Non-Usual Topologies on Space–Time and High-Energy Scattering

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Motivated by recently observed deviations from quantum electrodynamical theory, we study the possibility that our notions of space–time may need revision at small distances. In this work, we wish to call attention to certain techniques which are available for studying different space–time structures within the framework of topology. Our main effort is in the consideration of a non-usual topology on space–time in which is embedded an elementary length. By working separately in each \( n \)-particle subspace, the embedding is done in an inhomogeneous Lorentz invariant way, and we avoid any lattice structure in space–time. Particles in this topology are in general extended structures, and we find the surprising feature that, at high energies, the topology enhances backward and large-angle scattering. From these preliminary investigations, we are not as yet able to make more than qualitative comparison with experiment. Along the way, we have the opportunity to remark on ways of embedding an intrinsic breakdown of certain invariances (e.g., parity) in the topology of space–time.

INTRODUCTION

It has often been suspected that our common notions concerning the structure of space–time may break down for extremely small intervals. Indeed, it would be \textit{a priori} surprising if these notions were correct at all distances. A recent large-angle pair-production experiment\(^1\) in electrodynamics has raised the possibility that, even at presently available energies, we may in fact be entering into such a regime of breakdown. This disturbing notion finds indirect support, perhaps, in the various unsuccessful \textit{ad hoc} modifications of quantum electrodynamics\(^2\) (stimulated by the pair-production experiment), within the framework of a usual space–time structure. Although one may still hope to find an explanation of the data in essentially conventional terms, we have been motivated by the present situation to look into possible ways of changing the structure of space–time.

The structure of a space is most naturally studied through its topology.\(^3\) We have in mind here the use of topology to describe the local or microscopic structure of a space,\(^4\) rather than its global properties\(^5\) (such as torsion, macroscopic connectedness, etc.).

\(^1\) R. B. Blumenthal, D. C. Ehn, W. L. Faisstler, P. M. Joseph, L. J. Lanzerotti, F. M. Pipkin, and D. G. Stairs, Phys. Rev. \textbf{144}, 1199 (1966). A momentum-transfer of 6 BeV/c corresponds to a Compton wavelength of \( 3 \times 10^{-19} \) cm for the internal electron. If we believe that this experiment is just beginning to show the effects of the non-usual topology, then we might take this wavelength as a very rough indication (more likely an upper limit) for the elementary length \( \lambda \).


\(^4\) E. C. Zeeman (J. Math. Phys. \textbf{5}, 490 (1964); Cambridge University Preprint (1965)) has already considered a non-usual topology for space–time. [See, also in this connection, D. B. Wolf, Preprint, Computer Associates, London (1965)] Zeeman's and our approaches are basically different, but a marriage between them can be contrived, as briefly noted in the text.


\(^6\) In principle, of course, one would be willing to tolerate a breakdown of Lorentz invariance to order \( \lambda \), if \( \lambda \) were small enough.

\(^7\) Three recent papers by A. Das (J. Math. Phys. \textbf{7}, 45, 52, 61 (1966)) give an adequate referencing of existing theories with an elementary length.

\(^8\) A coarse topology is one that contains fewer than the usual open sets, while a fine topology contains more.
are, in some sense, giving up the ability to specify the relative coordinate of the two particles beyond a certain accuracy. However, the topology induced on either one-particle subspace is strictly finer than usual.

Following Zeeman, we define a trajectory to be a continuous map from some parameter space into our topological space. In this sense, all ordinary trajectories are excluded by the topology, and various interesting alternate possibilities arise. We find that, generally speaking, these trajectories imply that the two particles are extended in space over a minimum distance $\lambda$.

At the dynamical level, our job is to find equations of motion in the two-particle subspace. The difficulty involved in setting up operators on the space is the following: In one direction in our two-particle space the natural "derivative" is a difference operator, while in a perpendicular direction it is the usual differential operator. The operators along some intermediate ray are some unfamiliar "combination" of these two familiar operators. We avoid some of these problems by making the simple assumption that the equations of motion are separable between the sum and difference coordinate variables. In this way, the equations involve only differential and difference operators. It should be emphasized, however, that this procedure picks out only one from a large class of possible dynamics on the topology. These other dynamics would involve the use of the "fine" operators along an arbitrary ray; we do not have a great deal to say about them in this work.

We confine our considerations of classical mechanics on the new topologies to an analysis of the possible trajectories; then we proceed directly to quantum mechanics. In a simple momentum, energy, and probability conserving formalism, we infer the high-energy scattering of the theory. The surprising result is that at high energies the topology induces an extra effective "potential" between the interacting particles, which serves to enhance backward and high-energy scattering! This is certainly suggestive of the results of Blumenthal et al., but we have not yet calculated this effect quantitatively enough to allow more than a qualitative comparison with the data. Another interesting and surprising feature of the high-energy scattering is the presence of very high mass, but long-lived resonances. In fact, the higher the mass, the narrower the resonance. Certainly, there would be no mechanism in ordinary field theory or $S$-matrix theory to generate such particles.

We mention that the presence of the elementary length (and the corresponding damped high-energy behavior of the transformation functions to be discussed below) allows, in principle, the elimination of ultra-violet divergences in the theory. The detailed discussion of such a problem would require the choice of a particular theory on the topology (analogous to a choice of a particular Lagrangian in the usual topology); but we are content here in general with the deduction of what seems to be the high-energy behavior of any theory on the topology. By the same token, we do not discuss "intermediate energy" scattering on the topology, as this would also be "theory-dependent."

Along the way in our discussion, we have the opportunity to present various other topologies in which are embedded intrinsic violations of certain discrete symmetries, e.g., time reversal and/or parity invariance. In particular, we mention a topology in which, at a pre-dynamical level, some particles violate parity, and others do not. These topologies may be of some interest in their own right with regard to embedding certain features of the weak interactions in space-time itself.

The order of our presentation is as follows: In Sec. A, we consider a non-usual topology, and the possible particle trajectories, first in one spatial dimension, where we take care to retain translational invariance, then in three dimensions, where we must also keep rotational invariance, and finally in $1 + 3$ space-time, where we complete the embedding of the elementary length in an inhomogeneous Lorentz invariant manner. By building up in stages, we have illustrated the difficulties involved in incorporating an elementary length in more and more complicated spaces, and with progressively more stringent symmetry requirements. In Sec. B, we develop the simple separable dynamics, again in first one, then three, and finally in four dimensions. At the end of Sec. B, we discuss very briefly some of the problems involved in formulating a full field theory, allowing for the creation and destruction of particles.

A. TOPOLOGIES AND TRAJECTORIES ON THE TWO-PARTICLE SUBSPACE

For the purposes of orientation and simplicity, we first discuss intrinsically nonrelativistic topologies, taking a non-usual topology only on space, and leaving the usual topology on time. Things are in fact much simpler in one spatial dimension, and we can learn much in this simple case, which we accordingly discuss before going on to three dimensions. After that, we turn to a relativistic topologization, in which time and space are kept on an equal footing.

1. Nonrelativistic One-Dimensional Motion

Consider the space of two identical particles, located somewhere on a line, with coordinates $x_1$ and $x_2$. Ordinarily one assumes the usual (Euclidean)
topology on the two-dimensional space \( x_1 \otimes x_2 \), and on a time parameter, \( t \). In this section, we want to study another topology for this space, one which contains an elementary length, \( \lambda \), but which in no way implies an unesthetic lattice structure on the line itself. In particular, the topology is completely translationally invariant.

We define the non-usual topology by the base\(^9\):

\[
B^{(1)}_{x_1 \otimes x_2} = \{ X : a < X < b, n\lambda \leq x < (n + 1)\lambda \}, \quad (A1)
\]

where \( X = (x_1 + x_2)/2, x = x_1 - x_2, a \) and \( b \) are any real numbers, and \( n \) is any positive or negative integer.

As we have already indicated, \( \lambda \) is to be the elementary length in the theory. In words, we are taking, as the topology of \( x_1 \otimes x_2 \), the product of the usual topology on the sum variable \( X \), and an apparently coarse topology (actually one that is strictly incomparable with the usual topology) on the difference variable \( x \). We refer to this topological space as \( (x_1 \otimes x_2, \lambda)^{(1)} \), the superscript indicating that the individual particle spaces are one-dimensional. We can only separate two points (in the Hausdorff sense) if we can cover each of them with disjoint open sets. The coarseness of the \( x \) topology indicates that we are giving up, in some sense to be discussed below, the ability to specify the distance between the particles more accurately than \( \lambda \).

