Chapter 5

Scattering in Euler’s two-center problem

In the present chapter we apply the theory developed in Chapter 4 to the spatial Euler two-center problem. Our results show that this spatial problem has non-trivial scattering monodromy of two different types: pure and mixed scattering monodromy. We show that the second type is present also in the Kepler problem.

5.1 Euler’s two-center problem

The Euler problem of two fixed centers, also known as the Euler 3-body problem, is one of the most fundamental integrable problems of classical mechanics. It describes the motion of a point particle in Euclidean space under the influence of the Newtonian force field

\[ F = -DV, \quad V = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}. \]

Here \( r_i \) are the distances of the particle to the two fixed centers and \( \mu_i \) are the strengths (the masses or the charges) of these centers. We note that the Kepler problem corresponds to the special cases when the centers coincide or when one of the strengths is zero.

The (gravitational) Euler problem was first studied by L. Euler in a series of works in the 1760s [42,44]. He discovered that this problem is integrable by putting the equations of motion in a separated form. Elliptic coordinates, which separate the problem and which are now commonly used, appeared in his later paper [44] and, at about the same time, in the work of Lagrange [66]. The systematic use of elliptic coordinates in classical mechanics was initiated by Jacobi, who used a more general form of these coordinates to integrate, among other systems, the geodesic flow on a triaxial ellipsoid; see [55] for more details.

Since the early works of Euler and Lagrange, the Euler problem and its generalizations have been studied by many authors. First classically and then, since the
works of Pauli [82] and Niessen [80] in the early 1920s, also in the setting of quantum mechanics. We indicatively mention the works [15, 25, 31, 41, 89, 98, 100, 103]. For a historical overview we refer to [50, 81]. In what follows we will mainly be interested in the spatial Euler problem.

We observe that the Euler problem is a Hamiltonian system with two additional structures: it is a scattering system and it is also integrable in the Liouville sense. The structure of a scattering system comes from the fact that the potential

\[ V(q) \to 0, \quad ||q|| \to \infty, \]

decays at infinity sufficiently fast (is of long range; see Section 4.2). It allows one to compare a given set of initial conditions at \( t = -\infty \) with the outcomes at \( t = +\infty \). Liouville integrability comes from the fact that the system is separable; the three commuting integrals of motion are:

- the energy function — the Hamiltonian,
- the separation constant; see Section 5.2
- the component of the angular momentum about the axis connecting the two centers.

Separately these two structures of the Euler problem have been discussed in the literature. Scattering has been studied, for instance, in [60, 89]. The corresponding Liouville fibration has been studied in [100] — from the perspective of Fomenko theory [11, 49], action coordinates and Hamiltonian monodromy [27]. Following the point of view developed in Chapter 4, we shall consider both of the structures together and show that the Euler problem has non-trivial scattering invariants, which we shall call purely scattering and mixed scattering monodromy, cf. [5, 30, 38, 61, 72]. For completeness, the qualitatively different case of Hamiltonian monodromy will be also discussed.

The chapter is organized as follows. The problem is defined in Section 5.2. Bifurcation diagrams are given in Section 5.3. In Section 5.4 we discuss scattering monodromy of the problem. Hamiltonian monodromy is addressed in Subsection 5.4.3. Additional details are presented in the miscellaneous Section 5.5.

5.2 Separation procedure and regularization

We start with the 3-dimensional Euclidean space \( \mathbb{R}^3 \) and two distinct points in this space, denoted by \( o_1 \) and \( o_2 \). Let \( q = (x, y, z) \) be Cartesian coordinates in \( \mathbb{R}^3 \) and let \( p = (p_x, p_y, p_z) \) be the conjugate momenta in \( T^*_q \mathbb{R}^3 \). The Euler two-center problem can be defined as a Hamiltonian system on \( T^* (\mathbb{R}^3 \setminus \{o_1, o_2\}) \) with a Hamiltonian function \( H \) given by

\[ H = \frac{||p||^2}{2} + V(q), \quad V(q) = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}, \]  

(5.1)
where \( r_i : \mathbb{R}^3 \to \mathbb{R} \) is the distance to the center \( o_i \). The strengths of the centers \( \mu_i \) can be both positive and negative; without loss of generality we assume that the center \( o_1 \) is stronger, that is, \( |\mu_2| \leq |\mu_1| \).

**Remark 5.2.1.** When \( \mu_i > 0 \) (resp., \( \mu_i < 0 \)) the center \( o_i \) is attractive (resp., repulsive). The cases \( \mu_1 \neq \mu_2 = 0 \) and \( \mu_2 \neq \mu_1 = 0 \) correspond to a Kepler problem. In the case \( \mu_1 = \mu_2 = 0 \) the dynamics is trivial and we have the free motion \((t, q_0, p_0) \mapsto (q_0 + tp_0, p_0)\).

### 5.2.1 Separation and integrability

Without loss of generality we assume \( o_i = (0, 0, (-1)^i a) \) for some \( a > 0 \), so that, in particular, the fixed centers \( o_1 \) and \( o_2 \) are located on the \( z \)-axis in the configuration space. Rotations around the \( z \)-axis leave the potential function \( V \) invariant. It follows that (the \( z \)-component of) the angular momentum

\[
L_z = xp_y - yp_x
\]

commutes with \( H \), that is, \( L_z \) is a first integral. It is known [41][103] that there exists another first integral given by

\[
G = H + \frac{1}{2}(L^2 - a^2(p_x^2 + p_y^2)) + a(z + a)\frac{\mu_1}{r_1} - a(z - a)\frac{\mu_2}{r_2},
\]

where \( L^2 = L_x^2 + L_y^2 + L_z^2 \) is the squared angular momentum. The expression for the integral \( G \) can be obtained using separation in elliptic coordinates, as described below. It will follow from the separation procedure that the function \( G \) commutes both with \( H \) and with \( L_z \), which means that the problem of two fixed centers is Liouville integrable.

