Passivity Based Design of Sliding Modes for Optimal Load Frequency Control

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Abstract—This paper proposes a distributed sliding mode control strategy for optimal Load Frequency Control (OLFC) in power networks, where besides frequency regulation also minimization of generation costs is achieved (economic dispatch). We study a nonlinear power network of interconnected (equivalent) generators, including voltage and second-order turbine-governor dynamics. The turbine-governor dynamics suggest the design of a sliding manifold, such that the turbine-governor system enjoys a suitable passivity property, once the sliding manifold is attained. This work offers a new perspective on OLFC by means of sliding mode control, and in comparison with existing literature, we relax required dissipation conditions on the generation side and assumptions on the system parameters.

Index Terms—Load Frequency Control, economic dispatch, sliding mode control, incremental passivity, power systems stability.

I. INTRODUCTION

A power mismatch between generation and demand gives rise to a frequency in the power network that can deviate from its nominal value. Regulating the frequency back to its nominal value by Load Frequency Control (LFC) is challenging and it is uncertain if current implementations are adequate to deal with an increasing share of renewable energy sources [2]. Traditionally, the LFC is performed at each control area by a primary droop control and a secondary proportional-integral (PI) control. To cope with the increasing uncertainties affecting a control area and to improve the controller’s performance, advanced control techniques have been proposed to redesign the conventional LFC schemes, such as model predictive control (MPC) [3], adaptive control [4], fuzzy control [5], and sliding mode (SM) control [6]. However, due to the predefined power flows through the tie-lines, the possibility of achieving economically optimal LFC is lost [7]. Besides improving the stability and the dynamic performance of power systems, new control strategies are additionally required to reduce the operational costs of LFC [8]. In this paper, we propose a novel distributed optimal LFC (OLFC) scheme that incorporates the economic dispatch into the LFC loop, departing from the conventional tie-line requirements. An up-to-date survey on recent results on offline and online optimal power flows and OLFC can be found in [9]. We restrict ourselves here to a brief overview of online solutions to OLFC that are close to the presented work. Particularly, we focus on distributed solutions, in contrast to more centralized control schemes that have been studied in e.g. [10], [11], [12]. In order to obtain OLFC, the vast majority of distributed solutions appearing in the literature fit in one of two categories. First, the economic dispatch problem is distributively solved by a primal-dual algorithm converging to the solution of the associated Lagrangian dual problem [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. This approach generally requires measurements of the loads or the power flows, which is not always desirable in a LFC scheme. This issue is avoided by the second class of solutions, where a distributed consensus algorithm is employed to converge to a state of identical marginal costs, solving the economic dispatch problem in the unconstrained case [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37]. The proposed sliding mode controller design in this work is compatible with both approaches, although we put the emphasis on a distributed consensus based solution and remark on the primal-based dual approach.

A. Main contributions

Sliding mode control has been used to improve the conventional LFC schemes [38], possibly together with disturbance observers [39]. However, the proposed use of SM to obtain a distributed OLFC scheme is new and can offer a few advantages over the previous results on OLFC. Foremost, it is possible to incorporate the widely used second-order model for the turbine-governor dynamics that is generally neglected in the analytical OLFC studies. Since the generated control signals in OLFC schemes adjust continuously and in real-time the governor set points, it is important to incorporate the generation side in a satisfactory level of detail. In this paper, we adopt a nonlinear model of a power network, including voltage dynamics, having an arbitrarily complex and meshed topology. The generation side is modelled by an equivalent generator including voltage dynamics and second-order turbine-governor dynamics, which is standard in studies on conventional LFC schemes. We propose a distributed SM controller that is shown to achieve frequency control, while minimizing generation costs. The proposed control scheme continuously adjusts the
governor set point. Conventional SM controllers can suffer from the notorious drawback known as chattering effect, due to the discontinuous control input. To alleviate this issue, we incorporate the well known Suboptimal Second-Order Sliding Mode (SSOSM) control algorithm [40] leading to a continuous control input. To design the controllers obtaining OLFC, we recall an incremental passivity property of the power network [26] that prescribes a suitable sliding manifold. Particularly, the non-passive turbine-governor system, constrained to this manifold, is shown to be incrementally passive allowing for a passive feedback interconnection, once the closed-loop system evolves on the sliding manifold. The proposed approach differs substantially from two notable exceptions that also incorporate the turbine-governor dynamics ([41], [42]) and shows some benefits. In contrast to [41], we do not impose constraints on the permitted system parameters, and in contrast to [42] we do not impose dissipation assumptions on the generation side and allow for a higher relative degree (see also Remark 7). Furthermore, we believe that the chosen approach, where the design of the sliding manifold is inspired by desired passivity properties, offers new perspectives on the control of networks that have similar control objectives as the one presented, e.g. achieving power sharing in microgrids. As this paper is (to the best of our knowledge) the first to use sliding mode control to achieve OLFC, it additionally enables further studies to compare the performance with respect to other approaches found in the literature.

B. Outline

The present paper is organized as follows: In Section II the considered nonlinear model of the power network is introduced, including voltage and second-order turbine-governor dynamics. Particularly, we stress a useful incremental passivity property of the power network model, which we recall in Appendix A. In Section III we formulate the considered optimal Load Frequency Control problem, which aims besides frequency regulation also for an economic dispatch. In Section IV we propose a distributed Sliding Mode controller aiming at optimal Load Frequency Control, where we stress that the control signal to the governor is continuous, avoiding chattering. The stability of the power network in closed loop with the distributed sliding mode controller is studied in Section V, exploiting previously established passivity properties. Simulation results are reported and discussed in Section VI, where a small four-area power network is considered. Furthermore, a comparison with another controller suggested in the literature is performed. Finally, some conclusions and possible future research directions are gathered in Section VI.

C. Notation

Let \( \mathbf{0} \) be the vector of all zeros of suitable dimension and let \( \mathbf{1}_n \) be the vector containing all ones of length \( n \). The \( i \)-th element of vector \( x \) is denoted by \( x_i \). A steady state solution to system \( \dot{x} = \zeta(x) \), is denoted by \( \mathbf{\bar{x}} \), i.e., \( \mathbf{0} = \zeta(\mathbf{\bar{x}}) \). In case the argument of a function is clear from the context, we occasionally write \( \zeta(x) \) as \( \zeta \). Let \( A \in \mathbb{R}^{n \times m} \) be a matrix, then \( \text{im}(A) \) is the image of \( A \) and \( \ker(A) \) is the kernel of \( A \).

case \( A \in \mathbb{R}^{n \times n} \) is a positive definite (positive semi-definite) matrix, we write \( A > 0 \) (\( A \succeq 0 \)). The sign function is defined as

\[
\text{sgn}(x) := \begin{cases} 
-1 & \text{if } x < 0, \\
0 & \text{if } x = 0, \\
1 & \text{if } x > 0.
\end{cases}
\]

