Cigarette Smoke Up-regulates PDE3 and PDE4 to Decrease cAMP in Airway Cells

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Some Results on Exponential Synchronization of Nonlinear Systems

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Abstract—**Based on recent works on transverse exponential stability, we establish some necessary and sufficient conditions for the existence of a (locally) exponential synchronizing control law. We show that the existence of a structured synchronizer is equivalent to the existence of a stabilizer for the individual linearized systems (on the synchronization manifold) by a linear state feedback. This, in turn, is also equivalent to the existence of a symmetric covariant tensor field, which satisfies a control matrix function inequality. Based on this result, we provide the construction of such a synchronizer via backstepping approach. In some particular cases, we show how global exponential synchronization may be obtained.**

Index Terms—**Lyapunov stability, multi-agent systems, synchronization, lyapunov methods.**

I. Introduction

Controlled synchronization, as a coordinated control problem of a group of autonomous systems, has been regarded as one of the important group behaviors. It has found its relevance in many engineering applications, such as the distributed control of (mobile) robotic systems, the control and reconfiguration of devices in the context of Internet-of-Things, and the synchronization of autonomous vehicles (see, for example, [16]).

For linear systems, the solvability of this problem and, as well as, the design of controller have been thoroughly studied in literature. To name a few, we refer to the classical work on the nonlinear Goodwin oscillators [13], to the synchronization of linear systems in [23] and [25], and to the recent works in nonlinear systems [9]–[11], [21], [22]. For linear systems, the solvability of synchronization problem reduces to the solvability of stabilization of individual systems by either an output or state feedback. It has recently been established in [25] that for linear systems, the solvability of the output synchronization problem is equivalent to the existence of an internal model, which is a well-known concept in the output regulation theory.

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The generalization of these results to the nonlinear setting has appeared in the literature (see, for example, [8]–[11], [14], [15], [17], [18], [21], and [22]). In these works, the synchronization of nonlinear systems with a fixed network topology can be solved under various different sufficient conditions.

For instance, the application of passivity theory plays a key role in [8], [9], [14], [18], [21], and [22]. By using the input/output passivity property, the synchronization control law in these works can simply be given by the relative output measurement. Another approach for synchronizing nonlinear systems is by using the output regulation theory as pursued in [11], [15], and [17]. In these papers, the synchronization problem is reformulated as an output regulation problem, where the output of each system has to track an exogeneous signal driven by a common exosystem, and the resulting synchronization control law is again given by the relative output measurement. Finally, another synchronization approach that has gained interest in recent years is via incremental stability [6] or other related notions, such as convergent systems [17]. If we restrict ourselves to the class of incremental input-to-state stability (ISS), as discussed in [6], the synchronizer can again be based on the relative output/state measurement.

Despite assuming a fixed network topology, necessary and sufficient conditions for the solvability of synchronization problem of nonlinear systems is not yet established. Therefore, one of our main contributions of this paper is the characterization of controlled synchronization for general nonlinear systems with a fixed network topology. Using recent results on the transverse exponential contraction, we establish some necessary and sufficient conditions for the solvability of a (locally) exponential synchronization. It extends the work in [2] where only two interconnected systems are discussed. We show that a necessary condition for achieving synchronization is the existence of a symmetric covariant tensor field of order two whose Lie derivative has to satisfy a control matrix function (CMF) inequality, which is similar to the control Lyapunov function and detailed later in Section III.

This paper extends our preliminary work presented in [4]. In particular, we improve some results by relaxing some conditions (see the necessary condition section). Additionally, we present the backstepping approach that allows us to construct a CMF-based synchronizer as well as the extension of the local synchronization result to the global one for a specific case. Note that all proofs are given in the long version of this paper in [5].

This paper is organized as follows. We present the problem formulation of synchronization in Section II. In Section III, we present our first main results on necessary conditions to the solvability of the synchronization problem. Some sufficient conditions for local or global synchronization are given in Section IV. A constructive synchronizer design is presented in Section V, where a backstepping procedure is given for designing a CMF-based synchronizing control law.

Notation. The vector of all ones with a dimension N is denoted by $\mathbb{1}_N$. We denote the identity matrix of dimension n by I_n or I when no confusion is possible. Given $M_1, \dots, M_N$ square matrices,

$\mathtt{diag}\{M_1,\dots,M_N\}$ is the matrix defined as

$$\mathtt{diag}\{M_1,\dots,M_N\} = \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_N \end{bmatrix}.$$

Given a vector field f on $\mathbb{R}^n$ and a covariant two tensor $P : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, P is said to have a derivative along f denoted $\mathfrak{d}_f P$ if the following limit exists:

$$\mathfrak{d}_f P(z) \;=\; \lim_{h\to 0} \frac{P(Z(z,h)) - P(z)}{h} \tag{1}$$

where $Z(z,\cdot)$ is the flow of the vector field f with an initial state z in $\mathbb{R}^n$. In that case and when $m = n$ and f is C^1, $L_f P$ is the Lie derivative of the tensor along f, which is defined as

$$L_f P(z) = \mathfrak{d}_f P(z) + P(z)\frac{\partial f}{\partial z}(z) + \frac{\partial f}{\partial z}(z)^\top P(z)\,. \tag{2}$$

II. Problem Definition

A. System Description and Communication Topology

In this note, we consider the problem of synchronizing N identical nonlinear systems with $N \geq 2$. For every $i = 1,\dots,N$, the ith system Σ_i is described by

$$\dot{x}_i = f(x_i) + g(x_i)u_i\ ,\ i = 1,\dots,N \tag{3}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^p$, and the functions f and g are assumed to be C^2. In this setting, all the systems have the same drift vector field f and the same control vector field $g : \mathbb{R}^n \to \mathbb{R}^{n\times p}$, but not the same controls in $\mathbb{R}^p$. For simplicity of notation, we denote the complete state variables by $x = \begin{bmatrix} x_1^\top \dots x_N^\top \end{bmatrix}^\top$ in $\mathbb{R}^{Nn}$.