Consider prolate ellipsoidal coordinates \((\xi, \eta, \varphi)\):

\[
\xi = \frac{1}{2a}(r_1 + r_2), \quad \eta = \frac{1}{2a}(r_1 - r_2), \quad \varphi = \text{Arg}(x + iy).
\]

Here \( \xi \in [1, \infty), \eta \in [-1, 1] \), and \( \varphi \in \mathbb{R}/2\pi\mathbb{Z} \). Let \( p_\xi, p_\eta, p_\varphi = L_z \) be the conjugate momenta and \( l \) be the value of \( L_z \). In the new coordinates the Hamiltonian \( H \) has the form

\[
H = \frac{H_\xi + H_\eta}{\xi^2 - \eta^2},
\]

where

\[
H_\xi = \frac{1}{2a^2}(\xi^2 - 1)p_\xi^2 + \frac{1}{2a^2}\frac{l^2}{\xi^2 - 1} - \frac{\mu_1 + \mu_2}{a}\xi
\]

and

\[
H_\eta = \frac{1}{2a^2}(1 - \eta^2)p_\eta^2 + \frac{1}{2a^2}\frac{l^2}{1 - \eta^2} + \frac{\mu_1 - \mu_2}{a}\eta.
\]
Multiplying Eq. (5.5) by $\xi^2 - \eta^2$ and separating we get the first integral

$$G = \xi^2 H - H\xi = \eta^2 H + H\eta.$$  

In original coordinates $G$ has the form given in Eq. (5.3). Since $L_z = p_\phi$, the function $G$ commutes both with $H$ and with $L_z$.

### 5.2.2 Regularization

We note that in the case when one of the strengths is attractive, collision orbits are present and, consequently, the flow of $H$ on $T^*(\mathbb{R}^3 \setminus \{o_1, o_2\})$ is not complete. This complication is, however, not essential for our analysis since collision orbits, as in the Kepler case, can be regularized. More specifically, there exists a 6-dimensional symplectic manifold $(P, \omega)$ and a smooth Hamiltonian function $\tilde{H}$ on $P$ such that

1. $(T^*(\mathbb{R}^3 \setminus \{o_1, o_2\}), dq \wedge dp)$ is symplectically embedded in $(P, \omega)$,
2. $H = \tilde{H}|_{T^*(\mathbb{R}^3 \setminus \{o_1, o_2\})}$,
3. The flow of $\tilde{H}$ on $P$ is complete.

This result is essentially due to [60, Proposition 2.3], where a similar statement is proved for the gravitational planar problem. The planar problem in the case of arbitrary strengths can be treated similarly (note that collisions with a repulsive center are not possible). The spatial case follows from the planar case since collisions occur only when $L_z = 0$. We note that the integrals $L_z$ and $G$ can be also extended to $P$.

One important property of the regularization is that the extensions of the integrals to $P$, which will be also denoted by $H$, $L_z$ and $G$, form a completely integrable system. In particular, the Arnol’d-Liouville theorem [3] applies. In what follows we shall work on the regularized space $P$.

### 5.3 Bifurcation diagrams

Before we move further and discuss scattering in the Euler problem, we shall compute the bifurcation diagrams of the integral map $F = (H, L_z, G)$, that is, the set of the critical values of this map. We distinguish two cases, depending on whether $L_z$ is zero or different from zero. The bifurcation diagrams are obtained by superimposing the critical values found in these two cases. By a choice of units we assume that $a = 1$.

#### 5.3.1 The case $L_z = 0$

Since $L_z = 0$, the motion is planar. We assume that it takes place in the $xz$-plane. Consider the elliptic coordinates $(\lambda, \nu) \in \mathbb{R} \times S^1[-\pi, \pi]$ defined by

$$x = \sinh \lambda \cos \nu, \ z = \cosh \lambda \sin \nu.$$
5.3. BIFURCATION DIAGRAMS

The level set of constant $H = h, L_z = l = 0$ and $G = g$ in these coordinates is given by the equations

\[ p_\lambda^2 = 2h \cosh^2 \lambda + 2(\mu_1 + \mu_2) \cosh \lambda - 2g, \]
\[ p_\nu^2 = -2h \sin^2 \nu - 2(\mu_1 - \mu_2) \sin \nu + 2g, \]

where $p_\lambda$ and $p_\nu$ are the momenta conjugate to $\lambda$ and $\nu$. The value $(h, 0, g)$ is critical when the Jacobian matrix corresponding to these equations does not have full rank. Computation yields lines

\[ \ell_1 = \{ g = h + \mu_2 - \mu_1, l = 0 \}, \]
\[ \ell_2 = \{ g = h + \mu_1 - \mu_2, l = 0 \} \]

and \[ \ell_3 = \{ g = h + \mu, l = 0 \}, \] where $\mu = \mu_1 + \mu_2$.

Points that do not correspond to any physical motion must be removed from the obtained set (allowed motion corresponds to the regions where the squared momenta are positive).

**Remark 5.3.1.** The corresponding diagrams in the planar problem are given in Section 5.5; see Fig. 5.5 and 5.6. We note that in the planar case the set of the regular values of $F$ consists of contractible components and hence the topology of the regular part of the Liouville fibration is trivial. Interestingly, this is not the case if the dimension of the configuration space is $n = 3$.

We note that the singular Liouville foliation has non-trivial topology already in the planar case. The corresponding bifurcations, in the sense of Fomenko theory \[11,12,46,47,49\], have been studied in \[59,100\].