A question of paramount importance is of course the induced topology\(^10\) on the space of an individual particle. (This is the topology that should be compared with the usual situation.) One sees immediately that a base for the induced topology on, for example, the \( x_1 \) space, for fixed \( x_2 \), is

\[
B^{(1)}_{x_1}(x_2) = \{ x_1 : a < x_1 < b, x_2 + n\lambda \leq x_1 < b \}, \quad (A2)
\]

where \( a, b, \) and \( n \) are defined as in Eq. (A1). We call this topological space \( (x_1, \lambda; x_2)^{(1)} \). This notation emphasizes that the induced topology on \( x_1 \) is parametrized by \( x_2 \), indicating a pre-dynamical linkage between the two particles. Note that this topology is strictly finer (contains more open sets) than the usual topology. Despite the fact that, for fixed \( x_2 \), there is a set of "preferred" points—in that an open set\(^11\) extending to the right (or to the left, for that matter) from such a point may or may not contain the point—this topology is translationally invariant. This is because, in any translation, both particles are moved by the same amount, so that the preferred points in the \( x_1 \) space move also.

The fineness of the \( (x_1, \lambda; x_2)^{(1)} \) topology is mirrored in the enlarged set of continuous functions\(^12\) on \( (x_1, \lambda; x_2)^{(1)} \) into a space with the usual topology. Some single-valued continuous functions on \( (x_1, \lambda; 0)^{(1)} \) into \( f(x_1), U \) are shown in Fig. 1. This set is greater than the usual set of continuous functions, since it includes functions which may have arbitrary discontinuities (in the usual topology) at the preferred points \( n\lambda \).

Note that \((x_1 \otimes x_2, \lambda)^{(1)} \) is complicated from the topologist's point of view because, although it is normal and regular, it is non-Hausdorff,\(^13\) and is not even \( T_0 \). A space with an elementary length is non-Hausdorff in general (because in a Hausdorff space one can "distinguish" between any two points by means of disjoint open sets). By the same token, the space is not metrizable; but, as we see below, it is pseudo-metrizable. On the other hand, the topologies induced on the single particle spaces are Hausdorff and metrizable.

Possible Classical Trajectories in \((x_1 \otimes x_2, \lambda)^{(1)}\)

Following Zeeman, we take, as a natural definition of a trajectory in the topological space, a continuous map of a finite interval in \((\tau, U)\) into \((x_1 \otimes x_2, \lambda)^{(1)}\), where \( \tau \) is some invariant parameter and \( U \) is the usual topology. For these nonrelativistic topologies, the ordinary time will suffice as the parameter. (We discuss more restrictive definitions of trajectories below.) Note that this is the inverse of the prescription

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\(^9\) A subset \( B \) of a topology \( \tau \) is a base for \( \tau \) if each member of \( \tau \) is the union of members of \( B \).

\(^10\) By the induced topology on (say) \( x_1 \), we mean the relative topology on the space \( x_1 \) with respect to \((x_1 \otimes x_2, \lambda)^{(1)}\). The topology on a subset \( Y \) of \( X \) relative to the topological space \((X, \tau)\) is defined to be the family of all intersections of members of \( \tau \) with \( Y \) (see Ref. 3).

\(^11\) An alternate base for \((x_1, \lambda; x_2)^{(1)} \) would be the usual open intervals, plus the points \( x_4 + n\lambda \).

\(^12\) A continuous function of one space into another is a mapping of the first (or domain) space into the second (or range) space such that the inverse map of any open set in the range space is open in the domain space.

\(^13\) \((x_1 \otimes x_2, \lambda)^{(1)} \) is the topological product of the Hausdorff \((X, U)\) and the non-Hausdorff \((x, \lambda)\), so that our "information loss" is in the distance between the two particles.
giving the continuous functions on \((x_1 \otimes x_2, \lambda)^{(1)}\) into a function space with the usual topology.

In general, one can find trajectories in any direction in the \(x_1 \otimes x_2\) plane. Various of these are shown in Fig. 2. Trajectory (1) corresponds to both particles' moving with equal velocity, keeping at a constant distance from one another. Trajectories of this sort, in which the particles never change their relative distance "see" the usual topology (on \(X\)), and the particle motion appears quite ordinary. Trajectory (2) corresponds to two particles viewed in their center-of-mass system, with the coordinate origin midway between the particles. The topology "seen" by these trajectories is coarse. Trajectory (3) corresponds to the second particle's remaining fixed while the first particle moves. The topology seen by the moving particle is fine. The topology seen on a trajectory like (4)—with both particles moving—is in general fine. Note that, except for peculiar cases like (1), trajectories see non-usual induced topologies.

A representative sampling of the kinds of possible trajectories [corresponding to (3) in Fig. 2] in \(x_1\), for fixed \(x_2\), is given in Fig. 3. Trajectories like that shown in Fig. 3(i) correspond to particle two's remaining fixed while particle one jumps forward over each preferred line. In another frame, related by a Galilean transformation, particle 2 would move quite normally. (The topology allows one particle to move in an ordinary fashion, but not both.) Such a trajectory could be excluded by the requirement that the continuous map into \((x_1 \otimes x_2, \lambda)^{(1)}\) have an inverse in its range. That is, one would want to be able to tell at what time (or times) any particular position was occupied. With this requirement, we immediately exclude any one-to-one map: That is, the particles automatically have an extent in space. For example, "barb" trajectories like (ii) (iii) do have an inverse and are perhaps of some interest.

Trajectory (ii) corresponds to particle one's moving from left to right (while particle two remains fixed at the origin). As it approaches a preferred point, an image of the particle suddenly appears at the preferred point, and remains there while the particle itself passes through the point. The image survives after this for some time, and then it dies. Actually the images can live as long as one likes, but presumably the longer-lived ones would be less physical, and one would therefore want the dynamics to keep the image lifetimes short (i.e., to pick only such a sub-set of the trajectories). Trajectory (iii) is the time or parity reversed counterpart of (ii). Since any barb trajectory does have a time and/or parity reversed counterpart, it is evident that, unless the dynamics further limits the set of trajectories, the theory of a particle which moves (barb-like) past a fixed particle would be time reversal and parity invariant. (This applies also, of course, to all Galilean transformed configurations.) We see below that there also exist trajectories on this space with no mirror image and hence an intrinsic parity violation.

Other interesting barb-like trajectories can be found by mapping directly into \(x, X\). Some representative maps into \((x, \lambda)^{(1)}\) are shown in Fig. 4. Trajectory (i) is the analog of the barbs in \(x_1\)—except that this time the trajectory corresponds to two particles moving toward one another. Whenever \(x = n\lambda\), an image of each particle is suddenly created, and the two images persist for some time, after which they die simultaneously. Trajectory (ii) also involves barb-like motion on the part of both particles. In this case, when the distance between the two particles approaches \(n\lambda\), an image appears ahead of each particle and moves backwards, passing through the parent particle when the relative coordinate is \(n\lambda\), a short time after which each image dies. Note that all this is taking place in a momentum-conserving manner (since the images always co-exist and move in opposite directions with the same speed). In general, we find momentum-conserving trajectories most easily by mapping into \(x, X\).
This configuration is not allowed! The reason for this is that particle one must always include its left boundary, and particle two its right boundary. Thus a theory of these particles would be intrinsically parity-breaking. Such trajectories may be of interest if one wished to embed some of the features of the weak interactions directly in space–time. It is curious to note that, contrary to the case for the twin amoeba trajectories, parity need not be violated for amoeba-like trajectories in $x_1$, with $x_2$ constant. This is because these amoeba-like particles can be built to include their boundaries on either side. In three and four dimensions, we want to preserve rotational and Lorentz invariance, and we only consider a parity-invariant topological space.

It is worth mentioning in passing, however, that an intrinsic parity violation can be built into space–time, without any elementary length. For example, if one takes, on a one-particle subspace, the topology defined by the base

$$B = \{x: a \leq x < b\}$$

(A3)

for all real $a, b$, then one finds that the trajectories are of necessity extended in space, although the extension can be made as small as one pleases. If one picks the subset of trajectories that are bicontinuous, then the particles always include their left-hand, but not their right-hand boundaries. These trajectories then have no allowed mirror images. One might imagine that a theory could be set up in which neutrino trajectories, for example, were required to be bicontinuous between a parameter space and the topological space defined by Eq. (A3), while this requirement was not made for particles which interact strongly or electromagnetically. The way to break time-reversal invariance would be, of course, to take one of these one-sided topologies on the time-axis. Finally, we mention that our topology implies an “action at a distance.” For example, imagine two (twin) amoeboid particles, at rest, on opposite sides of the galaxy. By moving one particle a small distance, by means of conventional forces, during which it periodically emits and retracts pseudopods, we can cause the “fixed” particle to emit pseudopods simultaneously. (The “motion” of this particle never carries its center more than $\frac{1}{2} \lambda$ from its initial position.)

