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Existence of Solutions of N/D Equations*

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The manifold of solutions of certain classes of N/D equations is considered, under the restriction that the scattering amplitude have a uniform bound in all complex directions. Two cases are treated: (i) The N integral equation is Fredholm, but the associated homogeneous equation may have solutions; (ii) the equation has the marginally singular behavior characteristic of an asymptotically constant left-hand cut discontinuity. The existence theorem in the latter case proceeds by the construction of the appropriate resolvent.

1. INTRODUCTION

THE purpose of this paper is to investigate the existence of solutions of N/D equations.[1] In the cases in which the integral equations can be reduced to Fredholm form, the possibility that the associated homogeneous equation has a solution is considered.[2] The "marginally singular" behavior, in which $F(\nu) \sim \log \nu$, $\nu \to \infty$, is investigated by the explicit construction of the resolvent kernel of the resulting integral equation. Some properties of this resolvent, together with those of the solutions it generates, are adduced. This work complements that of Ref. 3, where similar resolvents are applied to certain physical problems.

It is unnecessary to consider equations with multiple subtractions, since it can be shown that an n-times-subtracted equation is equivalent to an unsubtracted equation with n *CDD* poles.[4] Accordingly, since *CDD* poles do not affect the Lebesgue classes of any of the functions or kernels considered in this paper, for brevity, all equations are written without such poles. Again, in an inelastic system, the Lebesgue classes are not altered if $\eta(\nu)$ or $R^{-1}(\nu)$ (functions describing the inelasticity[1,5]) have no finite zeros, and do not tend to zero too rapidly as $\nu \to \infty$.[4] Then an inelastic equation can be treated by the same methods as those given in this paper.

The question of zeros of the D function is not considered explicitly. They must occur neither in the complex plane, nor in the physical region. However, in the absence of essential singularities, there can only be a finite number of zeros, so that the additional constraints on the input have no implications regarding the Lebesgue classes of the permitted solutions.[6]

In Sec. 2, the possibilities are divided into the classes in which (a) the N equation is Fredholm; (b) the kernel is L^2 but the inhomogeneous term is not; (c) the equation is marginally singular. Sections 3 and 4 are devoted to this marginally singular case.

2. LEBESGUE CLASS OF N/D KERNELS

Consider the elastic scattering of two indistinguishable spinless bosons of mass unity; and let the partial-wave amplitude be $A_J(\nu)$, where ν is the momentum squared, and J the angular momentum. Suppose that the argument of $A_J(\nu)$ has limited total fluctuation in the physical region: $0 \leq \nu < \infty$. Then a decomposition

$$A_J(\nu) = \nu^{-J} N_J(\nu)/D_J(\nu) \tag{2.1}$$

exists,[7] in which $N_J(\nu)$ has the unphysical cut $-\infty < \nu \leq -1$, and $D_J(\nu)$ has the unitarity cut $0 \leq \nu < \infty$; moreover, neither N nor D has kinematical singularities. For simplicity, it is assumed that there are no elementary particles or bound states with the quantum numbers of the scattering channel, so that N has no poles and D has no zeros.

The function $N_J(\nu)$ satisfies

$$N_J(\nu) = F_J(\nu) + \frac{1}{\pi}\int_0^\infty d\nu'\, \nu'^J \rho(\nu') \times \frac{F_J(\nu') - F_J(\nu)}{\nu' - \nu} N_J(\nu'), \tag{2.2}$$

* This work was supported in part by the Air Force Office of Scientific Research, Grant No. AF-AFOSR-232-63.
[1] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).
[2] D. Morgan, Nuovo Cimento 36, 813 (1965).
[3] D. Atkinson and A. P. Contogouris, Nuovo Cimento 39, 1082 (1965). A. P. Contogouris and D. Atkinson, Nuovo Cimento 39, 1102 (1965).
[4] See D. Atkinson and D. Morgan, Nuovo Cimento 41, 559 (1966).
[5] G. Frye and R. L. Warnock, Phys. Rev. 130, 478 (1963).
[6] A. P. Contogouris (unpublished); A. P. Contogouris and D. Atkinson, "N/D Equations, Bethe-Salpeter Models and High Energy Scattering" (to be published).
[7] R. Omnes, Nuovo Cimento 8, 316 (1958).