5.3.2 The case $L_z \neq 0$

In order to compute the critical values in this case it is convenient to use the ellipsoidal coordinates $(\xi, \eta)$. (We note that for $L_z \neq 0$ the $z$-axis is inaccessible, so $(\xi, \eta)$ are non-singular.) The level set of constant $H = h, L_z = l$ and $G = g$ in these coordinates is given by the equations

\[ p_\xi^2 = \frac{(\xi^2 - 1)(2h\xi^2 + 2(\mu_1 + \mu_2)\xi - 2g) - l^2}{(\xi^2 - 1)^2}, \]
\[ p_\eta^2 = \frac{(1 - \eta^2)(-2h\eta^2 - 2(\mu_1 - \mu_2)\eta + 2g) - l^2}{(1 - \eta^2)^2}. \]

The value $(h, l, g)$ with $l \neq 0$ is critical when the corresponding Jacobian matrix does not have a full rank. Computation yields the following sets of critical values:

\[ \left\{ g = h(2\xi^2 - 1) + \frac{(\mu_1 + \mu_2)(3\xi^2 - 1)}{2\xi}, l^2 = -\frac{(\mu_1 + \mu_2 + 2h\xi)(-1 + \xi^2)^2}{\xi} \right\}, \]
CHAPTER 5. SCATTERING IN EULER’S TWO-CENTER PROBLEM

Figure 5.1: Positive energy slices of the bifurcation diagram for the spatial Euler problem, attractive case. The black points correspond to the critical lines $\ell_i$.

$$\left\{ g = h(2\eta^2 - 1) + \frac{(\mu_1 - \mu_2)(3\eta^2 - 1)}{2\eta}, l^2 = -\frac{(\mu_1 - \mu_2 + 2h\eta)(-1 + \eta^2)^2}{\eta} \right\},$$

where $\xi > 1$ and $-1 < \eta < 1$. As above, points that do not correspond to any physical motion must be removed.

Representative positive energy slices in the gravitational case $0 < \mu_2 < \mu_1$ are given in Fig. 5.1. The case of arbitrary strengths $\mu_i$ is similar. The structure of the corresponding diagrams can partially be deduced from the diagrams obtained in the planar case; see Section 5.5.

5.4 Scattering in Euler’s problem

In this section we study scattering in the Euler problem using the reference Kepler Hamiltonians

$$H_{r_1} = \frac{1}{2}p^2 - \frac{\mu_1 - \mu_2}{r_1}$$

and

$$H_{r_2} = \frac{1}{2}p^2 - \frac{\mu_2 - \mu_1}{r_2},$$

identified in Theorem 4.3.4. We shall show that the Euler problem has non-trivial scattering monodromy of two different types, namely, pure and mixed scattering monodromy, that the Hamiltonian and the mixed scattering monodromy remain in the limiting case of the Kepler problem, and that the Hamiltonian monodromy is present also in the spatial free flow.
5.4. SCATTERING IN EULER’S PROBLEM

5.4.1 Scattering map

Let $F = (H, L_z, G)$ denote the integral map of the Euler problem. Let $N$ be a submanifold of

$$NT = \{(h, l, g) \in \text{image}(F) \mid F^{-1}(h, l, g) \subset s\}.$$  \hspace{1cm} (5.6)

The manifold $F^{-1}(N)$ is an invariant submanifold of the phase space $P$, which contains scattering states only. Following the construction in Sections 4.2 and 4.3 we can define the scattering maps $S: B \rightarrow B$ with respect to $H$, the reference Kepler Hamiltonian $H_r = H_{r_1}$ or $H_r = H_{r_2}$, where

$$H_{r_1} = \frac{1}{2} p^2 - \frac{\mu_1 - \mu_2}{r_1} \quad \text{and} \quad H_{r_2} = \frac{1}{2} p^2 - \frac{\mu_2 - \mu_1}{r_2},$$

and $B = F^{-1}(N)/g^t_H$ as in Section 4.2.

Remark 5.4.1. We recall that the scattering map $S$ is defined by

$$S = (A^-)^{-1} \circ A^-_r \circ (A^+_r)^{-1} \circ A^+,$$

where

$$A^\pm = (\hat{p}^\pm, q^\pm): s^\pm / g^t_H \rightarrow AS \quad \text{and} \quad A^\pm_r = (\hat{p}^\pm, q^\pm_r): s^\pm_r / g^t_{Hr} \rightarrow AS$$

map $s^\pm \subset P$ and $s^\pm_r$ to the asymptotic states $AS$. Here the index $r$ refers to a reference system ($H_{r_1}$ or $H_{r_2}$ in our case).

Remark 5.4.2. We note that the potential

$$V = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}$$

of the Euler problem is of short range with respect to to $\tilde{V}(q) = -(\mu_1 + \mu_2)/\|q\|$, which is a Kepler potential. The reference potentials are Kepler potentials and are therefore rotationally symmetric. It follows that the decay Assumptions 4.2.3 are met.

5.4.2 Scattering monodromy

First, we consider the case of a gravitational problem ($0 < \mu_2 < \mu_1$) with $H_r = H_{r_2}$ as the reference Kepler Hamiltonian. The other cases can be treated similarly and will be addressed in Subsection 5.4.4; see Table 5.1.

For sufficiently large $h_0$ the $h = h_0$ slice of the bifurcation diagram has the form shown in Fig. 5.2 Let $\gamma_i, i = 1, 2, 3$, be a simple closed curve in

$$NT_{h_0} = \{(h, g, l) \in NT \mid h = h_0\}$$
that encircles the critical line $\ell_i$, where

$$
\ell_1 = \{ g = h + (\mu_2 - \mu_1), \ l = 0 \} , \ \ell_2 = \{ g = h + (\mu_1 - \mu_2), \ l = 0 \} \quad \text{and} \quad \\
\ell_3 = \{ g = h + (\mu_1 + \mu_2), \ l = 0 \} .
$$

For each $\gamma_i$, consider the torus bundle $F_i : E_i \rightarrow \gamma_i$, where the total space $E_i$ is obtained by gluing the ends of the fibers of $F$ over $\gamma_i$ via the scattering map $S$. We recall that scattering monodromy along $\gamma_i$ with respect to $H_r$ is defined as the usual monodromy of the torus bundle $F_i : E_i \rightarrow \gamma_i$; see Definition 4.3.7.