Action at a distance is not surprising in a nonrelativistic theory. We see below, however, that this characteristic can be preserved in a fully relativistic treatment.

2. Nonrelativistic Three-Dimensional Motion

The problem in three-dimensional motion is to propose a topology that contains an elementary length but is, at the same time, consistent with both translational and rotational invariance. The topology we
wish to study is a natural generalization of the $\lambda$-topology on the one-dimensional $x_1 \otimes x_2$. It is defined by the base:

$$B_{x_1 \otimes x_2}^{(3)} = \{ x, x : a < |x| < b \}$$

and all usual open cones in $X$, $n \lambda \leq |x| < (n + 1) \lambda$

and all usual open cones in $x$, where we have kept the same definitions of $X$, $x$ in terms of $x_1$, $x_2$ as in Eq. (A1), and where $a$, $b$ are any positive real numbers, $n$ any positive integer. Note particularly that in the relative topology of $X$, the point $x = 0$ is open.

We denote this (six-dimensional) topological space by $(x_1 \otimes x_2, \lambda)^{(3)}$. As before, the space is not Hausdorff, nor $T_0$. The reasons for this pathology are similar to those for $(x_1 \otimes x_2, \lambda)^{(1)}$, namely that, in taking a non-Hausdorff topology on $x$, we are giving up some ability to distinguish distances between the particles. Again we find that the space is only pseudo-metrizable. In that the coarsening is radial about either particle, the topology is rotationally invariant, and, of course, translationally invariant also.

The topology induced on the three-dimensional subspace of one particle is again strictly finer than the usual three-dimensional Euclidean topology. We can specify it by means of a base. For fixed $x_2$, a base for the relative topology on $x_1$ is the set of usual open $\epsilon$-spheres, centered about any point in the $x_1$ space, together with each point $x_1$ that satisfies $|x_1 - x_2| = n \lambda$, $n = 0, 1, 2, \ldots$. Hence this induced topology is the usual topology, plus a set of “preferred points,” just as in the one-dimensional case. In fact, the induced topology on any straight line, in the $x_1$ plane, running out of the point $x_1 = x_2$ is the topology of the positive $x_1$ axis in the one-dimensional example.

The (enlarged) set of continuous functions on $(x_1 \otimes x_2; \lambda)^{(3)}$ into $(\tau, U)$ is easily seen to contain, in addition to the usual $U$-continuous functions which may be $U$-discontinuous across the “preferred” spheres $|x_1 - x_2| = n \lambda$.

Trajectories in $(x_1 \otimes x_2, \lambda)^{(3)}$

Again we take Zeeman’s definition of a trajectory, as given above. A representative sampling of allowed trajectories that have inverses in their range is shown in Fig. 5. As in $(x_1, \lambda; x_2)^{(1)}$, so here the usual trajectories are in general excluded. Trajectory (i) is in fact usual: so long as a particle does not cross a preferred sphere, its trajectory is entirely ordinary. Note that, in another frame, particle two, originally at rest, will also appear to move normally. Trajectory (ii) is a normal barb trajectory, with images appearing on the preferred spheres. In another frame, particle two would move normally, while particle one would be barbed. The only situations in which both particles move normally are those in which the separation is never equal to an integral number of elementary lengths. Note also that the $x_1$ trajectories cannot be $U$-discontinuous in the angular variables about $x_2$.

In this topology, the barbed trajectories do not intrinsically violate parity invariance. In fact, we have not been able to find a rotationally invariant way of incorporating parity violation into a topology. Moreover, since the topology on the time-axis is still usual, there is no intrinsic violation of time-reversal invariance.

As in one dimension, the requirement of bicontinuity between the parameter space and $(x_1 \otimes x_2, \lambda)^{(3)}$ excludes all the barbed trajectories. Bicontinuous trajectories are easily generated by mapping into $x$ and $x$. These have the general form shown in Fig. 6 (in $x$ space). The trajectories correspond to a pair of amoeboid particles with a minimal radial spread (along their line of centers) of $\frac{1}{2} \lambda$ each. The angular extent of one particle about another can be made as small as one pleases. Motion along the line of centers
is accomplished by means of simultaneous, radial pseudo-pods, much as in the one-dimensional case. The angular motion of one particle about the other is entirely normal.

3. Four-Dimensional Relativistic Motion

The problem in four dimensions is to embed the elementary length in a way consistent with full inhomogeneous Lorentz invariance. This involves changing the topology along the time as well as the space axes. This enables one to break time-reversal invariance; but we choose not to do this: our topology is parity and this topology, we are sacrificing some knowledge of space axes. This enables one to break time-reversal invariance in a way consistent with full inhomogeneous Lorentz invariance. This involves

is clear that we have retained explicit Lorentz invariance; but we choose not to do this: our topology is parity and this topology, we are sacrificing some knowledge of space axes. This enables one to break time-reversal invariance in a way consistent with full inhomogeneous Lorentz invariance. This involves

We specify the non-Hausdorff space \((x_1 \otimes x_2, \lambda)^{(4)}\) by the base

\[
B^{(4)}_{x_1 \otimes x_2} = \left\{ X, x : \text{all usual open } \epsilon \text{-spheres in } X, \right. \\
\left. (x^2 \geq 0, n\lambda \leq x^2 < (n + 1)\lambda) \right. \\
\left. \text{and all usual open hyperbolic cones in } x \right\}
\]

Here we use the quadratic form \(x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2\). As before, \(n\) is any positive integer. With this topology, we are sacrificing some knowledge of the interval between two particle "observations." It is clear that we have retained explicit Lorentz invariance in our choice of open sets.

Suppose we fix \(x_2^2\) and consider the induced topology in \(x_1^2\) space. A base consists in all the usual open \(\epsilon\)-spheres, together with all the points lying on the "preferred hyperboloids" \((x_1 - x_2)^2 = n\lambda, n = -1, 0, 1, 2, \ldots\).

Note that, as it stands, the induced topology on the light cone \((x^2 = 0)\) is very coarse. In particular, in a \(1 + 1\) subspace, the only nontrivial open sets on the light cone would be the four quadrants of the light cone and the point \(x_1 = x_2\). This means that one could not separate points on a light ray sent between observers at \(x_1\) and \(x_2\). In the full four-dimensional space, however, it is easily seen that \(x_1\) can distinguish (by open sets) the direction of a light ray to \(x_2\), but cannot distinguish points on a given ray. The induced topology on the rest of the topological space \((x_1, \lambda; x_2)^{(4)}\)—that is, away from the light cone \(x^2 = 0\) is strictly finer than usual, so that in particular, a light ray from any point \(x_1\) aimed away from \(x_2\) travels over a very fine topology. This is also the case for general particle trajectories.

Actually, it is a simple matter to refine the topology on the light cone as far as one wishes, without essentially changing the induced topology on other subspaces.\(^{14}\) In this paper, we leave the question of the desirability of refining the light-cone topology as an open question.

The induced topology of most physical interest is on any subspace \(x_1^{(0)} = x_2^{(0)}\) (i.e., the two particles are usually considered at the same time). One sees easily that this subspace is, for all times \(t = x_1^{(0)} = x_2^{(0)}\), simply \((x_1 \otimes x_2, \lambda)^{(3)}\). Thus we have succeeded in embedding our three-dimensional topological space in space–time in a relativistically invariant way.

It is clear from the base \(B_{x_1 \otimes x_2}^{(4)}\) that we have embedded into the space some information about light-cone structure. If one wished to complete the job of embedding the light-cone structure into space–time, one would need to consider the ideas of Zeeman, according to which the topology is refined as far as is possible, consistent with the requirement that the relative topology on any time or space axis is usual. There seems no reason why this thoroughgoing refinement could not be combined with our notion of embedding an elementary length in a two-particle subspace. One would simply define the topology on \(x_1 \otimes x_2\) to be the finest consistent with the usual topology on \(X\) and an interval-coarsened topology on \(x\), considered along any space or time axis. We do not consider this idea any further here.