The synchronization manifold $\mathcal{D}$, where the state variables of different systems agree with each other, is defined by

$$\mathcal{D} = \{(x_1,\dots,x_N) \in \mathbb{R}^{Nn} \mid x_1 = x_2 = \cdots = x_N\}.$$

For every x in $\mathbb{R}^{Nn}$, we denote the Euclidean distance to the set $\mathcal{D}$ by $|x|_{\mathcal{D}}$.

The communication graph $\mathcal{G}$, which is used for synchronizing the state through distributed control u_i, $i = 1,\dots,N$, is assumed to be an undirected graph and is defined by $\mathcal{G} = (\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is the set of N nodes (where the ith node is associated to the system Σ_i) and $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is a set of M edges that define the pairs of communicating systems. Moreover, we assume that the graph $\mathcal{G}$ is connected.

Let us, for every edge k in $\mathcal{G}$ connecting node i to node j, label one end (e.g., the node i) by a positive sign and the other end (e.g., the node j) by a negative sign. The incidence matrix D that corresponds to $\mathcal{G}$ is an $N\times M$ matrix such that

$$d_{i,k} = \begin{cases} +1 & \text{if node } i \text{ is the positive end of edge } k \\ -1 & \text{if node } i \text{ is the negative end of edge } k \\ 0 & \text{otherwise.} \end{cases}$$

Using D, the Laplacian matrix L can be given by $L = DD^\top$ whose kernel, by the connectedness of $\mathcal{G}$, is spanned by $\mathbb{1}_N$.

B. Synchronization Problem Formulation

Using the description of the interconnected systems via $\mathcal{G}$, the state synchronization control problem is defined as follows.

Definition 1: The control laws $u_i = \phi_i(x)$, $i = 1\dots,N$ solve the *local uniform exponential synchronization* problem for (3) if the following conditions hold:

1) for all noncommunicating pair (i,j) (i.e., $(i,j) \notin \mathcal{E}$)

$$\frac{\partial\phi_i}{\partial x_j}(x) = \frac{\partial\phi_j}{\partial x_i}(x) = 0 \qquad \forall x \in \mathbb{R}^{Nn}$$

2) for all $x \in \mathcal{D}$, $\phi(x) = 0$ (i.e., ϕ is zero on $\mathcal{D}$); and
3) the manifold $\mathcal{D}$ of the closed-loop system

$$\dot{x}_i = f(x_i) + g(x_i)\phi_i(x),\ \ i = 1,\dots,N \tag{4}$$

is uniformly exponentially stable, i.e., there exist positive constants r, k, and $\lambda > 0$ such that for all x in $\mathbb{R}^{Nn}$ satisfying $|x|_{\mathcal{D}} < r$

$$|X(x,t)|_{\mathcal{D}} \leq k\exp(-\lambda t)\,|x|_{\mathcal{D}} \tag{5}$$

where $X(x,t)$ denotes the solution initialiged from x, holds for all t in the time domain of existence of solution.

When $r = \infty$, it is called the *global uniform exponential synchronization* problem. △

In this definition, the condition 1) implies that the solution u_i is a distributed control law that requires only a local state measurement from its neighbors in the graph $\mathcal{G}$.

An important feature of our study is that we focus on exponential stabilization of the synchronizing manifold. This allows us to rely on the study developed in [2] (or [3]), in which an infinitesimal characterization of exponential stability of a transverse manifold is given. As it will be shown in the following section, this allows us to formalize some necessary and sufficient conditions in terms of matrix functions ensuring the existence of a synchronizing control law.

III. Necessary Conditions

A. Infinitesimal Stabilizability Conditions

In [2], a first attempt has been made to give necessary conditions for the existence of an exponentially synchronizing control law for only two agents. In [3], the same problem has been addressed for N agents but without any communication constraints (all agents can communicate with all others). In both cases, it is shown that assuming some bounds on derivatives of the vector fields and assuming that the synchronizing control law is invariant by permutation of agents, the following two properties are necessary conditions.

IS *Infinitesimal Stabilizability:* The couple (f,g) is such that the n-dimensional manifold $\{\tilde{z} = 0\}$ of the transversally linear system

$$\dot{\tilde{z}} = \frac{\partial f}{\partial z}(z)\tilde{z} + g(z)\tilde{u} \tag{6a}$$

$$\dot{z} = f(z) \tag{6b}$$

with $\tilde{z}$ in $\mathbb{R}^n$ and z in $\mathbb{R}^n$ is stabilizable by a state feedback that is linear in $\tilde{z}$ (i.e., $\tilde{u} = h(z)\tilde{z}$ for some function $h : \mathbb{R}^n \to \mathbb{R}^{p\times n}$).