II. NONLINEAR POWER NETWORK MODEL

Throughout this work we consider a power network of \( n \) interconnected nodes that represent e.g. (equivalent) generators or control/coherent areas. To make the discussion explicit, we assume that the governing dynamics of the \( i \)-th node are described by the so called ‘single-axis model’. However, the upcoming controller design and presented results are expected to be also applicable to different models than the one presented in this section (see Remark 2). Following [43], the considered dynamics of the \( i \)-th node are\(^1\)

\[
\dot{\delta}_i = f_i^b, \\
T_{pi}f_i^b = -(j_i^b - f^n) \\
+ K_{pi} \left( P_{ti} - P_{di} + \sum_{j \in N_i} V_j V_j B_{ij} \sin(\delta_i - \delta_j) \right),
\]

\[
T_{V_i}\dot{V}_i = \mathbf{E}_{fi} - \left( (1 - (X_{di} - X_{di}^\prime)B_{di})V_i \right. \\
\left. - (X_{di} - X_{di}^\prime) \sum_{j \in N_i} V_j B_{ij} \cos(\delta_i - \delta_j),
\]

where \( N_i \) is the set of nodes connected to the \( i \)-th node by transmission lines. We assume that the network is lossless, which is generally valid in high voltage transmission networks where the line resistance is negligible. The voltage \( V \) generally corresponds to the \( q \)-axis internal voltage and we do not differentiate between the generator internal and terminal buses. Moreover, \( P_{ti} \) in (3) is the power generated by the \( i \)-th (equivalent) plant and can be expressed as the output of the

\(^1\) For notational simplicity, the dependency of the variables on time \( t \) is mostly omitted throughout this paper.
undirected graph $G=(\mathcal{V},\mathcal{E})$, where $\mathcal{V} = \{1,\ldots,n\}$ is the set of nodes and $\mathcal{E} = \{1,\ldots,m\}$ is the set of edges, representing the transmission lines connecting the nodes. The topology can be described by its corresponding incidence matrix $B \in \mathbb{R}^{n \times m}$. Then, by arbitrarily labeling the ends of edge $k$ with a + and a −, one has that

$$ B_{ik} = \begin{cases} +1 & \text{if } i \text{ is the positive end of } k, \\ -1 & \text{if } i \text{ is the negative end of } k, \\ 0 & \text{otherwise}. \end{cases} $$

To study the power network we write system (3) compactly for all nodes $i \in \mathcal{V}$ as

$$ \dot{\eta} = B^T f, $$

$$ T_p \dot{f} = -f + K_p (P_t - P_d - B \mathbf{\Gamma}(V) \sin(\eta)), $$

$$ T_V \dot{V} = -(X_d - X_d') E(\eta) V + \mathbf{E}_f, $$

and the turbine-governor dynamics in (4) as

$$ T_t \dot{P}_t = -P_t + P_g, $$

$$ T_g \dot{P}_g = -R^{-1} f - P_g + u, $$

where $f = f^b - f^n \mathbf{1}_n \in \mathbb{R}^n$ is the frequency deviation, $\eta = B^T \delta \in \mathbb{R}^m$ is vector describing the differences in voltage angles. Furthermore, $\mathbf{\Gamma}(V)_k = V_i V_j B_{ij}$, with $k \sim \{i,j\}$, i.e., line $k$ connects nodes $i$ and $j$. The components of the matrix $E(\eta) \in \mathbb{R}^{n \times m}$ are defined as

$$ E_{ii}(\eta) = \frac{1}{X_{di} - X_{di}'}, $$

$$ E_{ij}(\eta) = B_{ij} \cos(\eta_k) = E_{ji}(\eta) \quad k \sim \{i,j\} \in \mathcal{E}, $$

$$ E_{ij}(\eta) = 0 \quad \text{otherwise}. $$

The remaining symbols follow straightforwardly from (3) and (4), and are vectors and matrices of suitable dimensions.

In the remainder of this work we assume that there exists a (suitable) steady state solution to the power network model (5), (6).

**Assumption 1:** (Steady state solution) The unknown power demand (unmatched disturbance) $P_d$ is constant and for a given $\overline{P}_t$, there exist a $\overline{\eta}$ and state $(\overline{\eta}, \overline{f}, \overline{V}, \overline{P}_t, \overline{P}_g)$ that satisfies

$$ 0 = B^T \overline{f}, $$

$$ 0 = -\overline{f} + K_p (\overline{P}_t - P_d - B \mathbf{\Gamma}(\overline{V}) \sin(\overline{\eta})), $$

$$ 0 = -(X_d - X_d') E(\overline{\eta}) \overline{V} + \mathbf{E}_f, $$

and

$$ 0 = -\overline{P}_t + \overline{P}_g, $$

$$ 0 = -R^{-1} \overline{f} - \overline{P}_g + \overline{\eta}. $$

An important property of system (5) is that is incrementally (cyclo) passive (see Definition 1 in the Appendix A) with respect to a steady state solution $(\overline{\eta}, \overline{f}, \overline{V}, \overline{P}_t, \overline{P}_g)$ satisfying (8), (9). This has been established before in [26], and we recall the most important results in Appendix A at the end of this paper.

**Remark 1:** (Reactance and susceptance) For each (equivalent) generator $i \in \mathcal{V}$, the reactance is higher than the transient reactance, i.e. $X_{di} > X_{di}'$ [44]. Furthermore, the self-susceptance of node $i \in \mathcal{V}$ is given by $B_{ii} = \sum_{j \in \mathcal{N}_i} B_{ij}$ and the susceptance of a line satisfies $B_{ij} = B_{ji} < 0$. Consequently, $E(\eta)$ is a strictly diagonally dominant and symmetric matrix with positive elements on its diagonal and is therefore positive definite.

**Remark 2:** (Incremental passivity and applicability to other power network models) The focus of this work is to achieve OLFC by distributed sliding mode control for a nonlinear power network, explicitly taking into account the turbine-governor dynamics. Equations (5) are often adequately
enough to represent a power network for the purpose of frequency regulation and are often further simplified by assuming constant voltages, leading to the so called ‘swing equations’. To the controller design and the analysis in this paper, an incremental passivity property that is established in Appendix A is essential. This property has been established for various other models, including structure-preserving high voltage networks [41] or networks including sixth-order generator models [45]. Furthermore, underlying energy functions have been established for networks including internal and terminal generator buses and dynamic load models [46], [47]. It is therefore expected that the presented approach can straightforwardly be applied to a wider range of models than the one we consider in this paper.

III. OPTIMAL FREQUENCY REGULATION

We continue this paper by formulating the control objectives of optimal load frequency control. Before doing so, we first note that the steady state frequency \( \tilde{f} \) is generally different from zero without proper adjustments of the input \( \pi [26] \).

Lemma 1: (Steady state frequency) Let Assumption 1 hold, then necessarily \( \tilde{f} = \mathbb{1}_n f^* \) with

\[
f^* = \frac{\mathbb{1}_n^T (\pi - \bar{P}_d)}{\mathbb{1}_n^T (K_p^{-1} + R^{-1}) \mathbb{1}_n},
\]

where \( \mathbb{1}_n \in \mathbb{R}^n \) is the vector consisting of all ones.

This leads us to the first objective, concerning the regulation of the frequency deviation.

