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The Coulomb and Coulomb-like off-shell Jost functions

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Published in:
Journal of Mathematical Physics

DOI:
[10.1063/1.524161](https://doi.org/10.1063/1.524161)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
1979

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):
van Haeringen, H. (1979). The Coulomb and Coulomb-like off-shell Jost functions. *Journal of Mathematical Physics*, 20(6), 1109-1114. <https://doi.org/10.1063/1.524161>

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Citation: [Journal of Mathematical Physics](#) **20**, 1109 (1979); doi: 10.1063/1.524161

View online: <https://doi.org/10.1063/1.524161>

View Table of Contents: <http://aip.scitation.org/toc/jmp/20/6>

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The Coulomb and Coulomb-like off-shell Jost functions

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(Received 29 August 1978)*

The off-shell Jost functions are studied for a potential which is the sum of the Coulomb potential and an arbitrary local short-range central potential. We derive their singular on-shell behavior and their connection with the pure Coulomb off-shell Jost functions. For the latter we derive a large variety of interesting explicit analytic expressions which are useful for various purposes.

1. INTRODUCTION

In this paper we investigate the off-shell Jost functions $f_{cl}(k, q)$ for the Coulomb potential and the off-shell Jost functions $f_l(k, q)$ for a Coulomb plus short-range potential, $V = V_c + V_s$, where V_s is assumed to be local and central. As is now well known, these off-shell Jost functions are particularly interesting in connection with the transition matrices.

In Sec. 2 we show that $f_{cl}(k, q)$ is a basic constituent of $f_l(k, q)$. In particular, we prove that $f_l(k, q)$ has exactly the same singularity in $q = k$ as $f_{cl}(k, q)$. In order to obtain the most convenient formula for $f_{cl}(k, q)$, a regrouping of certain hypergeometric function expressions has to be performed. By doing this we supply the supplementary proof of the simple formula for $f_{cl}(k, q)$ that we have given before.¹ This formula contains Jacobi polynomials and certain polynomials of two variables, A_l .

In Sec. 3 we derive a large number of interesting expressions for these polynomials A_l . Each of these is useful for different purposes, as is clearly illustrated at the end of Sec. 3. We shall use the notation of Ref. 1.

2. THE OFF-SHELL JOST FUNCTIONS

In this section we will express the off-shell Jost function $f_l(k, q)$ for a Coulomblike potential in terms of the Coulomb off-shell Jost function $f_{cl}(k, q)$. By using this expression the on-shell behavior at $q = k$ is easily obtained. Further, we shall sketch the derivation of a simple closed expression for $f_{cl}(k, q)$.

We start by noting that²

$$f_l(k, q) = 1 + \frac{1}{2}\pi q(q/k)' f_l(k) \langle ql \downarrow | V_l | kl + \rangle. \quad (2.1)$$

Here $f_l(k)$ is the Jost function and $|kl + \rangle$ the "outgoing" scattering state, with energy k^2 , for the potential $V_l = V_{cl} + V_{sl}$. We use the Coulomb analog of Eq. (2.1) and apply the two-potential formalism. In this way we get the convenient expression,

$$f_l^{-1}(k) f_l(k, q) = f_{cl}^{-1}(k) f_{cl}(k, q) + \langle kl - | V_{sl} G_l | X_l \rangle. \quad (2.2)$$

Here G_l is the partial-wave Green operator for V_l , and $|X_l \rangle$ is defined by

$$|X_l \rangle = \frac{1}{2}\pi k G_{0l}^{-1} [(q/k)^{l+1} |ql \uparrow \rangle_0 - |kl \uparrow \rangle_0].$$