**Remark 5.4.3.** Alternatively, one can define $F_i : E_i \rightarrow \gamma_i$ by gluing the fibers of the original and the reference integral maps at infinity. Both definitions are equivalent in the sense that the monodromy of the resulting torus bundles are the same.

Consider a starting point $\gamma_i(t_0) \in \gamma_i$ in the region where $l > 0$. We choose a basis $(c_{\xi}, c_\eta, c_\phi)$ of the first homology group $H_1(F_i^{-1}(\gamma_i(t_0))) \simeq \mathbb{Z}^3$ as follows. The cycle $c_\xi = c_{\xi}^0 \cup c_{\xi}^r$ is obtained by gluing the non-compact $\xi$-coordinate lines
5.4. SCATTERING IN EULER’S PROBLEM

$c_\xi^o$ for the original and $c_\xi^r$ for the reference systems at infinity. In other words, for we glue the lines

$$p_\xi^2 = \frac{(\xi^2 - 1)(2h\xi^2 + 2(\mu_1 + \mu_2)\xi - 2g) - l^2}{(\xi^2 - 1)^2}$$

on $F^{-1}(\gamma_i(t_0)), \gamma_i(t_0) = (h, g, l)$, and

$$p_\xi^2 = \frac{(\xi^2 - 1)(2h\xi^2 + 2(\mu_2 - \mu_1)\xi - 2g) - l^2}{(\xi^2 - 1)^2}$$

on the reference fiber $F_r^{-1}(\gamma_i(t_0))$ at the limit points $\xi = \infty$, $p_\xi = \pm \sqrt{2h}$. The cycles $c_\eta$ and $c_\varphi$ are such that their projections onto the configuration space coincide with coordinate lines of $\eta$ and $\varphi$, respectively. In other words, the cycle $c_\eta$ on $F^{-1}(\gamma_i(t_0))$ is given by

$$p_\eta^2 = \frac{(1 - \eta^2)(-2h\eta^2 - 2(\mu_1 - \mu_2)\eta - 2g) - l^2}{(1 - \eta^2)^2}$$

and $c_\varphi$ is an orbit of the circle action given by the Hamiltonian flow of the momentum $L_z$. We have the following result.

**Theorem 5.4.4.** The scattering monodromy matrices $M_i$ along $\gamma_i$ with respect to the reference Hamiltonian $H_r$ and the natural basis $(c_\xi, c_\eta, c_\varphi)$ have the form

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Proof. Case 1**, loop $\gamma_1$. First we note that the cycle $c_\varphi$ is preserved under the parallel transport along $\gamma_1$. This follows from the fact that $L_z$ generates a free fiber-preserving circle action on $E_i$. The cycles $c_\xi$ and $c_\eta$ can be naturally transported only in the regions where $l \neq 0$. We thus need to understand what happens at the critical plane $l = 0$.

Let $R > 1$ be a sufficiently large number. Then

$$E_{1,R} = \{x \in E_1 \mid \xi(x) > R\}$$

has exactly two connected components, which we denote by $E_{1,R}^+ \text{ and } E_{1,R}^-$. We define a 1-form $\alpha$ on (a part of) $E_i$ by the formula

$$\alpha = pdq - \chi(\xi)p_\xi(h, g, l, \xi)d\xi,$$

where $\chi(\xi)$ is a bump function such that

(i) $\chi(\xi) = 0$ when $\xi < R;$
(ii) \( \chi(\xi) = 1 \) when \( \xi > 1 + R \).

The square root function \( p_{\xi}(h, g, l, \xi) \) is assumed to be positive on \( E_{1,R}^+ \) and negative on \( E_{1,R}^- \). By construction, the 1-form \( \alpha \) is well-defined and smooth on \( E_i \) outside collision points. Since

\[
d\alpha = dp \wedge dq = -\omega \quad \text{on} \quad F^{-1}(\gamma_i) \cup F_{r}^{-1}(\gamma_i) \subset E_i,
\]

we have that \( d\alpha = 0 \) on each fiber of \( F_i \).

Consider the modified actions with respect to the form \( \alpha \):

\[
I_\varphi = \frac{1}{2\pi} \int_{c_\varphi} \alpha, \quad I_\eta = \frac{1}{2\pi} \int_{c_\eta} \alpha \quad \text{and} \quad I_{\xi}^{mod} = \frac{1}{2\pi} \int_{c_\xi} \alpha.
\]

The modified actions are well defined and, in view of \( d\alpha = 0 \), depend only on the homology classes of \( c_\xi, c_\eta \) and \( c_\varphi \). It follows that \( I_\varphi \) and \( I_\eta \) coincide with the ‘natural’ actions (defined as the integrals over the usual 1-form \( pdq \)). We note that the ‘natural’ \( \xi \)-action

\[
I_\xi = \frac{1}{2\pi} \int_{c_\xi} pdq
\]

diverges, cf. [30]. From the continuity of the modified actions at \( l = 0 \) it follows that the corresponding scattering monodromy matrix has the form

\[
M_1 = \begin{pmatrix}
1 & 0 & m_1 \\
0 & 1 & m_2 \\
0 & 0 & 1
\end{pmatrix}.
\]

Since the modified actions do not have to be smooth at \( l = 0 \), the integers \( m_1 \) and \( m_2 \) are not necessarily zero. In order to compute these integers we need to compare the derivatives \( \partial_l I_\eta \) and \( \partial_l I_\xi \) at \( l \to \pm 0 \).