Objective 1: (Frequency regulation) The frequency deviation \( f \) asymptotically converges to zero, i.e.,

\[
\lim_{t \to \infty} f(t) = 0.
\]

From (10) it is clear that it is sufficient that \( \mathbb{1}_n^T (\pi - \bar{P}_d) = 0 \), to have zero frequency deviation at the steady state. Therefore, there is flexibility to distribute the total required generation optimally among the various (equivalent) generators. To make the notion of optimality explicit, we assign to every generator a strictly convex linear-quadratic cost function \( C_i(P_{ti}) \) related to the generated power \( P_{ti} \):

\[
C_i(P_{ti}) = \frac{1}{2} Q_i P_{ti}^2 + R_i P_{ti} + C_i \quad \forall i \in \mathcal{V}.
\]

Minimizing the total generation cost, subject to the constraint that allows for a zero frequency deviation can then be formulated as the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in \mathcal{V}} C_i(P_{ti}), \\
\text{subject to} & \quad \mathbb{1}_n^T (\pi - \bar{P}_d) = 0.
\end{align*}
\]

Note that the optimization problem above is convex, since no additional (tie-line) constraints on the power flows are considered. Indeed, it is common for optimal Load Frequency Control schemes to replace the line constraints in favour of an online economic dispatch of the generators. In case the resulting power flows are close to the line limits, feasibility of resulting steady state power flows can be guaranteed by relying e.g. on a primal-dual based approach (see Remark 8), where additional line constraint can be incorporated within the optimization problem (13) [15]. The lemma below makes the solution to (13) explicit [26]:

Lemma 2: (Optimal generation) The solution \( \overline{P}_t^{opt} \) to (13) satisfies

\[
\overline{P}_t^{opt} = \mathbb{Q}^{-1} (\overline{x}^{opt} - \mathcal{R}),
\]

where

\[
\overline{x}^{opt} = \mathbb{1}_n^T (P_d + \mathbb{Q}^{-1} \mathcal{R}) \mathbb{1}_n
\]

and \( \mathbb{Q} = \text{diag}(Q_1, \ldots, Q_n) \), \( \mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)^T \).

From (14) it follows that the marginal costs \( \mathbb{Q} \overline{P}_t^{opt} + \mathcal{R} \) are identical. Note that (14) depends explicitly on the unknown power demand \( P_d \). We aim at the design of a controller solving (13) without measurements of the power demand, leading to the second objective.

Objective 2: (Economic dispatch) The generated power \( P_t \) asymptotically converges to the optimal power generation, i.e.,

\[
\lim_{t \to \infty} P_t(t) = \overline{P}_t^{opt},
\]

with \( \overline{P}_t^{opt} \) as in (14).

In order to achieve Objective 1 and Objective 2 we refine Assumption 1 that ensures the feasibility of the objectives.

Assumption 2: (Existence of an optimal steady state) Assumption 1 holds when \( \tilde{f} = 0 \) and \( \overline{P}_t = \overline{P}_g = \overline{P}_t^{opt} \), with \( \overline{P}_t^{opt} \) as in (14).

IV. DISTRIBUTED SLIDING MODE CONTROL

In this section, we propose a distributed sliding mode controller to achieve Objective 1 and Objective 2 for the power network (5). To facilitate the upcoming discussion a few essential definitions of sliding mode control are gathered in Appendix B. Furthermore, in order to permit the controller design, the following assumption is made on the unknown demand (unmatched disturbance) and the available measurements:

Assumption 3: (Available information) The variables \( f_i, P_{ti} \) and \( P_{gi} \) are locally available\(^2\) at node \( i \). All the network parameters and the power demand \( P_d \) are constant and unknown, but with known bounds.

\(^2\)In case not all variables are locally available, Assumption 3 can be relaxed by implementing observers that estimate the unmeasured states in a finite time (see for instance [48]).
In Appendix A a passivity property of the power network (5) is recalled, with input $P_i$ and output $f$. Unfortunately, the turbine-governor system (6) does not immediately allow for a passive interconnection, since (6) is a linear system with relative degree two, when considering $-f$ as the input and $P_i$ as the output$^3$. This makes the controller design more challenging and is a major reason why the turbine-governor dynamics are generally neglected or approximated by a first-order system in analytical OLFC studies. To alleviate this issue, we propose a distributed Suboptimal Second-Order Sliding Mode (D–SSOSM) control algorithm that simultaneously achieves Objective 1 and Objective 2, by constraining (6) such that it enjoys a suitable passivity property, and by exchanging information on the marginal costs. As a first step (see also Remark 3 below), we augment the turbine-governor dynamics (6) with a distributed control scheme, resulting in:

$$
\begin{align*}
T_1 \dot{P}_t & = - P_t + P_g, \\
T_2 \dot{P}_g & = - R^{-1} f - P_g + u, \\
T_0 \dot{\theta} & = - \theta + P_t - A L^{\text{com}}(Q \theta + R).
\end{align*}
$$

(17)

Here, $QL + R$ reflects the ‘virtual’ marginal costs and $L^{\text{com}}$ is the Laplacian matrix corresponding to the topology of an underlying communication network. The diagonal matrix $T_0 \in \mathbb{R}^{n \times n}$ provides additional design freedom to shape the transient response and the matrix $A$ is suggested later to obtain a suitable passivity property. We note that $L^{\text{com}}(QL + R)$ represents the exchange of information on the marginal costs among the nodes. To guarantee an optimal coordination of generation among all the nodes the following assumption is made:

**Assumption 4: (Communication topology)** The graph corresponding to the communication topology is balanced and strongly connected$^4$.

We now propose a sliding function $\sigma(f, P_t, P_g, \theta)$ and a matrix $A$ for system (17), which will allow us to prove convergence to the desired state. The choices are motivated by the stability analysis in the next section, but are stated here for the sake of exposition. First, the sliding function $\sigma: \mathbb{R}^{4n} \rightarrow \mathbb{R}^n$ is given by

$$
\sigma(f, P_t, P_g, \theta) = M_1 f + M_2 P_t + M_3 P_g + M_4 \theta,
$$

(18)

where $M_1 > 0$, $M_2 \geq 0$, $M_3 > 0$ are diagonal matrices and $M_4 = -(M_2 + M_3)$. Therefore, $\sigma_1, i \in \mathcal{V}$, depends only on the locally available variables that are defined on node $i$, facilitating the design of a distributed controller (see Remark 5). Second, the diagonal matrix $A \in \mathbb{R}^{n \times n}$ is defined as

$$
A = (M_2 + M_3)^{-1} M_1 Q.
$$

(19)

$^3$ A linear system with relative degree two is not passive, as follows e.g. from the Kalman-Yakubovich-Popov (KYP) lemma.

$^4$ A directed graph is balanced if the (weighted) in-degree is equal to the (weighted) out-degree of every node and it is strongly connected if there is a directed path from any node to every other node. A balanced and strongly connected graph implies that $L^{\text{com}} + (L^{\text{com}})^T = L^{\text{com}} \succeq 0$ and that $\ker(L^{\text{com}}) = \text{im}(I_n)$. Any undirected and connected graph is balanced and strongly connected.