By inserting

$$\langle p | ql \uparrow \rangle_0 = 2(\pi q)^{-1} (p/q)^l (p^2 - q^2)^{-1},$$

we obtain a simple expression for $|X_l \rangle$ in the momentum representation,

$$\langle p | X_l \rangle = (p/k)^l (k^2 - q^2)/(p^2 - q^2). \quad (2.3)$$

Equation (2.2) is very interesting, since it clearly shows that $f_l(k, q)$ has exactly the same singularity in $q = k$ as $f_{cl}(k, q)$. As a matter of fact, by using Eq. (2.3) we have

$$\lim_{q \rightarrow k} \omega |X_l \rangle = 0, \quad k > 0,$$

and therefore,

$$\lim_{q \rightarrow k} \omega f_l(k, q) = f_l(k), \quad k > 0. \quad (2.4)$$

Here

$$\omega \equiv \left(\frac{q-k}{q+k} \right)^{i\gamma} \frac{e^{\pi\gamma/2}}{\Gamma(1+i\gamma)} = \frac{f_{co}(k)}{f_{co}(k, q)}.$$

Now we are going to summarily derive explicit expressions for $f_{cl}(k, q)$ [cf. Eqs. (4) and (7) of Ref. 1]. In order to evaluate $\langle ql \downarrow | V_{cl} | kl + \rangle$, which occurs in

$$f_{cl}(k, q) = 1 + \frac{1}{2}\pi q(q/k)' f_{cl}(k) \langle ql \downarrow | V_{cl} | kl + \rangle,$$

we use the well-known expressions,

$$\begin{aligned} \langle ql \downarrow | r \rangle &= (2/\pi)^{1/2} i^{-l} h^{(+)}(qr), \\ &= (-)^l (2/\pi)^{1/2} (qr)^{-l} e^{iqr} {}_2F_0(-l, l+1; (2iqr)^{-1}) \end{aligned}$$

and

$$\begin{aligned} \langle r | kl + \rangle &= (2/\pi)^{1/2} l! [f_{cl}(k) (2l+1)!]^{-1} (2ikr)^l e^{-ikr} \\ &\quad \times {}_1F_1(l+1-i\gamma; 2l+2; 2ikr). \end{aligned}$$

By using Ref. 3, p. 278, one obtains

$$\begin{aligned} \langle ql \downarrow | V_{cl} | kl + \rangle &= 2i\gamma l! [\pi q f_{cl}(k) (2l+1)!]^{-1} \\ &\quad \times \sum_{m=0}^l (m+1)_l (k/q)^m z^{l+1-m} \\ &\quad \times {}_2F_1(l+1+i\gamma, l+1-m; 2l+2; z), \end{aligned} \quad (2.5)$$

where $z = 2k/(q+k)$. The important step now is to separate off that part which contains the branch-point singularity in $q = k$. To this end we apply two transformations to the hypergeometric function ${}_2F_1$ on the right-hand side of Eq. (2.5) and find (Ref. 3, p. 47),

$$\begin{aligned} &{}_2F_1(l+1+i\gamma, l-m+1; 2l+2; z) \\ &= (1-z)^{m-i\gamma} \left(\frac{\Gamma(2l+2)\Gamma(i\gamma-m)}{\Gamma(l+1-m)\Gamma(l+1+i\gamma)} \right) z^{-2l-1} \end{aligned}$$

$$\begin{aligned} & \times {}_2F_1(m-l, -i\gamma-l; 1+m-i\gamma; 1-z) \\ & + \left(\frac{\Gamma(2l+2)\Gamma(m-i\gamma)}{\Gamma(l+1+m)\Gamma(l+1-i\gamma)} \right) z^{-2l-1} \\ & \times {}_2F_1(-m-l, i\gamma-l; 1-m+i\gamma; 1-z). \end{aligned}$$

The hypergeometric series for the ${}_2F_1$'s on the right-hand side break off. Therefore, these ${}_2F_1$'s can be rewritten in terms of Jacobi polynomials. One has, with $z = 2/(1+x)$,

$$\begin{aligned} & P_{l+m}^{(i\gamma-m, -i\gamma-m)}(x) \\ & = \binom{l+i\gamma}{l+m} z^{-m-l} {}_2F_1(-m-l, i\gamma-l; 1+i\gamma-m; 1-z), \end{aligned}$$

and so

$$\begin{aligned} & P_{l-m}^{(-i\gamma+m, i\gamma+m)}(x) \\ & = \binom{l-i\gamma}{l-m} z^{m-l} {}_2F_1(m-l, -i\gamma-l; 1-i\gamma+m; 1-z). \end{aligned}$$