A computation of the corresponding residues gives

\[
\lim_{l \to \pm 0} \partial_l I_\eta = \lim_{l \to \pm 0} \frac{1}{2\pi} \frac{1}{2\pi} \int_{c_\eta} pdq = \begin{cases} 
0, & \text{when } g < h + \mu_2 - \mu_1, \\
\mp 1/2, & \text{when } \mu_2 - \mu_1 < g - h < \mu_1 - \mu_2,
\end{cases}
\]

and

\[
\lim_{l \to \pm 0} \partial_l I_{\xi}^{mod} = \lim_{l \to \pm 0} \left( \frac{1}{2\pi} \frac{1}{2\pi} \int_{c_\xi} pdq - \frac{1}{2\pi} \frac{1}{2\pi} \int_{c_\xi} pdq \right) - \lim_{l \to \pm 0} \frac{1}{2\pi} \int_{c_\xi} \chi(\xi)p_{\xi}(h, g, l, \xi)d\xi = 0
\]
(for the two ranges of $g$). It follows that $m_1 = 0$ and $m_2 = 1$.

**Case 2**, loop $\gamma_2$. This case is similar to **Case 1**. The corresponding limits are given by

$$
\lim_{l \to \pm 0} (\partial_I I_\eta, \partial_I I^{\text{mod}}_\xi) = \begin{cases} 
(\mp 1/2, 0), & \text{when } \mu_2 - \mu_1 < g - h < \mu_1 - \mu_2, \\
(\mp 1, \pm 1/2), & \text{when } \mu_1 - \mu_2 < g - h < \mu_1 + \mu_1.
\end{cases}
$$

**Case 3**, loop $\gamma_3$. The computation in this case is also similar to **Case 1**. The corresponding limits are given by

$$
\lim_{l \to \pm 0} (\partial_I I_\eta, \partial_I I^{\text{mod}}_\xi) = \begin{cases} 
(\mp 1, \pm 1/2), & \text{when } \mu_1 - \mu_2 < g - h < \mu_1 + \mu_2, \\
(\mp 1, 0), & \text{when } h + \mu_1 + \mu_2 < g.
\end{cases}
$$

**Remark 5.4.5.** One difference between **Case 3** and the other cases is the topology of the critical fiber, around which scattering monodromy is defined. In **Case 3** the critical fiber is the product of a pinched cylinder and a circle, whereas in the other cases it is the product of a pinched torus and a real line. This implies, in fact, that **Case 3** is purely scattering, whereas in the other cases Hamiltonian monodromy is present.

Interestingly, Theorem 5.4.4 admits the following geometric proof in the purely scattering case.

**Proof for Case 3 of Theorem 5.4.4.** The action

$$
I'_\eta = \begin{cases} 
I_\eta, & \text{if } l \geq 0 \\
I_\eta - 2l, & \text{if } l < 0.
\end{cases}
$$

is smooth and globally defined (over $\gamma_3$). Moreover, the corresponding circle action extends to a free action in $F^{-1}_3(D_3)$, where $D_3 \subset NT_{h_0}$ is a 2-disk such that $\partial D_3 = \gamma_3$. Since there is also a circle action given by $I_\varphi$, the result can be also deduced from the general theory developed in [40,71].

**Remark 5.4.6.** We note that from the last proof it follows that the choice of a reference Hamiltonian does not affect the result in the purely scattering case. This agrees with the point of view presented in [38]. For the curves $\gamma_1$ and $\gamma_2$, when monodromy is mixed scattering, the two reference Kepler Hamiltonians give different results; see Table 5.1. Interestingly, for these curves, yet another result is obtained if one puts the problem inside an ellipsoidal billiard; in this case, the two monodromy matrices coincide with $M_1$ in Theorem 5.4.4; see [76].

We can now prove the following theorem, which describes the scattering map in the purely scattering case of the curve $\gamma_3$. 

Theorem 5.4.7. The scattering map $S: B_3 \to B_3$, where $B_3 = F^{-1}(\gamma_3)/g_H^t$, is a Dehn twist. The push-forward map is conjugate in $SL(3, \mathbb{Z})$ to

$$S_* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Proof. The proof is similar to the proof given in Theorem 4.4.1. The scattering map $S$ allows one to consider the compactified torus bundle

$$\text{Pr}: F^{-1}(\gamma_3)^c \to S^1 = \mathbb{R} \cup \{\infty\},$$

where $\mathbb{R}$ corresponds to the time. The torus bundle $F_3: E_3 \to \gamma_3$ has the same total space, but is fibered over $\gamma_3$. By Theorem 5.4.4 the monodromy of the bundle $F_3: E_3 \to \gamma_3$ is given by the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Then the monodromy of the first bundle $\text{Pr}: F^{-1}(\gamma_3)^c \to S^1$ is the same, for otherwise the total spaces would be different. The result follows.

Remark 5.4.8. It follows from the proof and our topological results on compact monodromy that Theorem 5.4.7 holds for any $\mu_i \neq 0$ and for any regular closed curve $\gamma \subset NT$ such that

1. The energy value $h$ is positive on $\gamma$;
2. $\gamma$ encircles the critical line $\{g = h + \mu_1 + \mu_2, l = 0\}$ exactly once and does not encircle any other line of critical values;
3. $\gamma$ does not cross critical values of $F$.

It can be shown that such a curve $\gamma$ always exists; an example is given in Fig. 5.3. We note that the third condition can be weakened in the case $-\mu_1 < \mu_2 < 0$. In this case the attraction of $\mu_1$ dominates the repulsion of $\mu_2$ and, as a result, bound motion coexists with unbound motion for a range of positive energies. Instead of $F^{-1}(\gamma)$ one may consider its unbounded component.