CMF *Control Matrix Function:* For all positive definite matrices $Q \in \mathbb{R}^{n\times n}$, there exist a continuous function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, whose values are symmetric positive definite matrices and strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that

$$\underline{p}\,\mathrm{I}_n \leq P(z) \leq \overline{p}\,\mathrm{I}_n \tag{7}$$

holds for all $z \in \mathbb{R}^n$, and the inequality (see (1) and (2))

$$v^\top L_f P(z) v \leq -v^\top Q v \tag{8}$$

holds for all (v,z) in $\mathbb{R}^n\times\mathbb{R}^n$ satisfying $v^\top P(z) g(z) = 0$.

An important feature of properties and CMF comes from the fact that they are properties of each individual agent, independent of the network topology. The first one is a local stabilizability property. The second one establishes that there exists a symmetric covariant tensor field of order two denoted by P whose Lie derivative satisfies a certain inequality in some specific directions. This type of condition can be related to the notion of control Lyapunov function, which is a characterization of stabilizability as studied by Artstein in [7] or Sontag in [24]. This property can be regarded as an Artstein-like condition. The dual of the CMF property has been thoroughly studied in [19] when dealing with an observer design (see [19, eq. (8)]; see also [1] or [2]).

B. Necessity of IS and CMF for Exponential Synchronization

We show that properties and CMF are still necessary conditions if one considers a network of agents with a communication graph $\mathcal{G}$ as given in Section II-A. Hence, as this is already the case for linear systems, we recover the paradigm, which establishes that a necessary condition for synchronization is a stabilizability property for each individual agent.

Theorem 1: Consider the interconnected systems in (3) with the communication graph $\mathcal{G}$ and assume that there exists a control law $u = \phi(x)$ where $\phi(x) = \left[\, \phi_1^\top(x) \;\ldots\; \phi_N^\top(x) \,\right]^\top$ in $\mathbb{R}^{Np}$ that solves the local uniform exponential synchronization for (3). Assume, moreover, that g is bounded, f, g, and the ϕ_is have bounded first and second derivatives, and the closed-loop system is complete. Then, properties and CMF hold.

Note that this theorem is a refinement of the result that is written in [4] since we have removed an assumption related to the structure of the control law.

The proof of this result can be found in [5].

In Section IV, we discuss the possibility to design an exponential synchronizing control law based on these necessary conditions.

IV. Sufficient Condition

A. Sufficient Conditions for Local Exponential Synchronization

The interest of the Property CMF, given in Section III-A, is to use the symmetric covariant tensor P in the design of a local synchronizing control law. Indeed, following one of the main results in [3], we get the following sufficient condition for the solvability of (local) uniform exponential synchronization problem. The first assumption is that, up to a scaling factor, the control vector field g is a gradient field with P as a Riemannian metric (see also [12] for similar integrability assumption). The second one is related to the CMF property.

Theorem 2 (Local Sufficient Condition): Assume that g is bounded and that f and g have bounded first and second derivatives. Also assume that there exists a C^2 function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ whose values are symmetric positive definite matrices and with a bounded derivative that satisfies the following two conditions:

1) there exist a C^2 function $U : \mathbb{R}^n \to \mathbb{R}$ that has bounded first and second derivatives, and a C^1 function $\alpha : \mathbb{R}^n \to \mathbb{R}^p$ that has bounded first and second derivatives such that
$$\frac{\partial U}{\partial z}(z)^\top = P(z)g(z)\alpha(z) \tag{9}$$
holds for all z in $\mathbb{R}^n$; and
2) there exist a symmetric positive definite matrix Q and positive constants $\underline{p}$, $\overline{p}$, and $\rho > 0$ such that (7) holds and
$$L_f P(z) - \rho \frac{\partial U}{\partial z}(z)^\top \frac{\partial U}{\partial z}(z) \leq -Q \tag{10}$$
holds for all z in $\mathbb{R}^n$.

Then, given a connected graph $\mathcal{G}$ with associated Laplacian matrix $L = (L_{ij})$, there exists a constant $\underline{\ell}$ such that the control law $u = \phi(x)$ with $\phi = \left[\, \phi_1^\top \;\ldots\; \phi_N^\top \,\right]^\top$ given by
$$\phi_i(x) = -\ell\alpha(x_i) \sum_{j=1}^{N} L_{ij} U(x_j) \tag{11}$$
with $\ell \geq \underline{\ell}$ solves the local uniform exponential synchronization of (3).

Remark 1: Assumption (10) is stronger than the necessary condition CMF. Note, however, that employing some variation on Finsler Lemma (see [3] for instance), it can be shown that these assumptions are equivalent when x remains in a compact set.