where

$$F_J(\nu) = \frac{1}{\pi}\int_{-\infty}^{-1} \frac{d\nu'}{\nu' - \nu} \operatorname{Im}\,[\nu'^{-J} A_J(\nu')] \tag{2.3}$$

and the phase-space factor is $\rho(\nu) = [\nu/(\nu+1)]^{\frac{1}{2}}$. The denominator function is given in terms of a solution of (2.2) by

$$D_J(\nu) = 1 - \frac{1}{\pi}\int_0^{\infty} d\nu'\, \nu'^J \rho(\nu') \frac{N_J(\nu')}{\nu' - \nu}. \tag{2.4}$$

Although the ensuing discussion is limited to integral J values, most of the results can be extended to any real J.

According to the unitarity condition for a physical amplitude,

$$\rho(\nu)\,|A_J(\nu)|^2 = \operatorname{Im} A_J(\nu) \le 1, \quad 0 \le \nu < \infty. \tag{2.5}$$

It is supposed that a similar limit exists on the left-hand cut, namely

$$\lim_{\nu\to\infty} \operatorname{Im} A_J(\nu) \le 1. \tag{2.6}$$

As may be shown from the Phragmén–Lindelöf theorem,[8] the following set of assumptions is sufficient to ensure that (2.5) implies (2.6):

$$\begin{aligned} A_J(\nu) &\sim \nu^{\alpha}(\log \nu)^{\beta}(\log\log \nu)^{\gamma} \cdots \\ &\quad \times (\log\log\cdots\log \nu)^{\omega}, \quad \nu \to \infty, \\ A_J(\nu) &\sim \nu^{\alpha'}(\log \nu)^{\beta'}(\log\log \nu)^{\gamma'} \cdots \\ &\quad \times (\log\log\cdots\log \nu)^{\omega'}, \quad \nu \to -\infty, \end{aligned} \tag{2.7}$$

$$|A_J(\nu)| < C_1 e^{(\pi-\epsilon)\nu}, \quad \epsilon > 0, \tag{2.8}$$

where $\alpha, \beta, \cdots, \omega, \alpha', \beta', \cdots, \omega'$, and C_1 are some constants, and (2.8) is supposed to hold in all complex directions.

Three classes which satisfy (2.6) are distinguished and treated separately:

$$\begin{aligned} &\text{(a)}\quad |A_J(\nu)| < C_2\nu^{-\frac{1}{2}-\epsilon} \\ &\text{(b)}\quad |A_J(\nu)| < C_3\nu^{-\epsilon} \end{aligned} \Big\}\ \text{some}\ \epsilon < 0,$$

$$\text{(c)}\quad |A_J(\nu)| \underset{\nu\to\infty}{\longrightarrow} \lambda \le 1. \tag{2.9}$$

In this paper the possibility that $A_J(\nu)$ tends to zero at $\nu = \infty$ more slowly than any power of ν is not considered; although the case $F_J(\nu) \sim 1/\log \nu$ can be solved by an extension of the methods of Sec. 3.

Class (a)

Both the kernel and the inhomogeneous term of (2.2) are square-integrable for any $J \ge 0$, because

$$\begin{aligned} |F_J(\nu)| &< C_4 \int_1^{\infty} \frac{d\nu'}{\nu' + \nu} \nu'^{-\frac{1}{2}-J-\epsilon}, \quad \nu \ge 0 \\ &\le C_5 \nu^{-\frac{1}{2}-\epsilon} \quad \text{all} \quad J \ge 0. \end{aligned} \tag{2.10}$$

From this, it follows that (2.2) is Fredholm, so that, unless there is a homogeneous solution, a unique square-integrable solution N_J exists. Moreover, the integral in (2.4) that defines D_J exists.

