_c \), where the operator \( \hat{T} \) has only continuous spectrum \( i\omega \) on the whole imaginary axis. However, as \( K \) further increases, the discrete eigenvalues emerge with real part \( x \neq 0 \) once \( K > K_c \). Imposing the critical condition \( x \to 0^\pm \), \( y \to y_j \) for
where $\omega_i$ is the eigenvalue of the operator $\hat{\Omega}$ to the foremost critical point for the stability change of the model, when we release the restriction about $\omega_i > 0$ for any $\omega_i$, as a result, in this situation, the system can never approach a stationary partially synchronized state $r > 0$. Accordingly, synchronization is inhibited and the only possible stationary state of the system is the incoherent state $r = 0$ [18,28,43].

### B. Landau damping effects

According to the above linear stability analysis, the incoherent state of the system Eq. (5) is only neutrally stable when $K < K_c$, however, as an already well-known result [31], Strogatz et al. showed that the order parameter $r(t)$ in this regime actually decays to zero in the long time limit ($t \to \infty$) in an exponential form in the classical Kuramoto model. In this subsection, we show that such an effect is far more generic even in the system where phase oscillators are coupled to the mean-field weighted by their natural frequencies. Addressing this question is nontrivial, since it has already been shown that the relaxation to the equilibrium is related to susceptibility of the system to external stimulations in statistical physics [16,34–36]. In particular, the decaying mechanism is remarkably similar to the famous Landau damping in plasma physics [36,37]. To this end, we report a general framework to determine the decay exponent, together with extensive numerical simulations that support the theoretical predictions.

According to Eq. (31), a solution of $\delta z_1(t, \omega)$ with an initial value $\delta z_1(0, \omega)$ is given by

$$\delta z_1(t, \omega) = e^{\hat{r} \cdot t} \cdot \delta z_1(0, \omega),$$

where the operator $e^{\hat{r} \cdot t}$ is calculated by means of the Laplace inversion formula, which yields

$$e^{\hat{r} \cdot t} = \lim_{\gamma \to \infty} \frac{1}{2\pi i} \int_{-\gamma-i\gamma}^{+\gamma+i\gamma} e^{s \cdot t} (s - \hat{r})^{-1} ds,$$
for $t > 0$ and $x > 0$. The resolvent $(s - \hat{T})^{-1}$ is obtained as follows: let $\phi(t)$ be an arbitrary function in the weighted Lebesgue space and note

$$\hat{R}(s)\phi = (s - \hat{T})^{-1}\phi = \left(s - i\omega - \frac{K}{2}f(\omega)\hat{P}\right)^{-1}\phi.$$ \hfill (39)

by multiplying $(s - i\omega - \frac{K}{2}f(\omega)\hat{P})$ to both sides, we have

$$(s - i\omega)\hat{R}(s)\phi = \phi + \frac{K}{2}f(\omega)\hat{P}\hat{R}(s)\phi = \phi + \frac{K}{2}f(\omega)(\hat{R}(s)\phi, P_0).$$ \hfill (40)

This is rearranged as

$$\hat{R}(s)\phi = (s - i\omega)^{-1}\phi + \frac{K}{2}(\hat{R}(s)\phi, P_0)(s - i\omega)^{-1}f(\omega),$$ \hfill (41)

taking the inner product with $P_0$, we obtain

$$(\hat{R}(s)\phi, P_0) = ((s - i\omega)^{-1}\phi, P_0)$$

$$+ \frac{K}{2}(\hat{R}(s)\phi, P_0)(s - i\omega)^{-1}f(\omega), P_0).$$ \hfill (42)

Reordering of the term then yields

$$(\hat{R}(s)\phi, P_0) = \frac{((s - i\omega)^{-1}\phi, P_0)}{1 - \frac{K}{2}((s - i\omega)^{-1}f(\omega), P_0)}.$$ \hfill (43)

Substituting Eq. (43) into Eq. (40) leads to

$$\hat{R}(s)\phi = (s - i\omega)^{-1}\phi$$

$$+ \frac{K}{2}\left((s - i\omega)^{-1}\phi, P_0\right)(s - i\omega)^{-1}f(\omega), P_0).$$ \hfill (44)