Remark 2: Note that for all $x = \mathbb{1}_N \otimes z = (z, \ldots, z)$ in $\mathcal{D}$ and for all (i, j) with $i \neq j$
$$\frac{\partial \phi_i}{\partial x_j}(x) = -\ell\alpha(z)L_{ij}\frac{\partial U}{\partial z}(z). \tag{12}$$
Hence, for all $x = \mathbb{1}_N \otimes z$ in $\mathcal{D}$, we get
$$\frac{\partial \phi}{\partial x}(x) = -\ell L \otimes \alpha(z)\frac{\partial U}{\partial z}(z)\,. \tag{13}$$

B. Sufficient Conditions for Global Exponential Synchronization

Note that in [3] with an extra assumption related to the metric (the level sets of U are totally geodesic sets with respect to the Riemannian metric obtained from P), it is shown that global synchronization may be achieved when considering only two agents that are connected. It is still an open question to know if global synchronization may be achieved in the general nonlinear context with more than two agents. However, in the particular case, in which the matrix $P(z)$ and the vector field g are constant, global synchronization may be achieved as this is shown in the following theorem.

Theorem 3 (Global Sufficient Condition): Assume that $g(z) = G$ and there exist a symmetric positive definite matrix P in $\mathbb{R}^{n\times n}$, a symmetric positive definite matrix Q and $\rho > 0$ such that
$$P\frac{\partial f}{\partial z}(z) + \frac{\partial f}{\partial z}(z)^\top P - \rho PGG^\top P \leq -Q\,. \tag{14}$$
Assume, moreover, that the graph is connected with Laplacian matrix L. Then, there exist constants $\underline{\ell}$ and positive real numbers $c_1, \ldots, c_N$ such that the control law $u = \phi(x)$ with $\phi = \left[\, \phi_1^\top \;\ldots\; \phi_N^\top \,\right]^\top$ given by
$$\phi_i(x) = -\ell\, c_i \sum_{j=1}^{N} L_{ij} G^\top P x_j \tag{15}$$
with $\ell \geq \underline{\ell}$ solves the global uniform exponential synchronization for (3).

Proof: Let $c_j = 1$ for $j = 2, \ldots, N$. Hence, only c_1 is different from 1 and remains to be selected. Let us denote $e = (e_2, \ldots, e_N)$ with $e_i = x_1 - x_i$ and $z = x_1$. Note that for $i = 2, \ldots, N$, we have the solution to the system (3) along with u defined in (15)
$$\begin{aligned}\dot{e}_i &= f(z) - \ell\, c_1 \sum_{j=1}^{N} L_{1j} GG^\top P x_j \\ &\quad - f(z + e_i) + \ell \sum_{j=1}^{N} L_{ij} GG^\top P x_j\,.\end{aligned}$$
Note that L being a Laplacian, we have, for all i in $[1, N]$, the equality $\sum_{j=1}^{N} L_{ij} = 0$. Consequently, we can add the term $\ell c_1 \sum_{j=1}^{N} L_{1j} GG^\top P x_1$ and substract the term $\ell \sum_{j=1}^{N} L_{ij} GG^\top P x_1$

in the preceeding equation above so that for $i = 2, \ldots, N$

$$\begin{aligned}\dot{e}_i &= f(z) - \ell c_1 \sum_{j=1}^{N} L_{1j} GG^\top P(x_j - x_1) \\ &\quad - f(z + e_i) + \ell \sum_{j=1}^{N} L_{ij} GG^\top P(x_j - x_1) \\ &= f(z) - f(z + e_i) - \ell \sum_{j=2}^{N} (L_{ij} - c_1 L_{1j}) GG^\top P e_j .\end{aligned}$$

One can check that these equations can be written compactly as

$$\dot{e} = \left[\int_0^1 \Delta(z, e, s) ds + \ell \left(A(c_1) \otimes GG^\top P \right) \right] e$$

with $A(c_1)$ being a matrix in $\mathbb{R}^{(N-1)\times(N-1)}$, which depends on the parameter c_1 and is obtained from the Laplacian as

$$A(c_1) = -\left[L_{2:N,2:N} - c_1 L_{1,2:N} \mathbb{1}_{N-1} \right]$$

where

$$L = \begin{bmatrix} L_{11} & L_{1,2:N} \\ L_{1,2:N}^\top & L_{2:N,2:N} \end{bmatrix}$$

and Δ is the $(N-1)n \times n$ matrix valued function defined as

$$\Delta(z, e, s) = \mathtt{Diag} \left\{ \frac{\partial f}{\partial z}(z - s e_2), \ldots, \frac{\partial f}{\partial z}(z - s e_N) \right\} .$$

The following Lemma shows that by selecting c_1 sufficiently small, the matrix A satisfies the following property. Its proof is given in the appendix.

Lemma 1: If the communication graph is connected, then there exist sufficiently small c_1 and $\mu > 0$ such that

$$A(c_1) + A(c_1)^\top \leq -\mu I.$$

With this lemma in hand, we consider now the candidate Lyapunov function $V(e) = e^\top P_N e$, where $P_N = (I_{N-1} \otimes P)$. Note that along the solution, the time derivative of this function satisfies

$$\dot{\widehat{V(e)}} = 2e^\top P_N \left[\int_0^1 \Delta(z, e, s) ds + \ell \left(A(c_1) \otimes GG^\top P \right) \right] e.$$