By regarding the sliding function (18) as the output function of system (5), (17), it appears that the relative degree of the system is one. This implies that a first-order sliding mode controller can be naturally applied [49] to attain in a finite time, the sliding manifold defined by $\sigma = 0$. However, the input $u$ to the governor affects the first time derivative of the sliding function, i.e., $u$ affects $\dot{\sigma}$. Since sliding mode controllers generate a discontinuous signal, we additionally require $\dot{\sigma} = 0$, to guarantee that the signal $u$ is continuous. Therefore, we define the desired sliding manifold as

$$
\{(\eta, f, V, P_t, P_g, \theta) : \sigma = \dot{\sigma} = 0\}.
$$

(20)

We continue in the next subsection with discussing a possible controller attaining the desired sliding manifold (20) while providing a continuous control input $u$.

**Remark 3: (First-order turbine-governor dynamics)** The rationale behind this seemingly ad-hoc choice of the augmented dynamics is that for the controlled first-order turbine-governor dynamics, where $u = \theta$ and $P_g = - R^{-1} f + \theta$, system

$$
\begin{align*}
T_1 \dot{P}_t & = - P_t - R^{-1} f + \theta, \\
T_0 \dot{\theta} & = - \theta - P_t - A L^{\text{com}}(Q \theta + R),
\end{align*}
$$

(21)

has been shown to be incrementally passive with input $-f$ and output $P_t$, and is able to solve Objective 1 and Objective 2 [41]. We aim at the design of $u$ and $A$ in (17), such that (17) behaves similarly as (21). This is made explicit in Lemma 4.

A. **Suboptimal Second-Order Sliding Mode controller**

To prevent chattering, it is important to provide a continuous control input $u$ to the governor. Since sliding mode controllers generate a discontinuous control signal, we adopt the procedure suggested in [40] and first integrate the discontinuous signal, yielding for system (17):

$$
\begin{align*}
T_1 \dot{P}_t & = - P_t + P_g, \\
T_2 \dot{P}_g & = - R^{-1} f - P_g + u, \\
T_0 \dot{\theta} & = - \theta + P_t - A L^{\text{com}}(Q \theta + R),
\end{align*}
$$

(22)

where $w$ is the new (discontinuous) input generated by a sliding mode controller discussed below. A consequence is

![Block diagram of the proposed Distributed Suboptimal Second-Order Sliding Mode (D–SSOSM) control strategy.](image-url)
that the system relative degree (with respect to the new control input $w$) is now two, and we need to rely on a second-order sliding mode control strategy to attain the sliding manifold (18) in a finite time [50]. To make the controller design explicit, we discuss a specific second-order sliding mode controller, the so-called ‘Suboptimal Second-Order Sliding Mode’ (SSOSM) controller proposed in [40]. We introduce two auxiliary variables $\xi_1 = \sigma \in \mathbb{R}^n$ and $\xi_2 = \sigma \in \mathbb{R}^n$, and define the so-called auxiliary system as:

$$
\dot{\xi}_1 = \xi_2, \\
\dot{\xi}_2 = \phi(\eta, f, V, P_t, P_g, \theta) + Gw.
$$

(Bearing in mind that $\dot{\xi}_2 = \dot{\sigma} = \phi + Gw$, the expressions for the mapping $\phi$ and matrix $G$ can straightforwardly be obtained from (18) by taking the second derivative of $\sigma$ with respect to time, yielding for the latter $G = M_1T^{-1}_g \in \mathbb{R}^{n \times n}$. We assume that the entries of $\phi$ and $G$ have known bounds

$$
|\phi_i| \leq \Phi_i, \quad \forall i \in \mathcal{V},
$$

(24)

$$
0 < G_{\text{min}} < G_i \leq G_{\text{max}}, \quad \forall i \in \mathcal{V},
$$

(25)

with $\Phi_i$, $G_{\text{min}}$, and $G_{\text{max}}$, being positive constants. Second, $w$ is a discontinuous control input described by the SSOSM control algorithm [40], and consequently for each node $i \in \mathcal{V}$, the control law $w_i$ is given by

$$
w_i = -\alpha_t W_{\text{max}}, \quad \sigma = \left(\xi_i - \frac{1}{2}\xi_{1,\text{max}_i}\right),
$$

(26)

with

$$
W_{\text{max}_i} > \max\left(\frac{\Phi_i}{\alpha^*_i G_{\text{min}_i}}, \frac{4\Phi_i}{3G_{\text{min}_i} - \alpha^*_i G_{\text{max}_i}}\right),
$$

(27)

$$
\alpha^*_i \in (0, 1] \cap \left(0, \frac{3G_{\text{min}_i}}{G_{\text{max}_i}}\right),
$$

(28)

$\alpha_t$ switching between $\alpha^*_i$ and 1, according to [40, Algorithm 1]. Note that indeed the input signal to the governor, $u(t) = \int_0^t w(\tau) d\tau$, is continuous, since the input $w$ is piecewise constant. The extremal values $\xi_{1,\text{max}_i}$ in (26) can be detected by implementing for instance a peak detection as in [51]. The block diagram of the proposed control strategy is depicted in Figure 2.

**Remark 4:** (Uncertainty of $\phi$ and $G$) The mapping $\phi$ and matrix $G$ are uncertain due to the presence of the unmeasurable power demand $P_d$ and voltage angle $\theta$, and possible uncertainties in the system parameters. In practical cases the bounds in (24) and (25) can be determined relying on data analysis and physical insights. However, if these bounds cannot be a-priori estimated, the adaptive version of the SSOSM algorithm proposed in [52] can be used to dominate the effect of the uncertainties.

**Remark 5:** (Distributed control) Given $A$ in (19), the dynamics of $\theta_i$ in (17) read for node $i \in \mathcal{V}$ as

$$
T_{\theta i} \dot{\theta}_i = -\theta_i + P_t + P_g - \frac{Q_i M_i}{2M_i + M_3 i} \sum_{j \in \mathcal{N}^{\text{com}}}_j (Q_i \theta_i + R_i - Q_j \theta_j - R_j),
$$

where $\mathcal{N}^{\text{com}}$ is the set of controllers connected to controller $i$. Furthermore, (26) depends only on $\sigma_i$, i.e., on states defined at node $i$. Consequently, the overall controller is indeed distributed and only information on marginal costs needs to be shared among connected controllers.

**Remark 6:** (Alternative SOSM controllers) In this work we rely on the SOSM control law proposed in [40]. However, to constrain system (5) augmented with dynamics (22) on the sliding manifold (20), where $\sigma = \dot{\sigma} = 0$, any other SOSM control law that does not need the measurement of $\dot{\sigma}$ can be used, e.g., the super-twisting control [53]. An interesting continuation of the presented results is to study the performance of various SOSM controllers within the setting of (optimal) LFC.

**Remark 7:** (Comparison with [41] and [42]) The controller proposed in [41] requires, besides a gain restriction in the controller, that

$$
4T_{p_t} T_{\theta i}^{-1} > 1, \\
K_{p_g}^{-1} T_{\theta i} T_{-1} > 1.
$$

(29)

In this work, we do not impose such restriction on the parameters. The result in [42] requires, besides some assumptions on the dissipation inequality related to the generation side, the existence of frequency dependent generation and load, whereas the generation/demand (output) depends directly (e.g. proportionally) on the frequency (input), avoiding complications arising from generation dynamics that have relative degree two when considering the input-output pair just indicated (see also Remark 9).