Remark 5.4.9. (Quantum scattering monodromy) As we noted before, Euler potential is not of short range with respect to the reference Kepler potentials given in Theorem 4.3.4. However, one can still produce a spectrum by considering a modified action difference, in a similar way to what we did in the planar case in Section 4.3.
Observe that the actions $I_\varphi = \frac{1}{2\pi} \int_{c_\varphi} \alpha$ and $I_\eta = \frac{1}{2\pi} \int_{c_\eta}$ are well-defined and coincide for the original and the reference systems. We choose the modified action difference as follows

$$I_{\text{mod}} = \lim_{R \to \infty} I_{\text{mod},R},$$

where

$$\pi I_{\text{mod},R} = \int_{\xi_0}^{R} \frac{\sqrt{2(\xi^2 - 1)(h\xi^2 + (\mu_1 + \mu_2)\xi - g) - l^2}}{(\xi^2 - 1)} d\xi$$

$$- \int_{\xi_0}^{R} \frac{\sqrt{2(\xi^2 - 1)(h\xi^2 + (\mu_1 - \mu_2)\xi - g) - l^2}}{(\xi^2 - 1)} d\xi - \frac{2\mu_2 \ln(2\sqrt{2hR})}{\sqrt{2h}}.$$
An analogue of the EBK quantization yields Fig 5.4, which shows the projection of a slice of the spectrum to the \((L_z, H)\)-plane. The slice is defined by

\[
I_\eta = \begin{cases} 
    \hbar(n_\eta + 0.5), & \text{if } l \geq 0 \\
    \hbar(n_\eta + 0.5) - 2l, & \text{if } l < 0.
\end{cases}
\]

The constants were chosen as follows:

\[\mu_1 = 0.7, \mu_2 = 0.3, \hbar = 1/200 \text{ and } n_\eta = 341.\]

5.4.3 Topology

As we have noted before, alongside non-trivial scattering monodromy, the Euler problem admits also non-trivial Hamiltonian monodromy, which is an intrinsic invariant of the system.

Here we consider the generic case of \(|\mu_1| \neq |\mu_2| \neq 0\) in the case of positive energies. The case of negative energies is similar — it has been discussed in detail in [100]. The critical cases can be easily computed from the generic case by
5.4. SCATTERING IN EULER’S PROBLEM

considering curves that encircle more than one of the singular lines

$$\ell_1 = \{ g = h + (\mu_2 - \mu_1), l = 0 \}, \; \ell_2 = \{ g = h + (\mu_1 - \mu_2), l = 0 \} \quad \text{and}$$

$$\ell_3 = \{ g = h + (\mu_1 + \mu_2), l = 0 \}.$$ 

Let $\gamma_i$ be a closed curve that encircles only the critical line $\ell_i$; see Fig. 5.3. The fibration $F: F^{-1}(\gamma_i) \to \gamma_i$ is a $T^2 \times \mathbb{R}$-bundle. The following theorem shows that the Hamiltonian monodromy is non-trivial along the curves $\gamma_1$ and $\gamma_2$ and is trivial along $\gamma_3$.

**Theorem 5.4.10.** The Hamiltonian monodromy of $F: F^{-1}(\gamma_i) \to \gamma_i$, $i = 1, 2$, is conjugate in $SL(2, \mathbb{Z}) \subset SL(3, \mathbb{Z})$ to

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Here the right-bottom $2 \times 2$ block acts on $T^2$ and the left-top $1 \times 1$ block on $\mathbb{R}$.

**Proof.** The result follows from the proof of Theorem 5.4.4. For completeness, we give an independent proof below.

After the reduction of the surface $H^{-1}(h_i)$ with respect to the flow $g'_t$ we get a singular $T^2$ torus fibration over a disk $D_i$, $\partial D_i = \gamma_i$, with exactly one focus-focus point. The result then follows from [67, 75, 105]. This argument applies to both of the lines $\ell_1$ and $\ell_2$. Since the flow of $L_z$ gives a global circle action, the monodromy matrix $M$ is the same in both cases; see [19].

**Theorem 5.4.11.** The Hamiltonian monodromy of $F: F^{-1}(\gamma_3) \to \gamma_3$ is trivial.

**Proof.** Observe that the Hamiltonian flows of $I_\varphi$,

$$I'_\eta = \begin{cases} I_\eta, & \text{if } l \geq 0 \\ I_\eta - 2l, & \text{if } l < 0. \end{cases}$$

and $H$ generate a global $T^2 \times \mathbb{R}$ action on $F^{-1}(\gamma_3)$. It follows that the bundle $F: F^{-1}(\gamma_3) \to \gamma_3$ is principal. Since $\gamma_3$ is a circle, it is also trivial.

We note that Hamiltonian monodromy is an intrinsic invariant of the Euler problem, related to the non-trivial topology of the integral map $F$. Interestingly, it is also present in the critical cases:

1. $\mu_1 = \mu_2$ (symmetric Euler problem) [100],
2. $\mu_1$ or $\mu_2 = 0$ (Kepler problem) [33] and
3. $\mu_1 = \mu_2 = 0$ (the free flow).

In the case of bound motion (1) and (2) are due to [100] and [33], respectively. We recall that from the scattering perspective, Hamiltonian monodromy is recovered if one considers the original Hamiltonian $H$ also as a reference.
These results (1)-(2) follow from the computation done in the generic case by taking the product of the corresponding matrices. For instance, for the free flow, the Hamiltonian monodromy is given by the product $M_0 = EM^2$, so that

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$  

This can also been proven by considering the quotient space with respect to the flow of $H = H_0$ and by applying the results of Chapters 1 and 2. Note that the number 2 is the Euler number given by the circle associated to $L_z$.

We also observe that in the case of the free flow we get a quadratic spherical pendulum (see Section 2.3.2) after reducing the Hamiltonian flow.