Note that we have

$$\begin{aligned}&P_N \Delta(z, e, s) \\ &\quad = \mathtt{diag} \left\{ P \frac{\partial f}{\partial z}(z - s e_2), \ldots, P \frac{\partial f}{\partial z}(z - s e_N) \right\}\end{aligned}$$

and

$$\begin{aligned}&2e^\top (I_{N-1} \otimes P)(A(c_1) \otimes GG^\top P)e \\ &\qquad = e^\top ([A(c_1) + A(c_1)^\top] \otimes PGG^\top P)e \\ &\qquad \leq -e^\top (\mu I_{N-1} \otimes PGG^\top P)e .\end{aligned}$$

Hence, we get $\dot{\widehat{V(e)}} \leq \int_0^s e^\top M(e, z, s) e \, ds$, where M is the $(N - 1)n \times (N - 1)n$ matrix defined as

$$M(e, z, s) = \mathtt{diag} \left\{ M_2(e, z, s), \ldots, M_N(e, z, s) \right\}$$

with, for $i = 2, \ldots, N$

$$\begin{aligned}M_i(e, z, s) = P \frac{\partial f}{\partial z}(z - s e_i) + \frac{\partial f}{\partial z}(z - s e_i)^\top P \\ - 2\ell\mu PGG^\top P.\end{aligned}$$

Note that by taking ℓ sufficiently large, with (14), this yields $M_i(e, z, s) \leq -Q$. This immediately implies that $\dot{\widehat{V(e)}} \leq -e^\top (I_{N-1} \otimes Q) e$. This ensures exponential convergence of e to zero on the time of existence of the solution. Let $\bar{x} = \mathtt{argmin}_{z \in \mathbb{R}^n} \sum_{i=1}^{N} |z - x_i|^2$. Note that we have

$$\begin{aligned}|e|^2 &\leq 2 \sum_{i=2}^{N} |x_i - \bar{x}|^2 + 2(N-1)|\bar{x} - x_1|^2 \\ &\leq 2(N-1)|x|_{\mathcal{D}}^2 \qquad (16)\end{aligned}$$

$$\begin{aligned}|x|_{\mathcal{D}}^2 &= \min_{z \in \mathbb{R}^n} \sum_{i=1}^{N} |z - x_i|^2 \\ &\leq \sum_{i=1}^{N} |x_1 - x_i|^2 = |e|^2. \qquad (17)\end{aligned}$$

This yields global exponential synchronization of the closed-loop system. ∎

In Section V, we show that the property CMF required to design a distributed synchronizing control law can be obtained for a large class of nonlinear systems. This is done via backstepping design.

V. Construction of an Admissible Tensor via Backstepping

A. Adding Derivative (or Backstepping)

As proposed in Theorem 2, a distributed synchronizing control law can be designed using a symmetric covariant tensor field of order 2, which satisfies (8). Given a general nonlinear system, the construction of such a matrix function P may be a hard task. In [20], a construction of the function P for observer based on the integration of a Riccati equation is introduced. Similar approach could be used in our synchronization problem. Note, however, that in our context, an integrability condition [i.e., (9)] has to be satisfied by the function P. This constraint may be difficult to address when considering a Riccati equation approach.

In the following, we present a constructive design of such a matrix P that resembles the backstepping method. This approach can be related to [26] and [27], in which a metric is also constructed iteratively. We note that one of the difficulty we have here is that we need to propagate the integrability property given in (9).

For outlining the backstepping steps for designing P, we consider the case in which the vector fields (f, g) can be decomposed as follows:

$$f(z) = \begin{bmatrix} f_a(z_a) + g_a(z_a) z_b \\ f_b(z_a, z_b) \end{bmatrix}$$

and

$$g(z) = \begin{bmatrix} 0 \\ g_b(z) \end{bmatrix} , \; 0 < \underline{g}_b \leq g_b(z) \leq \overline{g}_b$$

with $z = \begin{bmatrix} z_a^\top & z_b \end{bmatrix}^\top$, z_a in $\mathbb{R}^{n_a}$, and z_b in $\mathbb{R}$. In other words

$$\dot{z}_a = f_a(z_a) + g_a(z_a) z_b , \; \dot{z}_b = f_b(z) + g_b(z) u. \qquad (18)$$

Let $\mathcal{C}_a$ be a compact subset of $\mathbb{R}^{n_a}$. As in the standard backstepping approach, we make the following assumptions on the z_a-subsystem, where z_b is treated as a control input to this subsystem.

Assumption 1 (z_a-Synchronizability): Assume that there exists a C^∞ function $P_a : \mathbb{R}^{n_a} \to \mathbb{R}^{n_a \times n_a}$ that satisfies the following conditions.

1) There exist a C^∞ function $U_a : \mathbb{R}^{n_a} \to \mathbb{R}$ and a C^∞ function $\alpha_a : \mathbb{R}^{n_a} \to \mathbb{R}$ such that

$$\frac{\partial U_a}{\partial z_a}(z_a)^\top = \alpha_a(z_a) P_a(z_a) g_a(z_a) \tag{19}$$

holds for all z_a in $\mathcal{C}_a$.

2) There exist a symmetric positive definite matrix Q_a and positive constants $\underline{p}_a$, $\overline{p}_a$, and $\rho_a > 0$ such that

$$\underline{p}_a \, \mathrm{I}_{n_a} \le P_a(z_a) \le \overline{p}_a \, \mathrm{I}_{n_a} \qquad \forall z_a \in \mathbb{R}^{n_a} \tag{20}$$

holds and

$$L_{f_a} P_a(z_a) - \rho_a \frac{\partial U_a}{\partial z_a}(z_a)^\top \frac{\partial U_a}{\partial z_a}(z_a) \le -Q_a \tag{21}$$

holds for all z_a in $\mathcal{C}_a$.