When we insert all this in Eq. (2.5) we get a complicated expression. In order to simplify this expression we introduce the polynomials A_l ,

$$\begin{aligned} & A_l(q^2/k^2; \gamma^2) \\ & \equiv \sum_{m=0}^l \binom{l+m}{l} (-)^m \left(\frac{q}{k} \right)^{l-m} P_{l+m}^{(i\gamma-m, -i\gamma-m)} \left(\frac{q}{k} \right). \end{aligned} \quad (2.6)$$

Furthermore, we shall now prove that

$$\begin{aligned} & \sum_{m=0}^l \binom{l+m}{l} \left(\frac{k^2-q^2}{4kq} \right)^m P_{l-m}^{(m-i\gamma, m+i\gamma)} \left(\frac{q}{k} \right) \\ & = P_l^{(-i\gamma, i\gamma)} \left(\frac{q^2+k^2}{2kq} \right). \end{aligned} \quad (2.7)$$

For this proof we use

$$\begin{aligned} & P_n^{(\alpha, \beta)}(\xi) \\ & = \binom{n+\alpha}{n} {}_2F_1(-n, n+1+\alpha+\beta; 1+\alpha; \frac{1}{2} - \frac{1}{2}\xi), \end{aligned} \quad (2.8)$$

and the well-known integral representation,

$$\begin{aligned} & {}_2F_1(a, b; c; \xi) = \left(\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \right) \\ & \times \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-t\xi)^{-a} dt. \end{aligned} \quad (2.9)$$

The left-hand side of Eq. (2.7) then becomes

$$\begin{aligned} & \Gamma(l+1-i\gamma) [\Gamma(-i\gamma-l)\Gamma^2(l+1)]^{-1} \\ & \times \int_0^1 t^l (1-t)^{-i\gamma-l-1} [1 - \frac{1}{2}t(1-q/k)]^l \\ & \times \sum_{m=0}^l \binom{l}{m} [t^{-1} - \frac{1}{2}(1-q/k)]^{-m} \left(\frac{k^2-q^2}{4kq} \right)^m dt. \end{aligned}$$

By performing the summation and using again Eqs. (2.8) and (2.9) we obtain the desired expression, i.e.,

$$\begin{aligned} & \binom{l-i\gamma}{l} {}_2F_1(-l, l+1; 1-i\gamma; \frac{(q-k)^2}{4kq}) \\ & = P_l^{(-i\gamma, i\gamma)} \left(\frac{q^2+k^2}{2kq} \right). \end{aligned}$$

This completes the proof of Eq. (2.7).

By inserting the above expressions in Eq. (2.5) and using Eqs. (2.6) and (2.7) we obtain

$$\begin{aligned} & \langle ql | V_{cl} | kl + \rangle_c \\ & = 2c_{l\gamma} [\pi q f_{cl}(k)]^{-1} [-x^{-l} A_l(x^2; \gamma^2) \\ & + f_{c0}(k, q) P_l^{(-i\gamma, i\gamma)}(u)], \end{aligned} \quad (2.10)$$

cf. Eq. (7) of Ref. 1. Here $x = q/k$, $u = (q^2 + k^2)/(2kq)$,

$$f_{c0}(k, q) = \left(\frac{q+k}{q-k} \right)^{i\gamma},$$

and

$$c_{l\gamma} = \frac{\Gamma^2(l+1)\Gamma(1+i\gamma)\Gamma(1-i\gamma)}{\Gamma(l+1+i\gamma)\Gamma(l+1-i\gamma)}. \quad (2.11)$$

In Sec. 3 we will derive a large number of useful expressions for the polynomials A_l .

3. THE TWO-VARIABLE POLYNOMIALS A_l

In this section we shall derive a number of interesting explicit expressions for the polynomials A_l that occur in the formula (2.10) for the Coulomb off-shell Jost functions

$f_{cl}(k, q)$.