**Proposition 5.4.12.** Consider the integral map $F = (H, L_z, G)$ in the case when $\mu_1 = \mu_2 = 0$ (the free flow). Fix a positive energy and consider the integrable system obtained after the symplectic reduction of $F$ with respect to the free flow. Then the reduced system is given by a quadratic spherical pendulum.

**Proof.** The proposition follows from the expression of $L_z$ and $G$ in terms of the asymptotic direction and the impact parameter; see Eq. 4.1.  

---

Table 5.1: Scattering monodromy, general case.

<table>
<thead>
<tr>
<th></th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
<th>$\gamma_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generic</td>
<td>$</td>
<td>\mu_1</td>
<td>\neq</td>
</tr>
<tr>
<td>Critical</td>
<td>$\gamma_1 = \gamma_2 &lt; 0$</td>
<td>$m = -1, n = 1$</td>
<td>$m = 0, n = 1$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_1 = \gamma_2 &gt; 0$</td>
<td>$m = -1, n = 2$</td>
<td>$m = 1, n = 0$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_1 = \gamma_2 = 0$</td>
<td>$m = 0, n = 2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\gamma_1 &lt; \gamma_2$</td>
<td>$m = -1, n = 1$</td>
<td>$m = 1, n = 1$</td>
</tr>
</tbody>
</table>

Scattering monodromy w.r.t. $H_{r_2}$

|        | $\mu_1 \neq \mu_2$ | $m = 0, n = 1$ | $m = -1, n = 1$ | $m = 1, n = 0$ |
| Critical| $\gamma_1 = \gamma_2 < 0$ | $m = 0, n = 1$ | $m = -1, n = 1$ | $m = 1, n = 0$ |
|        | $\gamma_1 = \gamma_2 > 0$ | $n = 1$ | $m = 0, n = 1$ | $m = 0$, $m = 1$ |
|        | $\gamma_1 < \gamma_2$ | $m = 0, n = 1$ | $m = 1, n = 1$ | $m = 0, n = 1$ |
5.4.4 General case

Here we consider the case of arbitrary strengths $\mu_i$. We observe that the scattering monodromy matrices with respect to the reference Kepler Hamiltonians $H_{r_1}$ and $H_{r_2}$ are necessarily of the form

$$
\begin{pmatrix}
1 & 0 & m \\
0 & 1 & n \\
0 & 0 & 1
\end{pmatrix}
$$

for some integers $m$ and $n$. These integers (for different choices of the strengths $\mu_i$ and the critical lines $\ell_i$) are given in Table 5.1.

Remark 5.4.13. We note that one can compute the monodromy matrices in the critical cases from the matrices found in the generic cases. Specifically, it is sufficient to consider the curves that encircle more than one critical line $\ell_i$ and multiply the monodromy matrices found around each of these lines. For instance, the monodromy matrix around the curve $g = h$ in the free flow equals the product of the three monodromy matrices found in (any) generic Euler problem.

5.5 Miscellaneous

In this section we present the bifurcation diagrams for the planar Euler problem and prove Theorem 4.3.4.

5.5.1 Bifurcation diagrams for the planar problem

Here we give the bifurcation diagrams of the planar Euler problem in the case of arbitrary strengths $\mu_i$. The computation has been performed in Section 5.3, more details can be found in [25, 88, 100].

The computation of Section 5.3 yields the following critical lines

$$
\ell_1 = \{ g = h + \mu_2 - \mu_1 \}, \quad \ell_2 = \{ g = h + \mu_1 - \mu_2 \}
$$

and

$$
\ell_3 = \{ g = h + \mu \}, \quad \mu = \mu_1 + \mu_2,
$$

(5.7)

and the critical curves

$$
\{ g = \mu \cosh \lambda / 2, \quad h = -\mu / 2 \cosh \lambda \},
$$

$$
\{ g = (\mu_1 - \mu_2) \sin \nu / 2, \quad h = (\mu_2 - \mu_1) / 2 \sin \nu \}.
$$

Points that do not correspond to any physical motion must be removed from the obtained set. The resulting diagrams are given in Figs. 5.5 and 5.6. Here we distinguish two cases: generic case when the strengths $|\mu_1| \neq |\mu_2| \neq 0$ and the remaining critical cases.
We note that the critical cases occur when

$$|\mu_1| = |\mu_2| \text{ or when } \mu_1 \mu_2 = 0.$$  

In the case $\mu_1 = -\mu_2 \neq 0$, the attraction of one of the centers equalizes the repulsion of the other center, making the bifurcation diagram qualitatively different from the cases when $-\mu_1 < \mu_2 < 0$ or $0 < \mu_2 < -\mu_1$. However, we still have the three different critical lines $\ell_1$, $\ell_2$ and $\ell_3$. In the other critical cases collisions of the lines $\ell_i$ occur. For instance, $\mu_1 = 0$ implies that $\ell_1 = \ell_3$ and so on. A similar situation takes place in the spatial problem.

Figure 5.5: Bifurcation diagrams for the planar problem, generic cases $|\mu_1| \neq |\mu_2| \neq 0$. Top: attractive (left), repulsive (right). Bottom: mixed.
Figure 5.6: Bifurcation diagrams for the planar problem, non-generic cases $|\mu_1| = |\mu_2|$ or $\mu_1\mu_2 = 0$. From left to right, from top to bottom: symmetric attractive, anti-symmetric, symmetric repulsive, free flow, attractive Kepler problem, repulsive Kepler problem.
5.5.2 Kepler Hamiltonians

Here we prove the following result, which we have stated previously in Chapter 4. It shows that the Euler problem has two natural reference Hamiltonians when the strengths $\mu_1 \neq \mu_2$ and one otherwise.