**Remark 8:** (Primal-dual based approaches) Although the focus in this work is to augment the power network with consensus-type dynamics in (17), it is equally possible to augment the power network with a continuous primal-dual algorithm that has been studied extensively to obtain optimal LFC. This work provides therefore also means to extend existing results on primal-dual based approaches to incorporate the turbine-governor dynamics, generating the control input by a higher order sliding mode controller. The required adjustments follow similar steps as discussed in [41, Remark 9], and, for the sake of brevity, we directly state the resulting primal-dual based augmented system, replacing (17),

$$
T_{\phi} \dot{\phi}_i = -\phi_i + P_t + P_g,
$$

$$
T_g \dot{P}_g = -R^{-1} f - P_g + u,
$$

(26)

$$
T_{\theta} \dot{\theta}_i = -\theta_i + P_t - M_1 (M_2 + M_3) - \left(\nabla C(\theta) - \lambda\right),
$$

(30)

$$
0 = -B^T \dot{\lambda},
$$

(30)

$$
\dot{\lambda} = Bv - \theta + P_d.
$$

(30)
In this case only strict convexity of $C(\cdot)$ is required and the load $P_d$ explicitly appears in (30). The stability analysis of the power network, including the augmented turbine-governor dynamics (30), follows *mutatis mutandis*, the same argumentation as in the next section where the focus is on the augmented system (17). Some required nontrivial modifications in the analysis are briefly discussed in Remark 13.

V. STABILITY ANALYSIS AND MAIN RESULT

In this section we study the stability of the proposed control scheme, based on an enforced passivity property of (17) on the sliding manifold defined by (18). First, we establish that the second-order sliding mode controller (23)–(28) constrains the system in finite time to the desired sliding manifold.

**Lemma 3:** *(Convergence to the sliding manifold)* Let Assumption 3 hold. The solutions to system (5), augmented with (22), in closed loop with controller (23)–(28) converge in a finite time $T_r$ to the sliding manifold (20) such that

$$P_g = -M_3^{-1}(M_1 f + M_2 P_t + M_4 \theta) \quad \forall t \geq T_r.$$  \hspace{1cm} (31)

Proof: Following [40], the application of (23)–(28) to each (equivalent) generator ensures that $\sigma = \dot{\sigma} = 0$, $\forall t \geq T_r$. The details are omitted, and are immediate consequence of the used SSOSM control algorithm [40]. Then, from (18) one can easily obtain (31), where $M_3$ is indeed invertible. \hfill \Box

Exploiting relation (31), on the sliding manifold where $\sigma = \dot{\sigma} = 0$, the so-called equivalent system is as follows:

$$M_3 T_t \dot{P}_t = -(M_2 + M_3) P_t - M_4 \theta - M_1 f,$$
$$T_t \dot{\theta} = -\theta + P_t - A \hat{L}^{\text{com}}(Q \theta + \mathcal{R}).$$  \hspace{1cm} (32)

As a consequence of the feasibility assumption (Assumption 1), the system above admits the following steady state:

$$0 = -(M_2 + M_3) \hat{P}_t^{\text{opt}} - M_4 \overline{\theta} - M_1 f,$$
$$0 = -\overline{\theta} + \hat{P}_t^{\text{opt}} - A \hat{L}^{\text{com}}(\overline{Q \theta} + \mathcal{R}).$$  \hspace{1cm} (33)

Now, we show that system (32), with $A$ as in (19), indeed possesses a passivity property with respect to the steady state (33). Note that, due to the discontinuous control law (26), the solutions to the closed loop system are understood in the sense of Filippov. Following the equivalent control method [49], the solutions to the equivalent system are however continuously differentiable.

**Lemma 4:** *(Incremental passivity of (32))* System (32) with input $-f$ and output $P_t$ is an incrementally passive system, with respect to the constant $(\hat{P}_t^{\text{opt}}, \overline{\theta})$ satisfying (33).

Proof: Consider the following incremental storage function

$$S_2 = \frac{1}{2}(P_t - \hat{P}_t^{\text{opt}})^T M_1^{-1} M_3 T_t (P_t - \hat{P}_t^{\text{opt}}) + \frac{1}{2}(\theta - \overline{\theta})^T M_1^{-1} (M_2 + M_3) T_t (\theta - \overline{\theta}),$$  \hspace{1cm} (34)

which is positive definite, since $M_1 > 0, M_2 \geq 0$ and $M_3 > 0$. Then, we have that $S_2$ satisfies along the solutions to (32)

$$\dot{S}_2 = (P_t - \hat{P}_t^{\text{opt}})^T M_1^{-1} M_3 T_t \dot{P}_t + (\theta - \overline{\theta})^T M_1^{-1} (M_2 + M_3) T_t \dot{\theta},$$
$$= (P_t - \hat{P}_t^{\text{opt}})^T (-M_1^{-1}(M_2 + M_3) P_t - f - M_1^{-1} M_4 \theta) + (\theta - \overline{\theta})^T M_1^{-1} (M_2 + M_3) T_t (\theta - \overline{\theta}),$$
$$= (P_t - \hat{P}_t^{\text{opt}})^T (f - 0),$$

In view of $M_4 = -(M_2 + M_3), A = (M_2 + M_3)^{-1} M_1 Q$ and equality (33), it follows that

$$\dot{S}_2 = - (P_t - \theta)^T M_1^{-1} (M_2 + M_3) (P_t - \theta) - (Q \theta + \mathcal{R} - \overline{Q \theta} - \mathcal{R}) \hat{L}^{\text{com}}(Q \theta + \mathcal{R} - \overline{Q \theta} - \mathcal{R}) - (P_t - \hat{P}_t^{\text{opt}})^T (f - 0),$$

where $\hat{L}^{\text{com}} = \frac{1}{2} (L^{\text{com}} + (L^{\text{com}})^T) \geq 0$ (see Assumption 4).

Relying on the interconnection of incrementally passive systems, we can prove the main result of this paper concerning the evolution of the augmented system controlled via the proposed distributed SSOSM control strategy. Note that the proof exploits the incremental passivity property of the power network (5), which is derived in Appendix A.

**Theorem 1:** *(Main result: distributed OLFC)* Let Assumptions 1–6 hold. Consider system (5) and (17), controlled via (23)–(28). Then, the solutions to the closed-loop system starting in a neighbourhood of the equilibrium $(\bar{\eta}, \bar{f} = 0, \nabla, \hat{P}_t^{\text{opt}}, \overline{\theta}, \mathcal{R})$ approach the set where $\bar{f} = 0$ and $\hat{P}_t = \hat{P}_t^{\text{opt}}$, with $\hat{P}_t^{\text{opt}}$ given by (14).