In the case that a homogeneous solution of (2.2) does exist, i.e., there is a function $n_J(\nu)$ such that

$$n_J(\nu) = \frac{1}{\pi}\int_0^{\infty} d\nu'\, \nu'^J \rho(\nu') \frac{F_J(\nu') - F_J(\nu)}{\nu' - \nu} n_J(\nu'), \tag{2.11}$$

there will be, *in general*, no solution of (2.2) (the Fredholm alternative theorem). However, in such a case one can define a function

$$d_J(\nu) = -\frac{1}{\pi}\int_0^{\infty} d\nu'\, \nu'^J \rho(\nu') \frac{n_J(\nu')}{\nu' - \nu}. \tag{2.12}$$

This specifies a new decomposition

$$A_J(\nu) = \nu^{-J} n_J(\nu)/d_J(\nu) \tag{2.13}$$

satisfying the usual requirements, except that $d_J(\nu) \to_{\nu\to\infty} 0$. In this case $A_J(\nu)$ is said to belong to a *CDD* class of negative index.[4] The solution n_J of (2.11) need not be unique, although the number of linearly independent solutions must be finite (the multiplicity of the corresponding eigenvalue).

Finally, there is the possibility that there are solutions $\hat{n}_J(\nu)$ of the associated homogeneous equation, but that, for any such,

$$\int_0^{\infty} d\nu\, F_J(\nu)\hat{n}_J(\nu) = 0. \tag{2.14}$$

In this case a solution of (2.2) does exist, but it is not unique, since any homogeneous solution $n_J(\nu)$ may be added. This possibility is considered elsewhere[4] in connection with coincident zeros of N_J and D_J.

Class (b)

In this case

$$|F_J(\nu)| < C_6 \nu^{-J-\frac{1}{2}\epsilon}, \tag{2.15}$$

so that (2.2) is Fredholm for $J \ge 1$, and the considerations pertinent to class (a) apply. For $J = 0$, $F_J(\nu)$ may not be square integrable, so that (2.2) is not Fredholm. However, the kernel is still L_2, for

[8] E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, New York, 1950).

$$\left|\frac{F_0(\nu) - F_0(\nu')}{\nu - \nu}\right|$$
$$< C_7 \int_1^\infty d\nu'' \frac{\nu''^{-\epsilon}}{(\nu'' + \nu)(\nu'' + \nu')}, \quad \nu, \nu' \geq 0$$
$$< C_7 \int_1^\infty d\nu'' \frac{\nu''^{-\epsilon}}{(\nu'' + \nu)\nu''^{\frac{1}{2}(1-\epsilon)}\nu'^{\frac{1}{2}(1+\epsilon)}}$$
$$= C_8(\nu\nu')^{-\frac{1}{2}(1+\epsilon)}. \tag{2.16}$$

Hence (2.2) has a resolvent kernel; but a solution may not exist if the contraction of this resolvent with $F_0(\nu)$ fails to converge.

However, a solution may be found in this case by subtracting the D equation at, say, $\nu = 0$, and normalizing it to unity at this point. The modified integral equation is

$$N_0(\nu) = F_0(\nu) + \frac{1}{\pi}\int_0^\infty d\nu' \rho(\nu') \times \frac{\nu F_0(\nu) - \nu' F_0(\nu')}{\nu - \nu'} \frac{N_0(\nu')}{\nu'}, \tag{2.17}$$

with $F_0(\nu)$ defined as in (2.3), and $D_0(\nu)$ given by

$$D_0(\nu) = 1 - \frac{\nu}{\pi}\int_0^\infty d\nu' \rho(\nu') \frac{N_0(\nu')}{\nu'(\nu' - \nu)}. \tag{2.18}$$