Then, the order parameter $\delta z(t)$ reads

$$\delta z(t) = (\delta z(t, 0), P_0) = (e^{\hat{T}t}\delta z(0, 0), P_0)$$

$$= \lim_{y \to \infty} \int_{-iy}^{+iy} e^{s't}D(s)\phi ds,$$ \hfill (45)

where

$$D(s) = \int_{-\infty}^{\infty} \frac{g(\omega)\delta z(0, 0)}{s - i\omega} d\omega,$$ \hfill (46)

and

$$D'(s) = \int_{-\infty}^{\infty} \frac{g(\omega)f(\omega)}{s - i\omega} d\omega.$$ \hfill (47)

To invert the Laplace transform Eq. (45), we have to find the poles of $D(s)/(1 - KD'(s)/2)$, and the calculation should be analytically continued to the left half-complex plane where $\text{Re}(s) < 0$. Figure 1 presents a series of results from the numerical resolution of Eq. (5). In fact, from Eq. (45) when $K \to 0$, $\delta z(t) = \int_{-\infty}^{\infty} e^{s't}g(\omega)d\omega$, which is the Fourier transform of $g(\omega)$, that is, the oscillators rotate at angular frequencies given by their own natural frequencies. In terms of the Riemann-Lebesgue lemma, we obtain $\delta z(t) \to 0$ in the limit $t \to \infty$ as shown in Fig. 1(a). When the function $g(\omega)f(\omega)$ is a rational case, such as $g(\omega)$ is Lorentzian and $f(\omega) = \omega$, the resonance poles could be solved analytically Fig. 1(b). We have also conducted direct numerical simulations for other typical cases, where the inverse Laplace transform could not be calculated analytically, and the resonance poles could be calculated numerically, as shown in Fig. 1(c) and (d). It is found that the envelope of the order parameter follows the form of exponential decay approximately for the short time [28,36]. The above results suggest that the Landau damping effect is a generic phenomenon which is entirely due to the occurrence of resonance poles caused by analytic continuation, and the real parts of them control the exponential relaxation rate of the order parameter $\delta r(t)$ [31,33,36].

Recently, the Ott-Antonsen method has been proposed to obtain the low-dimensional dynamics of a large system of coupled oscillators [39,40], where the original set of differential equations can be reduced to the differential equations describing the temporal evolution of the order parameter $\delta z(t)$ alone, which makes it possible to depict the system in a global picture. However, the validity of this method needs several prerequisites. First, the first-order Fourier coefficient of the density function $\hat{F}(\theta, \omega, t)$ can be analytically continued from the real $\omega$ into the complex $\omega$-plane such that the continuation has no singularities. Second, to avoid divergence of the density function, the evolution of $\delta z(t, \omega)$ must satisfy $|\delta z(t, \omega)| \leq 1$ at any time in the invariant manifold. In the Appendix, we include the Ott-Antonsen method to study the relaxation dynamics of
both the incoherent state and the coherent state [15], provided that the above restrictions can be satisfied. It turns out that such a method is consistent with our theoretical formulations. For a more general case, where the preconditions are violated, the Ott-Antonsen method fails to treat properly the system, and hence it does not provide more substantial information than the traditional analysis here. Further investigation of the system utilizes the amplitude equation theory, which reveals the local bifurcation behaviors close to the critical point when the system satisfies the $O(2)$ group symmetry [41,42].

IV. CONCLUSION

To summarize, we extend the traditional Kuramoto model for the synchronization transition in an infinite large ensembles of globally coupled phase oscillators to the heterogeneously interacting scheme, where the mean-field coupling is weighted by their natural frequency characterized by a general function $f(\omega_i)$. Theoretically, the mean-field analysis, the linear stability analysis, the resonance poles method, and the Ott-Antonsen reduction have been carried out to obtain insights. Together with the numerical simulations, our study presented the following main results. First, we established the self-consistency equations [Eqs. (19) and (20)] in the main text. Substituting Eq. (A6) into Eq. (A5) yields the self-consistency equations [Eqs. (19) and (20)] in the main text.