Proof: Following Lemma 3, we have that the SSOSM control enforces system (17) to evolve $\forall t \geq T_r$ on the sliding manifold (20), resulting in the reduced order system (32). To study the obtained closed loop system, consider the overall incremental storage function $S = S_1 + S_2$, with $S_1$ given by (44) and $S_2$ given by (34). In view of Lemma 6, we have that $S$ has a local minimum at $(\bar{\eta}, \bar{f} = 0, \nabla, \hat{P}_t^{\text{opt}}, \overline{\theta})$ and satisfies (see Lemma 4 and Lemma 5) along the solutions to (5), (32)

$$\dot{S} = - f^T K_p^{-1} f - \hat{V}^T T_V (X_d - X_d')^{-1} \hat{V} - (P_t - \theta)^T M_1^{-1} (M_2 + M_3) (P_t - \theta) - (Q \theta + \mathcal{R} - \overline{Q \theta} - \mathcal{R}) \hat{L}^{\text{com}}(Q \theta + \mathcal{R} - \overline{Q \theta} - \mathcal{R}) \leq 0,$$

where $\hat{V} = T_V^{-1} \left(- (X_d - X_d') E(\eta) V + \mathcal{E}_f \right)$. Consequently, there exists a forward invariant set $\mathcal{Y}$ around $(\bar{\eta}, \bar{f} = 0, \nabla, \hat{P}_t^{\text{opt}}, \overline{\theta})$ and by LaSalle’s invariance principle the solutions that start in $\mathcal{Y}$ approach the largest invariant set contained in

$$\mathcal{Y} \cap \{ (\eta, f, V, P_t, \theta) : f = 0, V = ((X_d - X_d') E(\eta))^{-1} \mathcal{E}_f, \quad P_t = \theta, \theta = \overline{\theta} + Q^{-1} \mathcal{E}_f \},$$  \hspace{1cm} (35)
where \( \alpha \in \mathbb{R} \) is some scalar. On this invariant set the controlled power network satisfies
\[
\begin{align*}
\dot{\eta} &= B^T \theta, \\
0 &= K_p (\bar{\theta} + Q^{-1} \alpha - P_d - B \Gamma(V) \sin(\eta)), \\
0 &= -(X_d - X_d') E(\eta) V + E_f, \\
M_3 T_i \dot{P}_i &= 0, \\
T_d \dot{\theta} &= 0.
\end{align*}
\]
Pre-multiplying both sides of the second line of (36) with \( \mathbf{1}_n^T K_p^{-1} \) yields \( 0 = \mathbf{1}_n^T (\bar{\theta} + Q^{-1} \alpha - P_d) \). Since \( \bar{\theta} = \bar{\theta}_{opt} \), \( \mathbf{1}_n^T (\bar{\theta}_{opt} - P_d) = 0 \) and \( Q \) is a diagonal matrix with only positive elements, it follows that necessarily \( \alpha = 0 \). We can conclude that the solutions to the system (5) and (17), controlled via (23)–(28), indeed approach the set where \( \bar{\theta} = 0 \) and \( \bar{\theta}_{opt} = \bar{\theta}_{opt} \), with \( \bar{\theta}_{opt} \) given by (14). Furthermore, from (31) it follows that \( P_g \) approaches the set where \( P_g = P_t = \bar{\theta}_{opt} \).

\begin{remark} \textbf{(Reducing the relative degree)} \end{remark}

An important consequence of the proposed sliding mode controller (23)–(28) is that the relative degree of system (32) is one with input \( -f \) and output \( P_t \). This is in contrast to the ‘original’ system (6) that has relative degree two with the same input–output pair.

\begin{remark} \textbf{(Varying power demand)} \end{remark}

To allow for a steady state solution, the power demand (unmatched disturbance) is required to be constant. This is not needed to reach the desired sliding manifold, but is required only to establish the asymptotic convergence properties in Objective 1 and Objective 2. Furthermore, the proposed solution shows ([26, Remark 8]) the existence of a finite \( L_2 \)-to-\( L_\infty \) gain and a finite \( L_2 \)-to-\( L_\infty \) gain from a varying demand to the frequency deviation \( f \) [54], once the system evolves on the sliding manifold.

\begin{remark} \textbf{(Robustness to failed communication)} \end{remark}

The proposed control scheme is distributed and as such requires a communication network to share information on the marginal costs. However, note that the term \( -A C_{com} (Q \theta + \mathcal{R}) \) in (17) is not needed to enforce the passivity property established in Lemma 4, but is required to prove convergence to the economic efficient generation \( \bar{\theta}_{opt} \). In fact, setting \( A = 0 \) still permits to infer frequency regulation following the argumentation of Theorem 1.

\begin{remark} \textbf{(Region of attraction)} \end{remark}

LaSalle’s invariance principle can be applied to all bounded solutions. As follows from Lemma 2, we have that on the sliding manifold the considered incremental storage function attains a local minimum at the desired steady state, which allows us to show the existence of a region of attraction once the system evolves on the sliding manifold. Furthermore, the time to converge to the sliding manifold can be made arbitrarily small by properly initializing the system and choosing the gains of the SSOSM control algorithm. To characterize the region of attraction requires a careful analysis of the level sets associated to the incremental storage function \( S \), as well as of the trajectories outside of the sliding manifold. A preliminary (numerical) assessment indicates that the region of attraction is large, but a thorough analysis is left as a future endeavour.

\begin{remark} \textbf{(Stability of primal-dual based approaches)} \end{remark}

To accommodate the additional dynamics of states \( v \) and \( \lambda \) appearing in primal-dual based augmented system (30), an additional storage term is required in Lemma 6, namely:
\[
S_3 = \frac{1}{2} (v - \bar{v})^T (v - \bar{v}) + \frac{1}{2} (\lambda - \bar{\lambda})^T (\lambda - \bar{\lambda}),
\]
where \( \bar{\lambda} \) and \( \lambda \) satisfy the steady state equations
\[
\begin{align*}
0 &= -\bar{\theta} + \bar{\theta}_{opt} - M_1 (M_2 + M_3)^{-1} (\nabla C(\bar{\theta}) - \bar{\lambda}), \\
0 &= -B^T \bar{\lambda}, \\
0 &= B v - \bar{\theta} + P_d.
\end{align*}
\]
Consequently, \( S_2 + S_3 \) satisfies along the solutions to the system, constrained to the manifold \( \sigma = \dot{\sigma} = 0 \):
\[
\begin{align*}
\dot{S}_2 + \dot{S}_3 &= -(P_t - \theta)^T M_1^{-1} (M_2 + M_3) (P_t - \theta) \\
&\quad - (\theta - \bar{\theta})^T (\nabla C(\theta) - \nabla C(\bar{\theta})) \\
&\quad - (P_t - \bar{\theta}_{opt})^T (f - 0).
\end{align*}
\]
Note that, as a result of the mean value theorem, \( -(\theta - \bar{\theta})^T (\nabla C(\theta)) \leq 0 \), for some \( \bar{\theta} \in [\theta_i, \bar{\theta}_i] \), for all \( i \in \mathcal{V} \). The matrix \( \nabla C(\theta) \in \mathbb{R}^{n \times n} \) is positive definite due to the strict convexity of \( C(\cdot) \). The proof of Theorem 1 can now be repeated using the incremental storage function \( S = S_1 + S_2 + S_3 \).