**Theorem 5.5.1.** Among all Kepler Hamiltonians only

$$H_{r_1} = \frac{1}{2} p^2 - \frac{\mu_1 - \mu_2}{r_1} \quad \text{and} \quad H_{r_2} = \frac{1}{2} p^2 - \frac{\mu_2 - \mu_1}{r_2}$$

are reference Hamiltonians of $F = (H, L_z, G)$. In particular, the free Hamiltonian is a reference Hamiltonian of $F$ only in the case $\mu_1 = \mu_2$.

**Proof. Sufficiency.** Consider the Hamiltonian $H_{r_1}$. Let

$$G_{r_1} = H_{r_1} + \frac{1}{2} (L^2 - a^2 (p_z^2 + p_y^2)) + a (z + a) \frac{\mu_1 - \mu_2}{r_1}.$$

From Section 5.2.1 (see also Eq. (5.3)) it follows that the functions $H_{r_1}, L_z$ and $G_{r_1}$ Poisson commute. This implies that any trajectory $g_{t} H_{r_1}(x)$ belongs to the common level set of $F_{r_1} = (H_{r_1}, L_z, G_{r_1})$. For a scattering trajectory we thus get

$$F_{r_1} \left( \lim_{t \to +\infty} g_{t} H_{r_1} \right) = F_{r_1} \left( \lim_{t \to -\infty} g_{t} H_{r_1} \right).$$

A straightforward computation of the limit shows that also

$$F \left( \lim_{t \to +\infty} g_{t} H_{r_1} \right) = F \left( \lim_{t \to -\infty} g_{t} H_{r_1} \right).$$

The case of $H_{r_2}$ is completely analogous.

**Necessity.** Without loss of generality we assume $\mu_2 \leq \mu_1$. Let $H_r = \frac{1}{2} \|p\|^2 - \frac{\mu}{r}$, where $r : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is the distance to some point $o \in \mathbb{R}^3$, be a reference Hamiltonian of $F$. We have to show that

1. $\mu > 0$ implies $o = o_1$ and $\mu = \mu_1 - \mu_2$;
2. $\mu < 0$ implies $o = o_2$ and $\mu = \mu_2 - \mu_1$;
3. $\mu = 0$ implies $\mu_1 = \mu_2$.

**Case 1.** First we show that $o$ belongs to the $z$ axis. If this is not the case, then, due to rotational symmetry, we have a reference Hamiltonian $H_r$ with $o = (-b_0, 0, z_0)$ for some $b_0, z_0 \in \mathbb{R}, \ b_0 \neq 0$. This reference Hamiltonian $H_r$ has a trajectory $t \mapsto g_{t} H_r(x)$ that (in the configuration space) has the form shown in Figure 5.7. But for such a trajectory

$$L_z \left( \lim_{t \to +\infty} g_{t} H_{r_1} \right) = 0 \neq \sqrt{2\hbar} \cdot b_0 = L_z \left( \lim_{t \to -\infty} g_{t} H_{r_1} \right),$$

where $L_z$ is the angular momentum in the $z$ direction.
where \( h = H_r(x) > 0 \) is the energy of \( g_{H_r}^t(x) \). We conclude that \( o = (0, 0, b) \) for some \( b \in \mathbb{R} \).

Next we show that \( b \mu = a(\mu_1 - \mu_2) \). Consider a trajectory \( g_{H_r}^t(x) \) of \( H_r \) that has the form shown in Figure 5.8a. It follows from Eq. (5.3) that the function

\[
G_r = H_r + \frac{1}{2}(L^2 - b^2(p_x^2 + p_y^2)) + b(z + b)\frac{\mu}{r}
\]

is constant along this trajectory. Thus, for \( H_r \) to be a reference Hamiltonian we must have

\[
(G - G_r) \left( \lim_{t \to +\infty} g_{H_r,1}^t(x) \right) = (G - G_r) \left( \lim_{t \to -\infty} g_{H_r,1}^t(x) \right). 
\tag{5.8}
\]

In the configuration space, \( g_{H_r}^t(x) \) is asymptotic to the ray \( x = c, \ y = 0, \ z \geq 0 \) at \( t = +\infty \).

The other asymptote at \( t = -\infty \) gets arbitrarily close to the ray \( x = c, \ y = 0, \ z \leq 0 \) when \( c \to +\infty \). It follows that Eq. (5.8) is equivalent to

\[
a(\mu_1 - \mu_2) - b\mu = b\mu - a(\mu_1 - \mu_2) + \varepsilon,
\]

where \( \varepsilon \to 0 \) when \( c \to +\infty \). The remaining equality \( b = a \) can be proven using a trajectory \( g_{H_r}^t(x) \) that has the form shown in Figure 5.8b.
Figure 5.8: Kepler trajectories in the $y = 0$ plane.

Figure 5.9: The two branches ($z = z_0$ plane). In the repulsive case $\mu > 0$ a Kepler trajectory is represented by the convex branch.
Case 2. In this case trajectories $g_H^t(x)$ of the repulsive Kepler Hamiltonian $H_r$ do not project to the curves shown in Figs. 5.7, 5.8a, and 5.8b. However, each of these curves is a branch of a hyperbola. The ‘complementary’ branches are (projections of) trajectories of $H_r$; see Fig. 5.9. If the latter branches are used, the proof becomes similar to Case 1.

Case 3. In this case $H_r$ generates the free motion. Let

$$g_H^t(x) = (q(t), p(t)), \quad q(t) = (c, 0, t), \quad p(t) = (0, 0, 1).$$

Since $L^2$ and $(p_x, p_y, p_z)$ are conserved,

$$G \left( \lim_{t \to +\infty} g_{H_r}^t(x) \right) = G \left( \lim_{t \to -\infty} g_{H_r}^t(x) \right)$$

implies $a(\mu_1 - \mu_2) = a(\mu_2 - \mu_1)$ and hence $\mu_1 = \mu_2$. \hfill \Box