Relativistic Trajectories

For simplicity, we limit ourselves to mapping functions that are bicontinuous between the space \((x_1 \otimes x_2, \lambda)^{(4)}\) and \((\tau, U), \) where \(\tau\) is some invariant parameter, for example, the proper time of one of the two particles. This means that in the variable \(x\), with \(t_1 - t_2 = 0\), the trajectories are amoeba-like, just as we found above in \((x_1 \otimes x_2, \lambda)^{(3)}\). For identical particles, one would want to require the spread of each particle along the line connecting them to be the same in the center-of-mass frame.\(^{15}\) In this frame, the minimal radial spread of each particle is \(\frac{1}{2} \lambda.\)\(^{16}\) It follows that the minimum radial spread of one particle, in its rest frame, is \(\frac{1}{2} \lambda \gamma,\) where \(\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} > 1,\) and \(v\) is the velocity of the center-of-mass frame relative to this rest frame. Hence the spread of, say, the target particle in its rest frame increases (without bound) as the momentum of the incident particle increases. Again we see that the topology is coupling the particles at a pre-dynamical level.

Consider now particle one (say) in its rest frame. Suppose we attach synchronized clocks to different

\(^{14}\) Although if the light cone topology were so fine as to be discrete, this would add certain single point open sets to the relative topologies on lines in \(x_1 \otimes x_1\) that intersect \((x_1 - x_2)^2 = 0.\)

\(^{15}\) As in the one- and three-dimensional cases, one can find trajectories for which one particle is not extended in space, while the other is. This situation would persist in any Lorentz frame.

\(^{16}\) The angular spread of one particle about another can be made as small as one pleases. Note that the existence of a radial, but not an angular, spread in one frame ensures the same thing in all frames: in special relativity, straight lines always transform into straight lines.
parts of the (extended) particle. Then, in a moving frame we find that the particle is Lorentz contracted, and also that the leading edge is younger (earlier proper time) than the trailing edge. Of course, this must happen in any relativistic theory of extended particles. For example, it is presumably true for the dressed particles of ordinary field theory, but it is to be noted that in our case, even the bare particles in a field theory would in general be extended.

Note that, as mentioned in the one-dimensional nonrelativistic case, we still have action-at-a-distance (in the same sense as above), even though the open set structure is completely Lorentz invariant. Certainly, as we see in the section on dynamics, we can write equations of motion on the topology which are explicitly frame-independent. In this connection, it is important to remember that there is no contradiction between the Lorentz group and information transfer faster than light. Indeed, the Lorentz group admits of spacelike representation. Given (say) any “particle” of spacelike mass, one can always find Lorentz frames in which its velocity is infinite. (In our case these frames are the center-of-mass frames.) This of course is action-at-a-distance. Although causality is broken in our theory, the violation is in general only over intervals of order \( \lambda \).

Finally, we note that the topological space \((x_1 \otimes x_2, \lambda)\) is explicitly time-reversal and parity invariant (in the sense that open sets map into open sets under these transformations). This can be phrased, as above, in terms of trajectories and mirror trajectories, etc. An example of a relativistic topology which would intrinsically break time-reversal invariance can be defined by the base

\[ B = \{ X, x: \text{usual open } \varepsilon\text{-spheres in } X, \]

\[ \text{usual open } \varepsilon\text{-spheres in } x \text{ for } x^2 < 0, \]

and for \( x^2 \geq 0, \)

\[ n\lambda \leq x^2 < (n + 1)\lambda, \quad x_0 > 0, \]

\[ n\lambda < x^2 \leq (n + 1)\lambda, \quad x_0 < 0, \]

together with usual open hyperbolic cones. \( (A6) \)

In this space, the trajectories involving spatial extent for both particles would violate time-reversal invariance, while the trajectories for which one particle was a point particle would always go over into allowed time-reversed trajectories. [This is much like the situation with parity in \((x_1 \otimes x_2, \lambda)\).]

B. QUANTUM DYNAMICS ON THE NON-USUAL TOPOLOGIES

Our order of presentation in this section parallels that of Sec. A. After a brief discussion of the concept of metrization (explicitly only for the one-dimensional case), we discuss dynamics first in the one spatial dimensional case, then in higher dimensions. The metrization for each higher dimensional case is left to the corresponding section on dynamics.

1. Metrization

To study the dynamics of the particles on these topologies one must first seek a metric (or pseudometric) on the spaces. In general, each of the spaces discussed above is in fact only pseudometric— that is, if \( d(x, y) \) is the “best distance function” available on the space (compatible with its topology), then it is always necessary that for some distinct points \( x, y, d(x, y) = 0 \). As in Sec. A, we discuss first the one-dimensional motion and then work up to more dimensions.

What is a metric for \((x_1, \lambda; x_2)^{(1)}\)? Since this topological space contains the continuum of points \( x_1 \), the ordinary Euclidean metric is certainly a metric. It is not, however, the metric which metrizes the space, because the metric topology would be the usual topology, whereas \((x_1, \lambda; x_2)^{(1)}\) is strictly finer than usual. [The metric topology associated with a metric \( d(x, y) \) is that topology defined by the base of open \( \varepsilon \)-spheres \( d(x, y) < \varepsilon \).] We want to use a metric which metrizes the space, because this metric corresponds to making maximal use of our open sets. An \( \varepsilon \)-parametrized class of metrics with this property is \( (\varepsilon > 0) \)

\[ d(x_1, \hat{x}_1) = \begin{cases} |x_1 - \hat{x}_1| & \text{if neither } x_1 \text{ nor } \hat{x}_1 \text{ equals } n\lambda + x_2, \\ \text{equals } n\lambda + x_2, & \text{if either } x_1 \text{ or } \hat{x}_1 \text{ equals } n\lambda + x_2, \\ 0 & \text{whenever } x_1 = \hat{x}_1. \end{cases} \]

(B1)

To see what this class of metrics means physically, imagine the following thought experiment: An observer riding on the first particle \((x_1)\) watches the second particle (at point \( x_2 \)) as he approaches it. Because \( x_2 \) is a preferred point, the observer sees the particle at a distance \( |x_1 - x_2| + \varepsilon \). As \( x_1 \) approaches \( x_2 \), the observer sees the distance decreasing to \( \varepsilon \), then the distance changes abruptly to zero, and then to \( \varepsilon \) in the other direction, again abruptly. That is, the observer finds it impossible to approach the second particle smoothly. This is, of course, essentially the statement that the continuous functions on the space into \((t, U)\) may have \( U\)-discontinuities at the preferred points. In general the induced topology along any

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\(^{17}\) E. Wigner, (Lecture) Seminar on Theoretical Physics (Trieste, 1962), p. 64.

\(^{18}\) That is, the (pseudo-) metric topology should be the \( \lambda \)-topology.

\(^{19}\) Since \((x_1 \otimes x_2, \lambda)^{(1)}\) is a \( T \)-space, we are guaranteed that it is metrizable.

\(^{20}\) This emphasizes the difference between finitely curving a space (with the usual topology), and changing the topology.
direction in \( x_1 \otimes x_2 \) except (1) and (2) of Fig. 2 is fine and can be metrized in a similar fashion.

The space \((x, \lambda)^{(1)}\) cannot be metrized. However, a pseudo-metric which pseudo-metrizes the space is\(^{21}\)
\[
d(x, \bar{x}) = \lambda|x/\lambda| - \lambda|\bar{x}/\lambda|, \tag{B2}
\]
where \([x]\) is the number theory function meaning the largest integer not greater than \(x\). In particular, the pseudo-metric distance between two points that lie within the same preferred interval, say \(n\lambda \leq x < (n + 1)\lambda\), is zero. However, the triangular inequality is never violated, and \(d(x, \bar{x})\) is truly a pseudo-metric.

We find it convenient to set up a dynamics which is separable (in a sense to be explained below) in \(x, X\), since adequate operators are at hand (difference and differential operators, respectively). This is only a very simple way to proceed, but we envisage that a more thorough-going dynamics, freed from any such artificial separability constraint, would in general use some unfamiliar "combination" of differential and difference operators (such as, e.g., a \(U\)-discontinuity operator, which, when operating on continuous functions, would be different from the null operator in most directions). We do not discuss a classical mechanics on the topology, but, rather, go directly to a quantum mechanics. At this level we verify that our simple dynamics seems to describe quantized amoeboid trajectories.