The symmetrized form of (2.17) is

$$\frac{N_0(\nu)}{\nu^{\frac{1}{2}}} = \frac{F_0(\nu)}{\nu^{\frac{1}{2}}} + \frac{1}{\pi}\int_0^\infty d\nu' \rho(\nu') \times \frac{1}{(\nu\nu')^{\frac{1}{2}}} \frac{\nu F_0(\nu) - \nu' F_0(\nu')}{\nu - \nu'} \frac{N_0(\nu')}{\nu'^{\frac{1}{2}}}. \tag{2.19}$$

The inhomogeneous term is now square integrable, for

$$|F_0(\nu)/\nu^{\frac{1}{2}}| < C_6 \nu^{-\frac{1}{2}-\epsilon}, \tag{2.20}$$

while the Lebesgue class of the kernel has not been changed by the subtraction procedure (an invariable property of N/D kernels). Hence a solution exists even for the S wave, involving no CDD parameters.

Class (c)

This is the "marginally singular" possibility in which the discontinuity tends to a constant at infinity. This kernel singularity, which can be specified loosely by saying that the norm diverges logarithmically, is the most singular behavior tolerated by unitarity. The additional assumption

$$\operatorname{Im} A_J(\nu) = \lambda_J + r_J(\nu), \tag{2.21}$$

where $0 < \lambda_J \leq 1$ and $r_J(\nu) = O(\nu^{-\epsilon})$ with $\epsilon > 0$, is made.

For Class (c) amplitudes, it is more convenient to replace (2.1) by

$$A_J(\nu) = N_J(\nu)/D_J(\nu) \tag{2.22}$$

for all physical J, and then subtract both N_J and D_J equations at threshold. The resulting equations, which replace (2.2)–(2.4), are

$$\frac{N_J(\nu)}{\nu} = \frac{a_J}{\nu} + \frac{F_J(\nu)}{\nu} + \frac{1}{\pi}\int_0^\infty d\nu' \rho(\nu') \frac{F_J(\nu') - F_J(\nu)}{\nu' - \nu} \frac{N_J(\nu')}{\nu'}, \tag{2.23}$$

where

$$F_J(\nu) = \frac{\nu}{\pi}\int_{-\infty}^{-1} \frac{d\nu'}{\nu'(\nu' - \nu)} \operatorname{Im} A_J(\nu') \tag{2.24}$$

and

$$D_J(\nu) = 1 - \frac{\nu}{\pi}\int_0^\infty d\nu' \rho(\nu') \frac{N_J(\nu')}{\nu'(\nu' - \nu)}. \tag{2.25}$$

For $J \geq 1$, the subtraction constant

$$a_J = -\frac{1}{\pi}\int_{-\infty}^{-1} \frac{d\nu'}{\nu'^2} \operatorname{Im} A_J(\nu').$$

For $J > 1$ there are additional threshold conditions, while for $J = 0$, a_J constitutes an arbitrary parameter. Substituting (2.21) into (2.24) gives

$$F_J(\nu) = (\lambda_J/\pi)\log(\nu + 1) + C_J + O(\nu^{-\epsilon}), \tag{2.26}$$

where the constant C_J is given by

$$C_J = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'} r_J(\nu'). \tag{2.27}$$

If one writes

$$\begin{aligned} x &= \nu + 1, \\ \phi_J(x) &= N_J(\nu)/\nu, \\ \phi_J^0(x) &= \frac{a_J + F_J(\nu)}{\nu} \\ &= \frac{(\lambda_J/\pi)\log x + a_J + C_J + O(x^{-\epsilon})}{x - 1}, \\ \rho(x) &= 1 - \{(\nu + 1)^{\frac{1}{2}}[\nu^{\frac{1}{2}} + (\nu + 1)^{\frac{1}{2}}]\}^{-1}, \end{aligned} \tag{2.28}$$