\section{Case study}

In this section, the proposed control solution is assessed in simulation, by implementing a power network partitioned into four areas\(^6\). Three different scenarios are investigated and the topology of the considered power network is represented in Figure 3, together with the communication network (dashed lines). The line parameters are \( B_{12} = -5.4 \text{ p.u.}, B_{23} = -5.0 \text{ p.u.}, B_{34} = -4.5 \text{ p.u.} \) and \( B_{14} = -5.2 \text{ p.u.} \), while the network parameters and the power demand \( \Delta P_{di} \) of each area are provided in Table II, where a base power of 1000 MW is assumed. The matrices in (18) are chosen as \( M_i = \text{diag}(3.4, 2.7, 3.0, 3.2) \), \( M_2 = \text{diag}(1, 1.1, 1.2, 0.9) \), \( M_3 = \text{diag}(0.10, 0.09, 0.08, 0.11) \) and \( M_4 = -(M_2 + M_3) \), while the control amplitude \( W_{\max} \), and the parameter \( \alpha' \), in (26) are equal to 10 and 1, respectively, for all \( i \in \mathcal{V} \). Note that any other choice of \( M_1, \ldots, M_4 \), as defined in (18), is admissible.

\subsection{Scenario 1: Power demand variation}

The system is initially at the steady state. Then, at the time instant \( t = 1 \text{ s} \), the power demand in each area is increased according to the values reported in Table II. From Figure 4, \( \text{See e.g. [55]} \) on how the IEEE New England 39-bus system can be represented by a network consisting of four areas.
one can observe that the frequency deviations converge asymptotically to zero after a transient where the frequency drops because of the increasing load, while the voltages remain constant. Indeed, one can note from Figure 5 that the proposed controllers increase the power generation in order to reach again a zero steady state frequency deviation. Moreover, the total power demand is (optimally) shared among the areas, and the steady state marginal costs are identical, minimizing the total generation costs. Finally, Figure 6 shows the power flows through the power network and the sliding functions.

### B. Scenario 2: opening of a line

The system is initially at the steady state. Then, at the time instant $t = 1\text{ s}$, the line interconnecting Area 1 and Area 4 is opened. From Figure 7, one can observe that the frequency deviations converge asymptotically to zero after a transient where the frequency varies because of the opening

![Figure 3](image-url)  
**Fig. 3.** Scheme of the considered power network partitioned into four areas, where $P_{ij} = \frac{V_i V_j}{X_m} \sin(\delta_i - \delta_j)$. The solid arrows indicate the positive direction of the power flows through the power network, while the dashed lines represent the communication network. From the left, the configurations of the considered scenarios are represented, where the components that are failing/removed during the simulation are coloured red.

### TABLE II

**Nominal Network Parameters and Power Demand**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Area 1</th>
<th>Area 2</th>
<th>Area 3</th>
<th>Area 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{pi}$ (s)</td>
<td>21.0</td>
<td>25.0</td>
<td>23.0</td>
<td>22.0</td>
</tr>
<tr>
<td>$T_{t_i}$ (s)</td>
<td>0.30</td>
<td>0.33</td>
<td>0.35</td>
<td>0.28</td>
</tr>
<tr>
<td>$T_{b_i}$ (s)</td>
<td>0.080</td>
<td>0.072</td>
<td>0.070</td>
<td>0.081</td>
</tr>
<tr>
<td>$T_{\delta_i}$ (s)</td>
<td>5.54</td>
<td>7.41</td>
<td>6.11</td>
<td>6.22</td>
</tr>
<tr>
<td>$K_{pi}$ (s$^{-1}$ p.u.$^{-1}$)</td>
<td>120.0</td>
<td>112.5</td>
<td>115.0</td>
<td>118.5</td>
</tr>
<tr>
<td>$R_{t_i}$ (s$^{-1}$ p.u.$^{-1}$)</td>
<td>2.5</td>
<td>2.7</td>
<td>2.6</td>
<td>2.8</td>
</tr>
<tr>
<td>$X_{t_i}$ (p.u.)</td>
<td>1.85</td>
<td>1.84</td>
<td>1.86</td>
<td>1.83</td>
</tr>
<tr>
<td>$X_{b_i}$ (p.u.)</td>
<td>0.25</td>
<td>0.24</td>
<td>0.26</td>
<td>0.23</td>
</tr>
<tr>
<td>$E_{\delta_i}$ (p.u.)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$B_{\delta_i}$ (p.u.)</td>
<td>-13.6</td>
<td>-12.9</td>
<td>-12.3</td>
<td>-12.3</td>
</tr>
<tr>
<td>$T_{\delta_i}$ (s)</td>
<td>0.33</td>
<td>0.33</td>
<td>0.33</td>
<td>0.33</td>
</tr>
<tr>
<td>$Q_{i}$ (10$^4$ $\text{S}^{-1}$)</td>
<td>2.42</td>
<td>3.78</td>
<td>3.31</td>
<td>2.75</td>
</tr>
<tr>
<td>$R_{i}$ (10$^4$ $\text{S}^{-1}$)</td>
<td>0.11</td>
<td>0.10</td>
<td>0.13</td>
<td>0.12</td>
</tr>
<tr>
<td>$C_{i}$ (10$^4$ $\text{S}^{-1}$)</td>
<td>0.91</td>
<td>1.74</td>
<td>1.32</td>
<td>1.05</td>
</tr>
<tr>
<td>$P_{di}(0)$ (p.u.)</td>
<td>0.010</td>
<td>0.015</td>
<td>0.012</td>
<td>0.014</td>
</tr>
<tr>
<td>$\Delta P_{di}$ (p.u.)</td>
<td>0.010</td>
<td>0.015</td>
<td>0.012</td>
<td>0.014</td>
</tr>
</tbody>
</table>

![Figure 4](image-url)  
**Fig. 4.** Scenario 1. Time evolution of the frequency deviations and voltage dynamics, considering a power demand variation at the time instant $t = 1\text{ s}$.

![Figure 5](image-url)  
**Fig. 5.** Scenario 1. Time evolution of the turbine output powers and marginal costs, considering a power demand variation at the time instant $t = 1\text{ s}$.
C. Scenario 3: failing of a communication link

The system is initially at the steady state. Then, at the time instant \( t = 0.5 \) s the communication link between Area 1 and Area 2 fails, while at the time instant \( t = 1 \) s, the power demand in each area is increased according to the values reported in Table II. From Figure 4, one can observe that the frequency deviations converge asymptotically to zero after a transient where the frequency drops because of the increasing load, while the voltages remain constant. Indeed, one can note from Figure 5 that the proposed controllers increase the power generation in order to reach again a zero steady state frequency deviation. However, the total power demand is non optimally shared among the areas, and only the steady state marginal costs of Area 2, Area 3 and Area 4 are identical. This is due to the failing of the communication link, which prevents Area 1 from communicating with the other Areas. Finally, Figure 6 shows the power flows through the power network and the sliding functions.

D. Comparison with [41]

In this subsection, the proposed control scheme is compared with the controlled proposed in [41], which is given by

\[
T_\theta \dot{\theta} = -\theta + P_\gamma - (I_4 - R^{-1}) f - QL^{com}(Q\theta + R),
\]

\[u = \theta,
\]

\[Q_\theta + R\]
where we take $T_{\theta} = I_4$. We refer to [41] for the details. Here, we repeat Scenario 1 with the distributed controller (39). The resulting frequency deviations and turbine output powers are provided in Figure 13. In comparison with the proposed control scheme in this work (see Figure 4 and Figure 5), one can notice that the overall response when controller (39) is used, is slightly slower, with a larger frequency drop. On the other hand, the turbine output powers do not experience the overshoot that can be observed in Figure 5 for the control scheme that is proposed in this paper.