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APPENDIX: THE OTT-ANTONSEN METHOD FOR THE RELAXATION DYNAMICS

In this appendix, we provide the Ott-Antonsen method that allows to describe the generalized frequency-weighted Kuramoto model in the low-dimensional invariant manifold. Following this method, the density function $F(\theta,t,\omega)$ is sought in the form

$$F(\theta,t,\omega) = \frac{g(\omega)}{2\pi}(1 + F_+ + F_-),$$

(A1)

where

$$F_+ = \sum_{n=1}^{\infty} F_n(\omega,t)e^{in\theta},$$

(A2)

and $F_- = F_+^*$, with the additional ansatz

$$F_n = a^n(\omega,t).$$

(A3)

Substituting the ansatz into the continuity equation, one obtains the equation for $a(\omega,t)$

$$\frac{d\alpha}{dt} + i\omega \alpha + \frac{1}{2}(z \cdot \alpha^2 - z^*) = 0$$

(A4)

with the order parameter,

$$z(t) = \int g(\omega)a^*(\omega,t)d\omega.$$  

(A5)

Similar to the analysis in the mean-field theory, we look for a steady state $z(t) = re^{i\Omega t}$ with a constant order parameter $r$, a group velocity $\Omega$. By introducing a suitable change of the reference frame $\omega \rightarrow \omega + \Omega$ and set $\dot{\alpha} = 0$, we find $a_0(\omega)$ a solution:

$$\pm \sqrt{1 - \left(\frac{\omega}{Krf(\omega + \Omega)}\right)^2 - \frac{i\omega}{Krf(\omega + \Omega)}}, \quad |\omega| \leq Kr|f(\omega + \Omega)|,$$

$$-\frac{i\omega}{Krf(\omega + \Omega)} \left[1 - \sqrt{1 - \left(\frac{Krf(\omega + \Omega)}{\omega}\right)^2}\right], \quad \text{otherwise.}$$

(A6)

Substituting Eq. (A6) into Eq. (A5) yields the self-consistency equations [Eqs. (19) and (20)] in the main text. To investigate the relaxation dynamics, we consider a weak perturbation on the stationary state,

$$z(t) = r + \delta z(t), \quad \alpha(t) = a_0(\omega) + \delta \alpha(\omega,t),$$

(A7)

we obtain a linear equation for $\delta \alpha(\omega,t)$

$$\frac{d\delta \alpha}{dt} + i\omega \delta \alpha + \frac{1}{2}(2r\alpha_0 \delta \alpha + a_0^2 \delta z - \delta z^*) = 0,$$

(A8)
that must be solved self-consistently with

\[ \delta z(t) = \int g(\omega)\delta \alpha^*(\omega,t) d\omega. \]  

(A9)

Taking the Laplace transform of both sides of Eq. (A8), and reordering the terms then yields

\[ \delta \alpha(s,\omega) = \frac{\delta \alpha(t = 0) + \frac{Kf(\omega)}{2}(\delta \alpha^*(s) - \alpha_0^2 \delta z(s))}{s + i\omega + Kf(\omega)\alpha_0}. \]  

(A10)

Substituting Eq. (A10) into Eq. (A9) leads to

\[ \delta z(s) = \frac{B(s)}{1 - KA(s)/2}, \]  

(A11)

where

\[ B(s) = \int_{-\infty}^{\infty} g(\omega + \Omega) \frac{\delta \alpha(\omega, t = 0)}{s + i\omega + Kf(\omega + \Omega)\alpha_0} d\omega \]  

(A12)

and

\[ A(s) = \int_{-\infty}^{\infty} g(\omega + \Omega) \frac{\delta \alpha(\omega, t = 0)}{s + i\omega + Kf(\omega + \Omega)\alpha_0} d\omega, \]  

(A13)

the order parameter \( \delta z(t) \) is the inverse Laplace transform of \( \delta z(s) \). It is obvious that when the system is the incoherent state, \( \alpha_0 \equiv 0 \), and Eq. (A11) is the same as Eq. (45) in the main text.