Eq. (2.23) becomes

$$\phi_J(x) = \phi_J^0(x) + \frac{\lambda_J}{\pi^2}\int_1^\infty dx' \left[\frac{\log x'/x}{x' - x} + K_J(x, x')\right]\phi_J(x'), \tag{2.29}$$

where $K_J(x, x')$ is a square-integrable residual kernel. This singular equation may be written

$$\phi_J = \phi_J^0 + (\lambda_J/\pi^2)S\phi_J + (\lambda_J/\pi^2)K_J\phi_J, \tag{2.30}$$

where

$$S(x, x') = (\log x'/x)\cdot(x' - x)^{-1}$$

is a marginally singular kernel.

In the next section, it is shown that a resolvent kernel R_J exists and satisfies

$$R_J = \frac{1}{\pi^2} S + \frac{\lambda_J}{\pi^2} SR_J = \frac{1}{\pi^2} S + \frac{\lambda_J}{\pi^2} R_J S; \qquad (2.31)$$

moreover, this allows one to define solutions of (2.29) for any Fredholm kernel K_J.

3. CONSTRUCTION OF THE RESOLVENT

As the first stage in the solution of (2.29), set

$$\omega_J(x) = \phi_J^0(x) + \frac{\lambda_J}{\pi^2}\int_1^\infty dx'\, K_J(x, x')\phi_J(x') \qquad (3.1)$$

so that

$$\phi_J(x) = \omega_J(x) + \frac{\lambda_J}{\pi^2}\int_1^\infty dx'\, \frac{\log x'/x}{x' - x}\phi_J(x'). \qquad (3.2)$$

If one can construct a resolvent kernel R satisfying (2.31), then the general solution of (3.2), if it exists, satisfies

$$\phi_J(x) = \omega_J(x) + \lambda_J \int_1^\infty dx'\, R_J(x, x'; \lambda_J)\omega_J(x') + A\Psi_J(x), \qquad (3.3)$$

where A is an arbitrary constant, and $\Psi_J(x)$ is the general solution of the homogeneous equation

$$\Psi_J(x) = \frac{\lambda_J}{\pi^2}\int_1^\infty dx'\, \frac{\log x'/x}{x' - x}\Psi_J(x'). \qquad (3.4)$$

As may be verified from the properties of Legendre functions,[9] one solution of (3.4) is

$$\Psi_J(x) = P_{-s_0}(2x - 1), \qquad (3.5)$$

where

$$\lambda_J = \sin^2 \pi s_0$$

and

$$0 \leq \operatorname{Re} s_0 \leq \tfrac{1}{2}.$$

Suppose now that the "shifted Mehler transform" of $\omega_J(x)$ is defined by

$$\tilde{\omega}_J(s) = s \tanh \pi s \int_1^\infty dx\, \omega_J(x) P_{-\frac{1}{2}+is}(2x - 1). \qquad (3.6)$$

This has the unique inverse of

$$\omega_J(x) = \frac{1}{2}\int_{-\infty}^\infty ds\, \tilde{\omega}_J(s) P_{-\frac{1}{2}+is}(2x - 1). \qquad (3.7)$$

[9] *Higher Transcendental Functions*, A. Erdelyi, Ed., (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2.

Then, if $\omega_J(x)$ is such that $\tilde{\omega}_J(s)$ exists in part of the strip $-\frac{1}{2} < \operatorname{Im} s < \frac{1}{2}$, Eq. (3.2) can be diagonalized by Mehler transformation:

$$\tilde{\phi}_J(s) = \tilde{\omega}_J(s) + \lambda_J \operatorname{sech}^2 \pi s \tilde{\phi}_J(s). \qquad (3.8)$$