As a comparison to the standard backstepping method for stabilizing nonlinear systems in the strict-feedback form, the z_a-synchronizability conditions mentioned above are akin to the stabilizability condition of the upper subsystem via a control Lyapunov function. However, for the synchronizer design, as in the present context, we need an additional assumption to allow the recursive backstepping computation of the tensor P. Roughly speaking, we need the existence of a mapping q_a such that the metric P_a becomes invariant along the vector field $\frac{g_a}{q_a}$. In other words, $\frac{g_a}{q_a}$ is a Killing vector field.

Assumption 2: There exists a nonvanishing smooth function $q_a : \mathbb{R}^{n_a} \to \mathbb{R}$ such that the metric obtained from P_a on $\mathcal{C}_a$ is invariant along $\frac{g_a(z_a)}{q_a(z_a)}$. In other words, for all z_a in $\mathcal{C}_a$

$$L_{\frac{g_a(z_a)}{q_a(z_a)}} P_a(z_a) = 0\,. \tag{22}$$

Similar assumption can be found in [12] in the characterization of differential passivity.

Based on the Assumptions 1 and 2, we have the following theorem on the backstepping method for constructing a symmetric covariant tensor field P_b of the complete system (18).

Theorem 4: Assume that the z_a-subsystem satisfies Assumption 1 and Assumption 2 in the compact set $\mathcal{C}_a$ with a $n_a \times n_a$ symmetric covariant tensor field P_a of order two and a nonvanishing smooth mapping $q_a : \mathbb{R}^{n_a} \to \mathbb{R}$. Then, for all positive real number M_b, the system (18) with the state variables $z = (z_a, z_b) \in \mathbb{R}^{n_a+1}$ satisfies the Assumption 1 in the compact set $\mathcal{C}_a \times [-M_b, M_b] \subset \mathbb{R}^{n_a+1}$ with the symmetric covariant tensor field P_b be given by

$$P_b(z) = \begin{bmatrix} P_a(z_a) + S_a(z) S_a(z)^\top & S_a(z) q_a(z_a) \\ S_a(z)^\top q_a(z_a) & q_a(z_a)^2 \end{bmatrix}$$

where $S_a(z) = \frac{\partial q_a}{\partial z_a}(z_a)^\top z_b + \eta \alpha_a(z_a) P_a(z_a) g_a(z_a)$ and η is a positive real number. Moreover, there exists a nonvanishing mapping $q_b : \mathbb{R}^{n_a+1} \to \mathbb{R}$ such that P_b is invariant along $\frac{g}{q_b}$. In other words, Assumptions 1 and 2 hold for the complete system (18).

Remark 3: Note that with this theorem, since we propagate the required property, we are able to obtain a synchronizing control law for any triangular nonlinear system.

Proof: Let M_b be a positive real number and let $\mathcal{C}_b = \mathcal{C}_a \times [-M_b, M_b]$. Let $U_b : \mathbb{R}^{n_a+1} \to \mathbb{R}$ be the function defined by

$$U_b(z_a, z_b) = \eta U_a(z_a) + q_a(z_a) z_b$$

where η is a positive real number that will be selected later on. It follows from (19) that for all $(z_a, z_b) \in \mathcal{C}_b$, we have

$$\begin{aligned} \frac{\partial U_b}{\partial z}(z)^\top &= \begin{bmatrix} \eta \frac{\partial U_a}{\partial z_a}(z_a)^\top + \frac{\partial q_a}{\partial z_a}(z_a) z_b \\ q_a(z_a) \end{bmatrix} \\ &= \frac{1}{q_a(z_a)} P_b(z) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \alpha_b(z) P_b(z) g(z) \end{aligned}$$

with $\alpha_b(z) = \frac{1}{q_a(z_a) g_b(z)}$. Hence, the first condition in Assumption 1 is satisfied.

Consider z in $\mathcal{C}_b$ and let $v = \begin{bmatrix} v_a^\top & v_b \end{bmatrix}^\top$ in $\mathbb{R}^{n_a+1}$ be such that

$$v^\top P_b(z) g(z) = 0. \tag{23}$$

Note that this implies that

$$v_b = -v_a^\top \frac{S_a(z)}{q_a(z_a)}. \tag{24}$$

In the following, we compute the expression

$$v^\top L_f P_b(z) v = v^\top \mathfrak{d}_f P_b(z) v + 2 v^\top P_b(z) \frac{\partial f}{\partial z}(z) v\,.$$

For the first term, we have

$$\begin{aligned} v^\top \mathfrak{d}_f P_b(z) v &= v_a^\top \mathfrak{d}_{f_a} P_a(z_a) v_a + z_b v_a^\top \mathfrak{d}_{g_a} P_a(z_a) v_a \\ &\quad + v_a^\top \mathfrak{d}_f S_a(z) S_a(z)^\top v_a + 2 v_a^\top \mathfrak{d}_f S_a(z) q_a(z_a) v_b \\ &\quad + \mathfrak{d}_{f_a + g_a z_b} q_a(z_a)^2 v_b^2. \end{aligned}$$