2. One-Dimensional Quantum Mechanical Motion

We want to build a quantum mechanical description of the scattering of two particles of momentum and energy \((E_1, p_1)\) and \((E_2, p_2)\), respectively, on the topological space \((x_1 \otimes x_2, \lambda)^{(1)}\). The easiest way to do this is to suppose that the wavefunction can be written as the product
\[
\Psi(x_1, x_2, t; p_1, p_2) = \Psi(X) \psi(x) \exp\left\{\frac{i\hbar}{\lambda}(E_1 + E_2)t\right\}. \tag{B3}
\]
As ordinarily, we demand that the wavefunctions be continuous on their respective topologies into the usual topology. That is, \(\Psi(X)\) can be taken in the usual form
\[
\Psi(X) = \exp\left\{-\frac{i\hbar}{\lambda} P X\right\}, \quad P \equiv p_1 + p_2, \tag{B4}
\]
whereas \(\psi(x)\) is a block-type function.

Our job now is to set up a dynamics on \(x\). First we must seek a suitable momentum operator on the space.\(^{21}\) With the pseudo-metric discussed above, we clearly cannot define an ordinary differential operator. The closest analog to the usual correspondence \(p \rightarrow -i\hbar(d/dx)\) that we can define on the space is \(p \rightarrow -i\hbar\Delta\), where \(\Delta\) is the symmetric difference operator, such that for any continuous function \(f(x)\)
\[
Df(x) = \left[ f(x + \lambda) - f(x - \lambda) \right]/2\lambda. \tag{B5}
\]
This momentum is Hermitian with respect to the inner product
\[
\langle \psi_1, \psi_2 \rangle = \frac{1}{\lambda} \int_{-\infty}^{+\infty} \psi^*(x)\psi(x) \, dx = \lambda \sum_{n=-\infty}^{\infty} \psi^*(n\lambda)\psi(n\lambda). \tag{B6}
\]
Note that, with this inner product, the right and left difference operators are not Hermitian.

If we define a position operator, \(q\), with the diagonal representation
\[
\langle \psi, q \psi \rangle = \lambda|x/\lambda|, \tag{B7}
\]
then we find the following Lie algebra involving \(p\) and \(q\)
\[
\{q, p\} = \frac{i\hbar}{\lambda} \lambda, \quad \{q, q\} = \frac{i\lambda^2}{\hbar} \lambda, \quad \{p, q\} = 0, \tag{B8}
\]
where \(\lambda\), proportional to the commutator of \(p\) and \(q\), is an Hermitian averaging operator. It is defined in the \(q\)-diagonal representation by
\[
\text{af}(x) = \frac{i\hbar}{2}[f(x + \lambda) + f(x - \lambda)]. \tag{B9}
\]
Note that, as \(\lambda \rightarrow 0\), \(a \rightarrow 1\) (unity) and we recover the usual relations of quantum mechanics. In that the operators \(p, q\) are nonlocal (they couple functional values over a range \(2\lambda\)), one expects the particles to be spread in a way that involves both \(\hbar/m\) and \(\lambda\).

Suppose that \(\{q'\}\) is an eigenket of \(q\) with the eigenvalue \(q'\), and \(\{p'\}\) a ket in the dual (momentum) space with eigenvalue \(p'\). Then it is easy to show that
\[
\langle p' \mid q' \rangle = g(p')\exp\left\{\frac{i\lambda}{\hbar} \sin^{-1}\left(\frac{\lambda p'}{\hbar}\right)\right\}, \tag{B10a}
\]
where
\[
g(p') = \frac{\pi^\frac{1}{2}}{\lambda^\frac{1}{2}} \left(\frac{\hbar^2}{\lambda^2} - p'^2\right)^{-\frac{1}{4}}. \tag{B10b}
\]
This is a normalization function which guarantees the unitarity of this transformation function:
\[
\langle p' \mid p'' \rangle = \sum_{n=-\infty}^{\infty} \langle p' \mid n\lambda \rangle \langle n\lambda \mid p'' \rangle = \delta(p' - p''). \tag{B11a}
\]
\[
\langle m\lambda \mid n\lambda \rangle = \int_{-\infty}^{+\infty} \langle m\lambda \mid p \rangle \, dp \langle p \mid n\lambda \rangle = \delta_{mn}. \tag{B11b}
\]
The cut structure of \(\langle p' \mid q' \rangle\) and the evaluation of these sums are discussed in the Appendix.

One can guarantee that \(\langle p' \mid q' \rangle\) is also the wavefunction of a freely moving particle of energy\(^{22}\)
\[
\text{and the evaluation of these sums are discussed in the Appendix.}
\]

\(\text{One can guarantee that } \langle p' \mid q' \rangle \text{ is also the wavefunction of a freely moving particle of energy}\)

\(^{21}\) The pseudo-metric function \((B1)\) is not bicontinuous between its range and domain spaces. (The Euclidean metric is bicontinuous on the usual topology.) If one wanted to define a distance function, \(d(x, \bar{x})\) (on the usual topology) that was bicontinuous between the range and domain spaces, one would in fact be forced to use a multi-valued function, which is essentially an amoeboid trajectory (see Fig. 4) turned on its side. \(d(x, \bar{x})\) is not a metric or pseudometric (since, for example, the triangular inequality cannot be unambiguously satisfied—although the violation is always only of order \(\lambda\)). It is curious to note also that had we taken the base for \(x\) slightly differently, e.g., \((\alpha - 1)\lambda \leq x < \alpha \lambda\), then the space would not even be pseudo-metrizable.

\(^{22}\) Raising and lowering operators for \(q\) are \(s_+ = \alpha \pm \lambda k/\hbar\) which generate the eigenvalues \(nl\), \(n = \cdots -1, 0, +1, 2, \cdots \) starting from \(q = 0\). These are closely related to the right and left difference operators. With \(s_\pm\), one sees immediately that the Lie algebra \((B8)\) is reducible. In fact, \(\{q, s_\pm\} = \pm \lambda^2 k x, s_\pm = 0\). We are using the relative coordinate language; remember that
\[
p = \frac{d}{dq}(p_1 - p_2).\]
\[ E = \frac{p^2}{2m}, \quad m \text{ being the reduced mass, by taking as the Schrödinger equation} \]
\[ (p^2/2m)\psi = -(\hbar^2/2m)\nabla^2 \psi = E\psi. \quad (B12) \]

If one adds the time-dependence appropriate to the difference variable, it is easily seen that, with (essentially) usual assumptions at \( x = \pm \infty \), there is overall probability conservation.

\[ \frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx \psi(t, x)^2 = 0. \quad (B13) \]

This also holds if a time-independent potential is added to the Schrödinger equation (B12).

However, it is not possible to define a probability current that is meaningful to distances of order \( \lambda \).\(^{24}\)

That is, although there is never any probability loss or gain for regions large compared with \( \lambda \), one cannot watch the probability flux too carefully.

We turn now to a discussion of the potential-free motion of the two particles, in particular to a discussion of the function \( \langle q' | q' \rangle = \psi_p(q') \). For all momenta \( |p'| < \hbar/\lambda = p_c \) the particles move past one another freely, as in the usual topology. Above the critical energy \( E_c = \frac{p_c^2}{2m} \), \( \psi_p(q') \) has an exponential growth or decay in \( q' \). To keep probability conservation, we retain only the decay (in regions allowing arbitrarily large \( q' \)). For example, imagine two accelerators, at a distance \( x_0 \) from each other, directing beams of particles of equal energy at one another (we choose this center-of-mass experiment purely for simplicity). For clarity, suppose that all ordinary interactions between the particles are negligible (this is actually a high-energy approximation). At energies \( E < E_c \), the particles move past one another freely, so the transmission coefficient is unity. For \( E > E_c \), one finds backward scattering! In fact, the reflection and transmission coefficients depend on the location of the accelerators and detection apparatus (assumed located at the accelerators). The wavefunctions of the two beams\(^{25}\) are shown in Fig. 7. (The continuous curves of the figure should of course be the step-type functions appropriate to \( x \). For simplicity we forget this in the subsequent figures.) The absence of exponential tails outside \(-\frac{1}{2}x_0 < x < \frac{1}{2}x_0 \) is the boundary condition that the detecting apparatus stops any particle which reaches it. That is, the wavefunction of each beam decays exponentially after it leaves its accelerator. This is strikingly analogous to barrier penetration in ordinary quantum mechanics. Our situation is like having a strong repulsive potential of magnitude

\[ V = \frac{1}{2m} \left[ p^2 + \left( \frac{\hbar}{\lambda} \sinh^{-1} \left( \frac{2p}{\hbar} \right) \right)^2 \right] > E \quad (B14) \]

between the two particles. (Actually this “topological potential” fills the whole of space.) Thus, holding \( x_0 \) constant but increasing the beam momentum, one finds fewer and fewer particles from accelerator 1 reaching accelerator 2, and vice versa. The same thing happens for constant \( p \) as the distance of separation \( (x_0) \) increases. Thus, in this topology, experiments at very high energy become “configuration dependent” —observers close to the scattering region may observe different scattering patterns than more distant observers. All this is probability conserving, just as in the case of barrier penetration; that is, any particle from accelerator 1 not reaching accelerator 2 is found in the reflected beam detected at 1. Also, because the “barrier” is a function only of \( |x_1 - x_2| \), the scattering is momentum conserving, just as in ordinary quantum mechanics. Note, however, that at supercritical energies the back-scattering will take place independent of the distance of separation of the two particles. (The topology is effectively propagating information at infinity velocity.) This is action-at-a-distance again, just as discussed in Sec. A.