This equation can be solved for the Mehler transform $\tilde{\phi}_J(s)$. Inversion then yields

$$\phi_J(x) = \int_c ds\, \tilde{\omega}_J(s) \frac{P_{-\frac{1}{2}+is}(2x - 1)}{1 - \lambda_J \operatorname{sech}^2 \pi s}, \qquad (3.9)$$

where it may be verified that any contour C extending from $-\infty$ to $+\infty$ within the strip $-\frac{1}{2} < \operatorname{Im} s < \frac{1}{2}$ yields a solution of (3.2). By substituting the definition of $\tilde{\omega}_J(s)$ into (3.9), one has

$$\phi_J(x) = \omega_J(x) + \lambda_J \int_1^\infty dx'\, R_J(x, x'; \lambda_J)\omega_J(x'), \qquad (3.10)$$

where

$$R_J(x, x'; \lambda_J) = \frac{1}{2}\int_c ds\, \frac{s \tanh \pi s}{\cosh^2 \pi s - \lambda_J} \times P_{-\frac{1}{2}+is}(2x - 1)P_{-\frac{1}{2}+is}(2x' - 1). \qquad (3.11)$$

The integrand in (3.11) has poles whenever

$$\cosh^2 \pi s = \lambda_J. \qquad (3.12)$$

For $0 < \lambda_J < 1$, the general solution of (3.12) is

$$s = in \pm (i/\pi) \underset{(0,\frac{1}{2}\pi)}{\operatorname{arc\,cos}} \lambda_J^{\frac{1}{2}}, \qquad n = \cdots -1, 0, 1, 2, \cdots. \qquad (3.13)$$

There are two poles inside the strip $-\frac{1}{2} < \operatorname{Im} s < \frac{1}{2}$, obtained by setting $n = 0$ in (3.13).

A particular resolvent $R_J(x, x'; \lambda_J)$ may be picked out by defining the contour C in (3.11) to be the real axis $(-\infty, \infty)$. It is easy to verify that R_J satisfies (2.27). Moreover, homogeneous solutions are obtained by taking C to be a closed contour encircling either of the poles in the strip $-\frac{1}{2} < \operatorname{Im} s < \frac{1}{2}$. These are multiples of $P_{-s_0}(2x - 1)$ or $P_{-1+s_0}(2x - 1)$, where s_0 is defined in (3.5), depending on which pole is taken. However,

$$P_{-s_0}(2x - 1) = P_{-1+s_0}(2x - 1). \qquad (3.14)$$

Hence, there is only one independent homogeneous solution. If C is taken to be a contour surrounding a pole outside the strip $-\frac{1}{2} < \operatorname{Im} s < \frac{1}{2}$, then the result is not a homogeneous solution $\Psi_J(x)$, since the integral in (3.4) would fail to converge.

Finally, the general solution of (3.2) can be written

$$\phi_J(x) = \omega_J(x) + \lambda_J \int_1^\infty dx\, R_J(x, x'; \lambda_J)\omega_J(x') + AP_{-s_0}(2x - 1). \quad (3.15)$$

That (3.15) is in fact a solution can be verified by checking to see if, when using Eqs. (3.1) and (2.28), the integral converges uniformly with respect to x. This justifies the Mehler transformation.

As a function of λ_J, $R_J(x, x'; \lambda_J)$ has a branch point at $\lambda_J = 1$. This follows from the fact that the real contour $C \equiv (-\infty, \infty)$ in (3.11) is pinched at $s = 0$ when $\lambda_J \to 1$. The cut may be defined $(1 \leq \lambda_J < \infty)$. The function $P_{-s_0}(2x - 1)$ is entire in s_0; but s_0 has branch points at $\lambda_J = 0, 1$. However, $P_{-s_0}(2x - 1)$ has a branch point at $\lambda_J = 0$ only: the branch cut $(1 \leq \lambda_J < \infty)$ vanishes identically, as a result of the relation (3.14). According to (3.15), the general solution will have branch cuts $(-\infty < \lambda_J \leq 0)$ and $(1 \leq \lambda_J < \infty)$ (assuming that the arbitrary factor A is chosen to be holomorphic in λ_J). However, one solution, defined by $A = 0$, has just one finite branch point $\lambda_J = 1$: this solution is analytic at $\lambda_J = 0$ and has a power series expansion about that point.