\section{VII. Conclusions}

A Distributed Suboptimal Second-Order Sliding Mode (D-SSOSM) control scheme is proposed to solve an optimal load frequency control problem in power systems. In this work, we adopted a nonlinear model of a power network, including voltage dynamics, where each node is represented by an (equivalent) generator including second-order turbine-governor dynamics. Based on a suitable chosen sliding manifold, the controlled turbine-governor system, constrained to this manifold, possesses an incremental passivity property that is exploited to prove that the frequency deviation asymptotically approaches zero and an economic dispatch is achieved. Designing the sliding modes, based on passivity considerations, appears to be powerful and we will pursue this approach within different settings, such as achieving power sharing in microgrids. Additionally, we would like to compare the performance of the proposed sliding mode based control scheme in greater detail with other approaches to OLFC appearing in the literature. Since the underlying communication network plays a critical role for the distributed controller, future research directions should also focus on possible delays, discrete...
time communication, optimal topologies and larger classes of directed networks.

APPENDIX

A. Incremental passivity of the power network

Incremental passivity has been shown to play an outstanding role in the analysis of power networks and related controller designs. Particularly, for system (5) a useful passivity property has been established before in [26], and we recall some essential results for the sake of completeness. To facilitate the discussion, we first define ‘incremental passivity’.

**Definition 1:** (Incremental passivity) System

\[
x = \zeta(x, u), \\
y = h(x),
\]

\(x \in \mathbb{R}^n, u, y \in \mathbb{R}^m\), is incrementally passive with respect to a constant triplet \((\pi, \tau, \bar{y})\) satisfying

\[
0 = \zeta(\pi, \tau), \\
\bar{y} = h(\pi),
\]

if there exists a continuously differentiable function \(S : \mathbb{R}^n \rightarrow \mathbb{R}^+\), such that for all \(x \in \mathbb{R}^n, u \in \mathbb{R}^m\) and \(y = h(x), \bar{y} = h(\pi)\)

\[
\dot{S} = \frac{\partial S}{\partial x} \zeta(x, u) + \frac{\partial S}{\partial \pi} \zeta(\pi, \tau) \leq -W(y, \bar{y}) + (y - \bar{y})^T(u - \tau).
\]

In case \(W(y, \bar{y}) > 0\), the system is called ‘output strictly incrementally passive’. In case \(S\) is not lower bounded, the system is called ‘incrementally cyclo-passive’.

To state an incremental passivity property of (5), we make use of the following storage function [26], [56]:

\[
S_1(\eta, f, V) = \frac{1}{2} f^T \tau p + \frac{1}{2} V^T E(\eta)V,
\]

that can also be interpreted as a Hamiltonian function of the system [15].

**Lemma 5:** (Incremental cyclo-passivity of (5)) System (5) with input \(P_t\) and output \(f\) is an output strictly incrementally cyclo-passive system, with respect to the constant \((\bar{\eta}, \bar{f}, \bar{V})\) satisfying (8).

**Proof:** For notational convenience we define \(x = (\eta, f, V)\). A tedious but straightforward evaluation of (note the use of a calligraphic \(S\))

\[
S_1(x) = S_1(x) - S_1(\tau) - \nabla S_1(\tau)^T(x - \tau),
\]

shows that \(S_1(x)\) satisfies [26], [56]

\[
\dot{S}_1(x) = -(f - \bar{f})^T K_p^{-1}(f - \bar{f}) - \bar{V}^T \tau (X_d - X_d')^{-1}V + (f - \bar{f})^T (P_t - \bar{P}_t),
\]

along the solutions to (5).

For the stability analysis in Section V the following technical assumption is needed on the steady state that eventually allows us to infer boundedness of solutions.\(^8\)

**Assumption 5:** (Steady state voltages and voltage angles) Let \(\bar{V} \in \mathbb{R}^n_{2,0}\) and let differences in steady state voltage angles satisfy

\[
\bar{\eta}_k \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \forall k \in E.
\]

Furthermore, for all \(i \in V\) it holds that

\[
\frac{1}{X_{di} - X_{di}'} - B_{ii} + \sum_{k \sim i,j} B_{ij}(\bar{V}_i + \bar{V}_j \sin^2(\bar{\eta}_k)) > 0.
\]

\(\diamondsuit\)

The assumption above holds if the generator reactances are small compared to the line reactances and the differences in voltage (angles) are small [56]. It is important to note that this holds for typical operation points of the power network. The main consequence of Assumption 5 is that the incremental storage function \(S_1\) now obtains a strict local minimum at a steady state satisfying (8).

**Lemma 6:** (Local minimum of \(S_1\)) Let Assumption 3 hold. Then, the incremental storage function \(S_1\) has a local minimum at \((\bar{\eta}, \bar{f}, \bar{V})\) satisfying (8).

**Proof:** Under Assumption 5, the Hessian of (43), evaluated at \((\bar{\eta}, \bar{f}, \bar{V})\), is positive definite [26, Lemma 2], [56, Proposition 1]. Consequently, \(S_1\) is strictly convex around \((\bar{\eta}, \bar{f}, \bar{V})\). The incremental storage function (44) is defined as a Bregman distance [57] associated with (43) for the points \((\eta, f, V)\) and \((\bar{\eta}, \bar{f}, \bar{V})\). Due to the strict convexity of \(S_1\) around \((\bar{\eta}, \bar{f}, \bar{V})\), (44) has a local minimum at \((\bar{\eta}, \bar{f}, \bar{V})\).

\(\blacksquare\)

B. Sliding mode control

In this subsection we recall some definitions that are essential to sliding mode control. To this end, we consider system

\[
x = \zeta(x, u),
\]

\(x \in \mathbb{R}^n, u \in \mathbb{R}^m\).

**Definition 2:** (Sliding function) The sliding function \(\sigma(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m\) is a sufficiently smooth output function of system (48).

**Definition 3:** (r-sliding manifold) The \(r\)-sliding manifold\(^9\) is given by

\[
\{x \in \mathbb{R}^n, u \in \mathbb{R}^m : \sigma = L_\zeta \sigma = \cdots = L^{(r-1)}_\zeta \sigma = 0\},
\]

\(\diamondsuit\) In case boundedness of solutions can be inferred by other means, Assumption 5 can be omitted.

\(\diamondsuit\) For the sake of simplicity, the order \(r\) of the sliding manifold is omitted in the following.
where $L^r \sigma^r(x)$ is the $(r-1)$-th order Lie derivative of $\sigma(x)$ along the vector field $\zeta(x, u)$. With a slight abuse of notation, we also write $L^r \sigma(x) = \dot{\sigma}(x)$.

**Definition 4:** $(r$–sliding mode) An $r$–order sliding mode is enforced from $t = T_r \geq 0$, when, starting from an initial condition $x(0) = x_0$, the state of $(48)$ reaches the $r$–sliding manifold $(49)$, and remains there for all $t \geq T_r$.

Furthermore, the order of a sliding mode controller is identical to the order of the sliding mode that it is aimed at enforcing.

**REFERENCES**


[10] S. Trip and C. De Persis, “Communication requirements in a master-enforcing. identical to the order of the sliding mode that it is aimed at enforcing.