With (24), it yields

$$\begin{aligned} &v_a^\top \mathfrak{d}_f S_a(z) S_a(z)^\top v_a + 2 v_a^\top \mathfrak{d}_f S_a(z) q_a(z_a) v_b \\ &\quad + \mathfrak{d}_{f_a + g_a z_b} q_a(z_a)^2 v_b^2 = 0. \end{aligned}$$

Hence

$$v^\top \mathfrak{d}_f P_b(z) v = v_a^\top \mathfrak{d}_{f_a} P_a(z_a) v_a + z_b v_a^\top \mathfrak{d}_{g_a} P_a(z_a) v_a\,.$$

On the other hand, for the second term, we have

$$P_b(z) = \begin{bmatrix} P_a(z_a) & 0 \\ 0 & 0 \end{bmatrix} + \frac{P_b(z) g(z) g(z)^\top P_b(z)}{(q_a(z_a) g_b(z))^2}.$$

Hence, with (23), it yields

$$\begin{aligned} v^\top P_b(z) \frac{\partial f}{\partial z}(z) v &= \begin{bmatrix} v_a^\top & -v_a^\top \frac{S_a(z)}{q_a(z_a)} \end{bmatrix} P(z) \\ &\quad \times \begin{bmatrix} \frac{\partial f_a}{\partial z_a}(z_a) + \frac{\partial g_a}{\partial z_a}(z_a) z_b & g_a(z_a) \\ \frac{\partial f_b}{\partial z_a}(z_a, z_b) & \frac{\partial f_b}{\partial z_b}(z_a, z_b) \end{bmatrix} \begin{bmatrix} v_a \\ -\frac{S_a(z)^\top}{q_a(z_a)} v_a \end{bmatrix} \\ &= v_a^\top P_a(z_a) \frac{\partial f_a}{\partial z_a}(z_a) v_a + z_b v_a^\top P_a(z_a) \frac{\partial g_a}{\partial z_a}(z_a) v_a \\ &\quad - \frac{\eta}{\alpha_a(z_a) q_a(z_a)} \left| \frac{\partial U_a}{\partial z_a}(z_a) v_a \right|^2 \\ &\quad - \frac{z_b}{q_a(z_a)} v_a^\top P_a(z_a) g(z_a) \frac{\partial q_a}{\partial z_a}(z_a). \end{aligned}$$

Hence, we get

$$\begin{aligned} v^\top L_f P_b(z) v = v_a^\top L_{f_a} P_a(z_a) v_a \\ - \frac{2\eta}{\alpha_a(z_a) q_a(z_a)} \left| \frac{\partial U_a}{\partial z_a}(z_a) v_a \right|^2 \\ + z_b v_a^\top \left[\mathfrak{d}_{g_a} P_a(z_a) + P_a(z_a) \frac{\partial g_a}{\partial z_a}(z_a) \right. \\ \left. -2 z_b v_a^\top P_a(z_a) \frac{g(z_a)}{q_a(z_a)} \frac{\partial q_a}{\partial z_a}(z_a) \right] v_a \ . \end{aligned}$$

Let η be a positive real number such that

$$\rho_a \leq \frac{2\eta}{\alpha_a(z_a) q_a(z_a)} \qquad \forall z_a \in \mathcal{C}_a \ .$$

Using (21) in Assumption 1 and (22) in Assumption 2, it follows that for all z in $\mathcal{C}_b$ and all v in $\mathbb{R}^{n_a+1}$

$$\begin{aligned} & v^\top P_b(z_a) g(z) = 0 \\ & \Rightarrow v^\top \mathfrak{d}_f P_b(z) v + 2 v^\top P_b(z) \frac{\partial f}{\partial x}(z) v \leq -v^\top Q_a v. \end{aligned}$$

Employing Finsler theorem and the fact that $\mathcal{C}_b$ is a compact set, it is possible to show that this implies the existence of a positive real number ρ_b such that for all z in $\mathcal{C}_b$

$$L_f P(z) - \rho_b \frac{\partial U_b}{\partial z}(z)^\top \frac{\partial U_b}{\partial z}(z) \leq -Q_b \tag{25}$$

where Q_b is a symmetric positive definite matrix.

To finish the proof, it remains to show that the metric is invariant along g with an appropriate control law. Note that if $q_b(z) = q_a(z_a) g_b(z)$, then it follows that this function is also nonvanishing. Moreover, we have

$$\begin{aligned} L_{\frac{g}{q_b}} P_b(z) = \mathfrak{d}_{\frac{g}{q_b}} P_b(z) - \frac{P(z)}{q_a(z_a)^2} \begin{bmatrix} 0 & 0 \\ \frac{\partial q_a}{\partial z_a}(z_a) & 0 \end{bmatrix} \\ - \begin{bmatrix} 0 & \frac{\partial q_a}{\partial z_a}(z_a)^\top \\ 0 & 0 \end{bmatrix} \frac{P(z)}{q_a(z_a)^2}. \end{aligned}$$