It is instructive to display the sheet structure of $R_J(x, x'; \lambda_J)$ in the λ_J variable. In fact, R_J is uniform in the variable s_0, defined as in (3.5), or equivalently,

$$s_0 = (i/\pi) \log [(1 - \lambda_J)^{\frac{1}{2}} + i\lambda_J^{\frac{1}{2}}]. \quad (3.16)$$

The first sheet of λ_J, cut $(-\infty, 0)$, $(1, \infty)$, is defined to map onto the strip $0 < \text{Im } s_0 < \frac{1}{2}$. In Fig. 1 the complete λ_J structure is shown mapped into the s_0 plane. The discontinuity of R_J across the cut $(1 \leq \lambda_J < \infty)$ is just the difference between the values of R_J at some point $\lambda_J = \frac{1}{2} + it_0$ (say) and at $\lambda_J = \frac{1}{2} - it_0$. The corresponding discontinuity across $(-\infty < \lambda_J \leq 0)$, the difference calculated from $\lambda_J = it_0$ to $\lambda_J = -it_0$ is zero, a consequence of the symmetry of the integrand in (3.11) with respect to s. This means that there is no cut $(-\infty, 0)$ on sheet I, a fact that has already been noted. Sheet II is defined to be the sheet connected to sheet I across the cut $(1 \leq \lambda_J < \infty)$: it maps onto the strip $\frac{1}{2} \leq \text{Im } s_0 \leq 1$. On this sheet there is a branch cut $(-\infty < \lambda_J \leq 0)$ as well as the cut $(1 \leq \lambda_J < \infty)$. This is true of all sheets except the first. As can be seen from Fig. 1, a double circuit of $\lambda = 1$ which does not encircle $\lambda = 0$ brings one back onto sheet I. On the other hand, repeated circuits enclosing $\lambda = 0$ and $\lambda = 1$ plunge one into lower and lower sheets. The branch points may be said to be individually like square roots, but together like a logarithm.

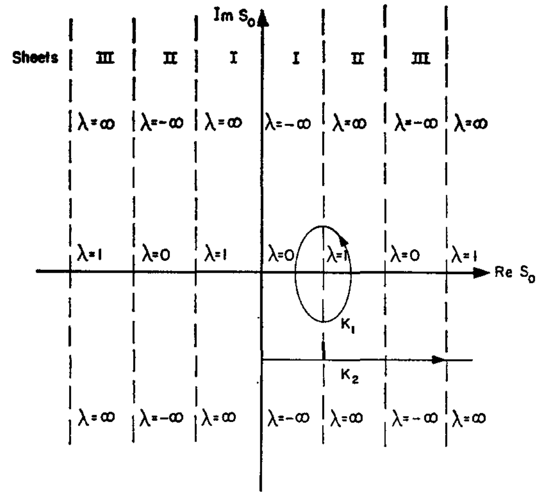

FIG. 1. Map of the infinite-sheeted λ plane into the entire s_0 plane. The contour K_1 is the map of a double circuit of $\lambda = 1$, not enclosing $\lambda = 0$. The line K_2 is the map of an infinite encirclement of both $\lambda = 0$ and $\lambda = 1$.