We can say a few things qualitatively about scattering in the presence of an ordinary potential [to be added to Eq. (B12)]. We do not study any particular potentials (although the Schrödinger difference equation is in general no more difficult to solve than the corresponding differential equation), but it is interesting to note the qualitative effect of attractive and repulsive potentials on the “topological scattering.” Consider a scattering in which the free wave approaches an attractive potential step with a somewhat sub-critical momentum; on the far side of the step the momentum \( p = [2m(E - V)]^{1/2} \) is larger than on the near side. If it exceeds \( p_c \), the effect is repulsion, or, more accurately, backward scattering of the attractive step. This occurs for a small step in ordinary potential theory, but in our topology, the repulsive effect is enhanced by taking the attraction stronger! On the other hand, in the vicinity of a repulsive potential, the topological scattering is not set in until \( E = \frac{p_c^2}{2m} + V > \frac{p_c^2}{2m} \).

\(^{24}\) This sort of difficulty is common to any nonlocal theory. See, for example, P. Kristenson and C. Møller, Dan. Mat. Fys. Medd. 27, (1952), and C. Bloch, ibid., 27, (1952).

\(^{25}\) We assume for clarity that the particles of each accelerator are distinguishable, so that we can talk of individual wavefunctions.
We also note that our quantum mechanics does indeed seem to describe quantized amoeboid trajectories (rather than quantized barbs or quantized discrete trajectories). To see this, one tries to construct the tightest possible wave packets. Clearly, these cannot be tightened below \( \lambda \), regardless of how large a momentum spread is allowed. Moreover, reattaching the correct time dependence, we see that the "particle" is always present (i.e., does not disappear and reappear rapidly, as it would for a discrete trajectory).

Finally, we mention that the wave equation (B12), taken together with (B3) and (B4), does not allow the calculation of a one-particle wavefunction or wave equation (say, for \( x_1 \) independent of \( x_2 \)). The topology has inextricably interwoven the two particles. This peculiarity will carry over into four dimensions.

3. Three-Dimensional Quantum Mechanical Motion

In the three-dimensional case, we again separate off the (ordinary) dynamics in \( X \) as above, and pseudometrize the difference variable \( x \) in analogy with the one-dimensional situation:

\[
d(x, x') = [(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2]^{1/2},
\]

\[
x_1 = \lambda [r/\lambda] \sin \theta \cos \varphi,
\]

\[
x_2 = \lambda [r/\lambda] \sin \theta \sin \varphi,
\]

\[
x_3 = \lambda [r/\lambda] \cos \theta,
\]

(B15)

and similarly for the primed quantities.

It turns out that it is not possible to find a set of three commuting momenta in this space. For example, define

\[
P_{x_1} = \hbar \left[ \cos \theta \partial_{x_1} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \right], \quad \lambda \equiv \lambda [r/\lambda],
\]

(B16)

where \( \partial_x \) is the symmetric difference operator

\[
\partial_x f(s, \theta, \varphi) = \begin{cases} [f(s + \lambda, \theta, \varphi) - f(s - \lambda, \theta, \varphi)]/2\lambda & \text{if } \neq 0, \\ [f(s + \lambda, \theta, \varphi) - f(s, \theta, \varphi)]/\lambda & \text{if } = 0
\end{cases}
\]

(B17)

(and similarly for \( p_{x_2}/p_{x_3} \) taking \( \partial/\partial r \rightarrow \partial_x \) in the usual spherical polar expressions). Although these operators are Hermitian with respect to the usual inner product, they do not commute. Physically, this is because they are not really generators of orthogonal translations. (This in turn is because the \( r \) translation cannot be infinitesimal.) On the other hand these "momenta" commute for very large \( r \) and can be thus used to classify "plane" waves\(^{28}\) according to a momentum vector \( p \), at least very far away from the target.

One can build parallel translation operators on the space in the following manner: Define \( D_{x_1} \), the translation operator in \( x_1 \), as that operator which, having changed \( r \) by \( \lambda \), readjusts \( \theta, \varphi \) so that \( x_2, x_3 \) are in the end unchanged.\(^{27}\) One finds that \( D_{x_1} \) depends on \( x_2 \) and \( x_3 \), which means that it fails to commute with \( D_{x_2}, D_{x_3} \) (defined in a similar way). Physically, this means that, for example, a translation in \( x_1 \) followed by a translation in \( x_2 \) is not equivalent to the operations in the opposite order—i.e., the difference space \( x \) is curved. The noncommutativity of \( D_{x_1}, D_{x_2} \) is illustrated in Fig. 8.

We take, as the coarsened form of the free Schrödinger equation on the difference variable \( (p = \hbar k) \),

\[
\frac{\partial^2 \chi}{\partial s^2} + \frac{1}{s^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial \chi}{\partial \theta} + \frac{1}{s^2 \sin^2 \theta} \frac{\partial^2 \chi}{\partial \varphi^2} + k^2 \chi = 0,
\]

(B18)

where \( \chi = \psi \). Note that, although the energy \( E = p^2/2m \) is well defined, the solutions are not (except at very large \( r \)) eigenfunctions of \( p_{x_1} \). Thus, although we can classify each wave according to its (asymptotic) momentum, we cannot, in the interaction region, resolve \( E \) accurately into the sum of the squares of any momenta. This is a general feature of spaces with noncommuting momenta. There are of course other ways of coarsening the Schrödinger equation, but the qualitative results we extract from the dynamics are independent of the particular dynamics we choose. With essentially the usual boundary conditions on the wavefunction, one can show that overall probability is conserved with this wave equation.

We turn now to solutions of the wave equation. It is easy to find spherical waves in the topology. The waves

\[
\chi(s) = \exp \left( \pm \frac{iE}{\hbar} s \sin^{-1} \left( \frac{p_1}{\hbar} \right) \right), \quad E = \frac{p^2}{2m}
\]

are exact solutions of (B18). Thus, for super-critical

\(^{28}\)We see the form these take below.

\(^{27}\)These operators do not in general approach the usual translation operators for large \( r \). That is, for example, far out on the \( x_2 \) axis, the only allowed translations in \( x_1 \) are very large. Moreover, these operators are not Hermitian.
momenta, waves trying to approach the scattering center will tend to be reflected, and waves attempting to radiate from the center will tend to be reflected back towards the center.\textsuperscript{28}

Of more physical interest are the analogs of plane waves on the topology. These are much more difficult to obtain. Towards this end, we guess a solution to Eq. (B19) of the form

\[ \chi(r, \theta, \varphi) = r \exp \left( \frac{ib}{\lambda} \sin^{-1} [\lambda f(\theta)] \right). \] (B20)

This function solves Eq. (B18) up to terms of order \( (\lambda/r) \) with the proviso that

\[ \left( \frac{d\varphi}{d\theta} f(\theta) \right)^2 = (1 - f^2)(1 - k^2 f^2). \] (B21)

Thus \( f \) is an inverse elliptic function. We content ourselves with a discussion of \( f \) in certain energy ranges.