However, since we have

$$\mathfrak{d}_g P_b(z) = \begin{bmatrix} \frac{\partial q_a}{\partial z_a}(z_a)^\top \frac{S_a(z)}{q_a(z_a)} + \frac{S_a(z)^\top}{q_a(z_a)} \frac{\partial q_a}{\partial z_a}(z_a) & \frac{\partial q_a}{\partial z_a}(z_a)^\top \\ \frac{\partial q_a}{\partial z_a}(z_a) & 0 \end{bmatrix}$$

and

$$\begin{aligned} & \frac{P_b(z)}{q_a(z_a)^2} \begin{bmatrix} 0 & 0 \\ \frac{\partial q_a}{\partial z_a}(z_a) & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{\partial q_a}{\partial z_a}(z_a)^\top \\ 0 & 0 \end{bmatrix} \frac{P_b(z)}{q_a(z_a)^2} \\ & = \begin{bmatrix} \frac{\partial q_a}{\partial z_a}(z_a)^\top \frac{S_a(z)}{q_a(z_a)} + \frac{S_a(z)^\top}{q_a(z_a)} \frac{\partial q_a}{\partial z_a}(z_a) & \frac{\partial q_a}{\partial z_a}(z_a)^\top \\ \frac{\partial q_a}{\partial z_a}(z_a) & 0 \end{bmatrix}. \end{aligned}$$

Then, the claim holds. ■

B. Illustrative Example

As an illustrative example, consider the case in which the vector fields f and g are given by

$$f(z) = \begin{bmatrix} -z_{a1} + \sin(z_{a2})\cos(z_{a1}) + z_{a2} \\ 2 + \sin(z_{a1})] z_b \\ 0 \end{bmatrix}, \ g(z) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This system may be rewritten with $z_a = (z_{a1}, z_{a2})$ as

$$\dot{z}_a = f_a(z_a) + g_a(z_a) z_b \ , \ \dot{z}_b = u$$

with

$$f_a(z_a) = \begin{bmatrix} -z_{a1} + \sin(z_{a2})\cos(z_{a1}) + z_{a2} \\ 0 \end{bmatrix}$$

$$g_a(z_a) = \begin{bmatrix} 0 \\ 2 + \sin(z_{a1}) \end{bmatrix}.$$

Consider the matrix $P_a = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Note that if we consider $U_a(z_a) = z_{a1} + 2 z_{a2}$, then (19) is satisfied with $\alpha_a = \frac{1}{2+\sin(z_{a1})}$. Moreover, note that we have $v^\top \frac{\partial U_a}{\partial z_a}(z_a) = 0 \Leftrightarrow v_1 + 2v_2 = 0$. Moreover, we have

$$\begin{aligned} & [-2 \ 1] \, P_a \frac{\partial f_a}{\partial z_a}(z_a) \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -3 \left[-2 \frac{\partial f_{a1}}{\partial z_{a1}} + \frac{\partial f_{a1}}{\partial z_{a2}} \right] \\ & = -3 \\ & [-2(-1 + \sin(z_{a2})\sin(z_{a1})) - \cos(z_{a1})\cos(z_{a2}) + 1] \\ & = -3. \\ & [3 - \sin(z_{a2})\sin(z_{a1}) - \cos(z_{a1} - z_{a2})] \\ & \leq -3. \end{aligned}$$

The function $\frac{\partial f_a}{\partial z_a}(z_a)$ being periodic in z_{a1} and z_{a2}, we can assume that z_{a1} and z_{a2} are in a compact subset denoted $\mathcal{C}_a$. This implies employing Finsler Lemma that there exist ρ_a and Q_a such that inequality (21) holds. Consequently, the z_a subsystem satisfies Assumption 1. Finally, note that Assumption 2 is also trivially satisfied by taking $q_a(z_a) = 2 + \sin(z_{a1})$. From Theorem 4, it implies that there exist positive real numbers ρ_b and η such that with $U(z) = \eta(z_{a1} + 2z_{a2}) + \frac{z_b}{2+\sin(z_{a1})}$ with $\alpha(z) = 2 + \sin(z_{a1})$, (9) and (10) are satisfied. Hence, from Theorem 2, the control law given in (15) solves the local exponential synchronization problem for the N identical systems that exchange information via any undirected communication graph $\mathcal{G}$, which is connected.

VI. Conclusion

In this paper, based on recent results in [3], we have presented necessary and sufficient conditions for the solvability of local exponential synchronization of N identical affine nonlinear systems through a distributed control law. In particular, we have shown that the necessary condition is linked to the infinitesimal stabilizability of the individual system and is independent of the network topology. The existence of a symmetric covariant tensor of order two, as a result of the infinitesimal stabilizability, has allowed us to design a distributed synchronizing control law. When the tensor and the controlled vector field g are both constant, it is shown that global exponential synchronization may be achieved. Finally, a recursive computation of the tensor has been also discussed.

Appendix
Proof of Lemma 1

The matrix L, being a balanced Laplacian matrix, is positive semidefinite and its eigenvalues are real and satisfy $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. Consequently, the principal submatrix $L_{2:N,2:N}$ of L is also symmetric positive semidefinite (by the Cauchy's interlacing theorem). Moreover, by Kirchhoff's theorem, the matrix $L_{2:N,2:N}$, which is a minor of the Laplacian, has a determinant strictly larger than 0 since the graph is connected. Hence, $L_{2:N,2:N}$ is positive definite. Consequently, there exists c_1 sufficiently small such that $A(c_1)$ is negative definite.

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