It should be noted that the continuation of $\phi_J(x)$ onto a higher sheet is not a solution of (3.2) in general, because the integral would diverge. The $N_J(\nu)$ and $D_J(\nu)$ functions that can be defined from the continued $\phi_J(x)$ correspond in fact to N/D solutions of higher CDD classes.[4]

4. COMPLETE SOLUTION

The problem to be investigated in this section is the existence of a solution of Eq. (2.29), corresponding to each solution (3.15) of (3.2). Suppose one writes Eq. (3.15) in operator form:

$$\phi_J = \omega_J + \lambda_J R_J \omega_J + AP_{-s_0}. \quad (4.1)$$

Then, substituting for ω_J according to (3.1),

$$\phi_J = \phi_J^0 + (\lambda_J/\pi^2)K_J\phi_J + \lambda_J R_J \phi_J^0 + (\lambda_J^2/\pi^2)K_J R_J \phi_J + AP_{-s_0}. \quad (4.2)$$

It was shown in Ref. 3 that R_J satisfies the family of majorizations

$$|R_J(x, x'; \lambda_J)| \leq B_1 x^{-\frac{1}{2}-p} x'^{-\frac{1}{2}+p} \quad (4.3)$$

for any $-\frac{1}{2} + s_0 \leq p \leq \frac{1}{2} - s_0$, with B_1 some constant. It follows from (2.26) and (2.28) that

$$|K_J(x, x')| \leq B_2(xx')^{-\frac{1}{2}-\frac{1}{2}\epsilon}. \quad (4.4)$$

Then, from (4.3), (4.4), and (2.28),

$$\phi_J^0 + \lambda_J R_J \phi_J^0$$

is a square-integrable function and

$$K_J + \lambda_J K_J R_J \tag{4.5}$$

is a square-integrable kernel. Hence, in the case $A = 0$, (4.2) is a Fredholm equation, and a solution almost always exists. However,

$$P_{-s_0}(2x - 1) \sim x^{-s_0}, \qquad 0 < s_0 < \tfrac{1}{2}, \tag{4.6}$$

so that if $A \neq 0$, the inhomogeneous term in (4.2) is not square-integrable, and, in general, a solution will not exist. In fact, suppose that the resolvent of the Fredholm kernel $K_J + \lambda_J K_J R_J$ is $\mathfrak{F}_J$. If the contraction $\mathfrak{F}_J P_{-s_0}$ converges, then a solution of (4.2), and hence of (2.29), exists for any constant A.

If, for a given λ_J, the condition (2.21) can be strengthened to

$$r_J(\nu) = o(\nu^{-s_0-\epsilon}) \quad \text{for some} \quad \epsilon > 0, \tag{4.7}$$

one can show that $\mathfrak{F}_J P_{-s_0}$ always converges, so that there is always a one-parameter infinity of solutions. For in this case

$$|K_J(x, x')| < B_3 x^{-\frac{1}{2}-\eta} x'^{-\frac{1}{2}+\eta-s_0-\epsilon} \tag{4.8}$$

for any $-\frac{1}{2} + s_0 < \eta < \frac{1}{2}$. Then it follows from (4.3) and (4.8) that

$$\left| K_J(x, x') + \lambda_J \int_1^\infty dx'' K_J(x, x'') R_J(x'', x') \right| < B_4 x^{-\frac{1}{2}-p-s_0} x'^{-\frac{1}{2}+p} \tag{4.9}$$

for any $-\frac{1}{2} + s_0 \leq p \leq \frac{1}{2} - s_0$. If it is supposed that the Fredholm resolvent $\mathfrak{F}_J$, which is itself Fredholm, satisfies some power bound,

$$|\mathfrak{F}_J(x, x')| < B_4 x^{-u} x'^{-v} \quad \text{with} \quad u, \quad v > \tfrac{1}{2}, \tag{4.10}$$

then (4.9) implies that, in fact, $u = \frac{1}{2} + p + s_0$, $v = \frac{1}{2} - p$, with $-\frac{1}{2} + s_0 \leq p \leq \frac{1}{2} - s_0$, as in (4.9). By choosing $p = -\frac{1}{2} + s_0$, one sees immediately that $\mathfrak{F}_J P_{-s_0}$ converges, so that (4.2) always has a solution.

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