To second order in \( \lambda k \), one finds

\[ \psi(r, \theta, \varphi) \sim \exp \left( \frac{ir}{\lambda} \sin^{-1} (\lambda k \cos \theta) \right). \] (B22)

which becomes more and more like a plane wave as \( \lambda k \) tends to zero. If we loosely define a current\textsuperscript{29} as proportional to \( \psi^* \tilde{\nabla} \psi - \psi \tilde{\nabla} \psi^* \), where \( \tilde{\nabla} \) is a coarsened gradient in spherical coordinates [as in Eq. (B16)], then one finds that the flux lines corresponding to the almost plane-wave solution (B22) bend into the scattering center as shown in Fig. 9, but straighten out again as usual for very large \( r \). Physically, this is like having a potential which is attractive for large \( |x| \) and repulsive for small \( |x| \), balanced in such a way that there is no scattering in the absence of a real potential. In the presence of such an ordinary potential, it is clear that scattering would be enhanced in the lower partial waves\textsuperscript{30} (since the effective impact parameter of each flux line is reduced by the “topological potential”).

At supercritical energies the solution (B22) is also valid so long as \( \theta \) is in the cone \( |\theta - \pi| \ll 1/\lambda k \). In this range the \( \sin^{-1} \) is imaginary and we find damping in this cone. That is, part of the incoming beam is back-scattered through the angles in this cone. Presumably this cone of backward (and large angle) scattering first appears at the critical energy with zero solid angle along the backward \( z \) axis, the solid angle of the back-scattering increasing as the energy increases. We have not yet been able to calculate the exact dependence of the solid angle on energy. Finally we note that, as the energy goes to infinity, the space becomes more and more opaque to the scattering wave, which is back-scattered completely—just as in the one-dimensional case.

At this point in our discussion, it is worth considering a variant of the present model. In particular, suppose our non-usual topology extended only out to a radius (say) \( b \) in \( x \) space. (This could roughly simulate an elementary length which decreased to zero for larger particle separations.) In terms of the topological potential, we would be chopping off most or all of its long-range attraction, leaving only the repulsive core. In this case, there would be scattering in the absence of an ordinary potential; the scattering would be like that from a soft repulsive sphere of radius \( b \) and hardness proportional to \( (\lambda k)^2 \).\textsuperscript{31} The overall scattering effect is shown in Fig. 10. We have in mind a smooth joining of flux tangents at the sphere boundary, but we do not go into the difficult questions associated with a rigorous embedding of the non-usual sphere in the usual topology.

As the scattering energy increases further, the bending effect becomes more pronounced, until finally, at the critical energy, a cone of large angle scattering opens up. (The cone begins purely backward as above, but subtends a larger solid angle with increasing energy.) At still higher energies, the embedded sphere

\textsuperscript{28} This latter situation is reminiscent of a bound-state wavefunction—the binding being done by the topological potential.

\textsuperscript{29} Recall that probability currents cannot be believed over distances of order \( \lambda \).

\textsuperscript{30} As we see below, the breakup into partial waves is essentially normal.

\textsuperscript{31} We learn more about the structure of this topological potential in the discussion below on partial wave analysis.
becomes totally opaque, large angle-scattering everything that hits it. As usual for hard-sphere scattering, there will be a forward diffraction peak as well at high energies.

We can discuss some other interesting properties of this model in terms of the individual partial waves. Because our coarsening of the space is spherically symmetric, and the angular part of the Schrödinger equation is unchanged, we can expand the wavefunction in the usual Legendre polynomials:

$$\psi(r, \theta) = \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \theta) \psi_l(r).$$  \hspace{1cm} (B23)

The (free) partial wave equation for \( \psi_l(\theta) = i \psi_l(\theta) \) is

$$D_l^2 \psi_l(\theta) + \left[ k^2 - l(l+1) \right] \psi_l(\theta) = 0.$$  \hspace{1cm} (B24)

If we keep the topology throughout all \( \theta \), the asymptotic solutions for large \( \theta \) are

$$\psi_l(\theta) \sim \exp \left\{ \pm i \frac{\theta}{\lambda} \sin^{-1}(\lambda k) \right\}.$$  \hspace{1cm} (B25)

The partial wave equation is very much like the one-dimensional wave equation discussed above—but this time with a repulsive centrifugal barrier. It is interesting to discuss the interplay of this barrier with the topology. For this it is convenient to go to the embedded case. At super-critical energies, the free solutions will look much as in Fig. 11. That is, the wave (of energy \( E > E_0 \)) propagates freely inward to \( b \), at which point it senses a repulsive thick spherical shell. This shell extends into roughly

$$r_0 = [(l+1)/2m(E-E_0)]^{\frac{1}{2}}$$  \hspace{1cm} (B26)

at which point the momentum \( p \) becomes again less than critical and the wave propagates freely (again) until it hits the centrifugal barrier itself—after this it decays into the barrier as usual. As the energy increases at fixed \( l \), the spherical shell becomes thicker and higher, the height growing with \( E \) essentially as in Eq. (B14). At fixed energy, the shell is thicker for smaller \( l \). Thus this effective potential induced on the topology is both energy-dependent and nonlocal.\(^{32}\)

A very curious feature of the shell is its hollow center, in which resonances (of energy) can easily be trapped. (Actually, they are trapped between the inside of the topological potential barrier and the centrifugal barrier.) The really peculiar thing about these topological resonances is their long lifetime: Because the shell is thicker and tougher at higher energies, the higher mass a resonance has, the longer it lives! Certainly there is no mechanism in ordinary dynamics which could produce such a particle.

### 4. Relativistic Quantum Mechanical Motion

Analogously to the one- and three-dimensional cases, we pseudo-metrize the difference variable \( x \), in such a way that the interval \( x^2 \) is discretized. Moreover, we for the most part content ourselves with checking that the equations in a \( 1+1 \) space (one space and one time dimension) still yield the back-scattering effect.

A serious difficulty in a naive time quantization is always that probability may leak in time. For example, if one quantizes a single particle’s time \( (t) \) with the “Schrödinger” equation

$$(i\hbar D_t \psi = E\psi, \hspace{1cm} (B27)$$

then one finds the solutions decaying in time for \( E > E_c \). This is highly undesirable, and we want to check that it is not occurring in our “difference-interval” quantization.

We study the scattering of two free bosons of mass \( \mu \) on \((x_1 \otimes x_2, \lambda)\). To get a suitable wave equation, we first factor the solution of the Klein–Gordon equation\(^{33}\) on the usual topology

$$(\Box^2 + \Box^2 + 2\mu^2)\psi(x_1, x_2) = 0 \hspace{1cm} (B28)$$

into a product of a center-of-mass factor \( \exp (i\mathbf{P} \cdot \mathbf{X}) \) and a solution of the equation in the relative coordinate:

$$(\Box^2 + \mu^2 - \frac{\Box}{\lambda^2})\psi(x) = 0, \hspace{1cm} (B29)$$

where \( s = (p_1 + p_2)^2 \). Our main interest is in the system as it appears to some observer whose time is \( t_0 \) i.e., \( t_1 = t_2 = t_0 \) and \( t_1 - t_0 = 0 \). Thus we specialize for the moment to \( x^2 \) spacelike. In particular, for spatial \( x \) positive, the equation in the difference variable can be written

$$\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \sigma \frac{\partial \varphi(\sigma, \theta)}{\partial \sigma} - \frac{1}{\sigma^2} \frac{\partial^2}{\partial \theta^2} \varphi(\sigma, \theta) + (\frac{\Box}{\lambda^2} - \mu^2)\varphi(\sigma, \theta) = 0,$$

where \( \sigma = (-x^2)^{\frac{1}{2}} \) and \( \theta = \tanh^{-1} (x_0/x_1) \). We can separate off the angular dependence by the “partial wave analysis”

$$\varphi(\sigma, \theta) = \frac{1}{\sigma^\frac{1}{2}} \int_0^\infty d\eta e^{-\eta|\sigma|} R_n(\eta; \sigma) R_n^* (\sigma)$$

\[ + \left( \frac{\Box}{\lambda^2} - \frac{n^2 + \frac{\Box}{\lambda^2}}{\sigma^2} \right) R_n(\sigma) = 0, \hspace{1cm} (B31)\]

\( ^{32} \) In the sense that the partial waves are decoupled, the induced potential is still central.

\( ^{33} \) Our quadratic form is \( \mathbf{P}^2 = \rho_k^2 - |\mathbf{p}|^2 = \mu^2. \)
where a “prime” means differentiation with respect to \( \sigma \). With this form in mind, we are in a position to write a suitable generalization of the 2 particle Klein–Gordon equation on our space. We define \( \varphi(x_1x_2) \) as the product of \( \exp (iPX) \) and \( \varphi(\sigma, 0) \) as shown in (B30), but this time with \( R_n \) satisfying

\[
\mathcal{D}_n^2 R_n(\sigma) + \left( \left( \left( 1 - \mu^2 \right) - \left( n^2 + \frac{3}{2} \right) \right) R_n(\sigma) = 0. \right. \tag{B32}
\]

This is of the form of the coarsened partial wave Schrödinger equations (B24) in three dimensions—but with

\[
\varphi_{2n} \sim \frac{1}{2} s - \mu^2, \quad \varphi(l + 1) \sim n^2 + \frac{3}{2}
\]

we have in mind the same procedure for each of the other three orbits about the light cone; that is, coarsened only the “partial wave” equations.

As in the three-dimensional case, we find that there is a critical point in Eq. (B32) for increasing \( s \). In fact (ignoring for the moment the “centrifugal barrier”), the propagation is no longer free for energies \( s \) greater than

\[
s_e = 4|\mu^2 + (\hbar c/\lambda)^2|.
\]

(B33)

Physically, this means that the aforementioned observer sees at any time \( t_0 \) a system wavefunction like Fig. 7—i.e., the particles are back-scattering off each other’s topological potential—just as in the non-relativistic cases. This is not surprising—as we pointed out in Sec. A, the topology induced on this observer’s subspace at any time is exactly \((x_1 \otimes x_2, \lambda)^{(a)}\). In particular, the “bounce” of the two particles is simultaneous in the center-of-mass frame. In this frame then the “forces of topological repulsion” propagate at infinite velocity (although in other frames there is in general a time lag between the bounce of one and the bounce of the other). As mentioned in Sec. A, there is no contradiction between the Lorentz group and action-at-a-distance. Of course, the “centrifugal barrier” introduces the same qualitative features discussed in the three-dimensional section. Note that the behavior of the system with \( t_0 \) is entirely independent of \( t_1 - t_2 \). Thus, our coarsening of the difference variable has avoided any loss of probability with \( t_0 \).\(^{24}\)

5. Many Free Particles and Interactions

Thus far we have treated only the two-particle subspace with our non-usual topology. The analog of our relativistic topology on the \( n \)-particle subspace would be to coarsen the topology on some subset of difference variables. Requiring that all \( n(n - 1) \) difference variables are coarse would be too restrictive:

For large \( n \), there would be in general only one available configuration for the particles (collinear) even at low energies. This precludes a coarsening that puts all \( n \) particles on an equal footing.

In electrodynamics one could certainly, for example, coarsen the topology between every electron and positron that were created together. In this way, every electron (or positron) would have a “memory”—showing up only when the relative momentum of the pair was supercritical: If one could obtain, say, the electron of a pair created on the other side of the galaxy, then, by raising the electron’s momentum above critical (relative to the positron), one could affect the positron. (For example, firing the electron towards the positron, they would scatter off one another at a large angle.) In this way, one could transfer information over large distances instantaneously. Notice that these long distance effects could be avoided by cutting off the topology at a relativistic radius analogous to \( b \) in the three-dimensional case.

In the case of the recent pair production experiments, one would obtain enhancement of large angle production due to the “topological repulsion” between the electron and the positron. On the other hand, there would be no effect on ordinary electron–positron scattering (unless they were originally created together). Putting the new topology only between \( e^+e^- \) and not between \( \mu^+\mu^- \) could distinguish between these two situations, leaving the \( \mu \) case normal.

Because the new topology is always on a two-particle subspace, and one-particle wave equations (say, for \( x_1 \) independent of \( x_2 \)) cannot be found, it does not seem possible to formulate a one-particle-equation-of-motion type of field theory, or for that matter, to write down a Lagrangian in any simple sense. It seems to us that the simplest way to construct a Lorentz-invariant theory allowing for particle creation would be in terms of the generalized unitarity equations of axiomatic field theory.\(^{35}\) For example, consider the equations for the retarded functions. These equations are ordinarily written in terms of difference variables. One could then use the coarsened Klein–Gordon operators (discussed above) in the (appropriate electron-positron) difference variables. For \( \Delta_n(u - v) \) one would want to use the solution to our coarsened difference Klein–Gordon equation

\[
\Delta_n(x) \sim \int_{e,c} \frac{d^4k}{k^2 - \mu^2} \phi_k(x),
\]

(B34)

\(^{35}\) See, for example, K. Nishijima, Phys. Rev. 119, 485 (1960); 122, 298 (1961); 124, 255 (1961).
where $c_\phi$ is a counterclockwise circle about $k_0 = +(k^2 + \mu^2)^{1/2}$ for $x_\phi > 0$, and a clockwise circle about $k_0 = -(k^2 + \mu^2)^{1/2}$ for $x_\phi < 0$—with the proviso that when a pole meets a branch point of $\phi_k(x)$, it moves onto that side of the cut with exponentially decreasing behavior. Thus one can begin the usual iterative solution of these equations. Of course there will be the usual equal-time ambiguities at each order—which can be used as usual to specify the “interaction.” To obtain the scattering amplitudes from the retarded formulas, one would want to use the usual formulas, only this time replacing appropriate pairs of Fourier transform factors by solutions of our coarsened Klein–Gordon equation. Detailed discussion of such a program is beyond the scope of the present work. However, it should be emphasized that, whatever the interaction chosen, at ultra-high energies the elastic back-scattering will dominate, in that it will in general prevent the particles from reaching the interaction region.

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APPENDIX
The transformation function from configuration to momentum space [Eq. (B10a)], can be written

$$\langle p \mid n\lambda \rangle = \frac{(1 - \lambda^2 p^2/\hbar^2)^{1/2} + i(\lambda p/\hbar)^n}{\pi^{3/2}(\hbar^2/\lambda^2 - p^2)^{1/2}}.$$  

This function is defined to be cut for $(-\infty < p \leq -\hbar/\lambda)$ and $(\hbar/\lambda \leq p < \infty)$, and the first (“physical”) sheet is specified by

$$-\pi < \arg ((1 - \lambda^2 p^2/\hbar^2)^{1/2} + i(\lambda p/\hbar)^n) < \pi.$$  

It can be seen easily that, for $n > 0$, $\langle p \mid n\lambda \rangle$ tends to zero as $p \to \pm \infty + i\epsilon$, $\epsilon > 0$, while it is unbounded as $p \to \pm \infty - i\epsilon$. For $n < 0$, the same statements hold if the sign of $\epsilon$ is changed. Hence we make the rule that, whenever, an integral over $-\infty < p < \infty$ of an integrand involving $\langle p \mid n\lambda \rangle$ has to be performed, we take the integration contour just above/below the cuts in the $p$-plane for $n > / < 0$. With this prescription it is easy to demonstrate Eqs. (B11).

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Modification of the Ehrenfest Model in Statistical Mechanics

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The Ehrenfest model has been used to explain the “irreversibility” of thermodynamics and statistical mechanics. The modification described in this paper allows transitions to occur in both directions between the two “boxes” at each step of the model procedure. The equilibrium probability distribution is given in the form of a finite product, or in an iterated form particularly suitable for machine calculation. The analysis is illustrated by a simple model of an ionization–recombination process.

The Ehrenfest model of heat exchange between two bodies has been used successfully by Kac in discussing the relationship between “irreversibility” of thermodynamic laws and statistical mechanics. In this model two boxes represent two bodies in thermal contact, and their temperatures are represented by a number of balls contained in each box. At successive intervals of time, a single ball moves from one box to the other according to a probabilistic law: the probability of transition is simply the ratio of the present number of balls in the box to the total number of balls in both boxes. Since the sum of these two ratios is unity, there is always exactly one transition at each step of the model procedure.

This note is concerned with a slightly more complicated set of transition probabilities in which transitions occur both ways between the two boxes at each step of the model procedure. Furthermore, the transition probabilities are not necessarily the ratios of the number of balls in each box. To illustrate, let $p_{1\rightarrow 2}(k)$ be the probability of a ball going from box 1 to box 2 at a given step of the procedure when there are exactly $k$ balls in box 1 prior to the transfer. Similarly, let $p_{2\rightarrow 1}(k)$ be the probability of a ball going from box 2 to box 1 at a given step of the procedure when there are exactly $k$ balls in box 1 (not box 2) prior to the

\begin{footnotesize}
\begin{enumerate}
\item P. Ehrenfest and T. Ehrenfest, Z. Physik 8, 311 (1907).
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