To start with, we have

$$A_l \equiv A_l(x^2; \gamma^2) = \sum_{n=0}^l \binom{l+n}{l} (-)^n x^{l-n} P_{l+n}^{(i\gamma-n, -i\gamma-n)}(x) \quad (3.1)$$

[Eq. (2.6)], where $x = q/k$. Substitution of

$$\begin{aligned} & P_{l+n}^{(i\gamma-n, -i\gamma-n)}(x) \\ & = \Gamma(l+1+i\gamma) [\Gamma(l+n+1)\Gamma(l-n+1)\Gamma(i\gamma-l)]^{-1} \\ & \times \int_0^1 t^{l-n} (1-t)^{i\gamma-l-1} [1 - \frac{1}{2}t(1-x)]^{l+n} dt \end{aligned}$$

yields

$$\begin{aligned} & A_l = \Gamma(l+1+i\gamma) [\Gamma^2(l+1)\Gamma(i\gamma-l)]^{-1} \\ & \times \sum_{n=0}^l \binom{l}{n} \int_0^1 (-tx)^{-n} [1 - \frac{1}{2}t(1-x)]^n (tx)^l \\ & \times [1 - \frac{1}{2}t(1-x)]^l (1-t)^{i\gamma-l-1} dt. \end{aligned}$$

The summation is easily carried out. We then get

$$A_l = 4^{-l} i\gamma c_{l\gamma}^{-1} \int_0^1 (1-t)^{i\gamma-l-1} [(2-t)^2 - x^2 t^2]^l dt, \quad (3.2a)$$

where $c_{l\gamma}$ is given by Eq. (2.11). The polynomials A_l can also be expressed in terms of Gegenbauer polynomials $C_n^{-\lambda}$. Indeed, by introducing $\tau = 1-t$ we get from Eq. (3.2a),

$$\begin{aligned} & A_l = \left(\frac{1-x^2}{4} \right)^l i\gamma c_{l\gamma}^{-1} \int_0^1 \tau^{i\gamma-l-1} \\ & \times \left(1 + \tau^2 - 2\tau \frac{x^2+1}{x^2-1} \right)^l d\tau. \end{aligned} \quad (3.2b)$$

It is well known that

$$(1 - 2\tau\xi + \tau^2)^\lambda = \sum_{n=0}^{\infty} C_n^{-\lambda}(\xi) \tau^n, \quad |\tau| < 1, \quad \lambda \neq 0.$$

Because of

$$C_n^{-l} \equiv 0, \quad n = 2l + 1, 2l + 2, \dots,$$

we can apply the above expansion to Eq. (3.2b), the result being,

$$A_l = \left(\frac{1-x^2}{4} \right)^l i\gamma c_{l\gamma}^{-1} \sum_{n=0}^{2l} \frac{1}{n-l+i\gamma} C_n^{-l} \left(\frac{x^2+1}{x^2-1} \right). \quad (3.3a)$$

By using

$$C_{l-n}^{-l}(\xi) \equiv C_{l+n}^{-l}(\xi), \quad -l \leq n \leq l,$$

we recast the above sum in the more convenient form,

$$\begin{aligned} \sum_{n=0}^{2l} \frac{1}{n-l+i\gamma} C_n^{-l} \left(\frac{x^2+1}{x^2-1} \right) \\ = -i\gamma \sum_{n=0}^l \frac{\epsilon_n}{n^2 + \gamma^2} C_{l \pm n}^{-l} \left(\frac{x^2+1}{x^2-1} \right). \end{aligned}$$

Here ϵ_n is the Neumann symbol,

$$\epsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n = 1, 2, 3, \dots \end{cases}$$

In this way we obtain from Eq. (3.3a),

$$A_l = \left(\frac{1-x^2}{4} \right)^l c_{l\gamma}^{-1} \gamma^2 \sum_{n=0}^l \frac{\epsilon_n}{n^2 + \gamma^2} C_{l \pm n}^{-l} \left(\frac{x^2+1}{x^2-1} \right). \quad (3.3b)$$

This expression can be rewritten in terms of the Jacobi polynomials $P_l^{(n, -n)}$. By using

$$\begin{aligned} C_n^{\lambda} \left(\frac{x^2+1}{x^2-1} \right) \\ = (\lambda)_n (n!)^{-1} \left(\frac{x+1}{x-1} \right)^n {}_2F_1 \left(\lambda, -n; 1-\lambda-n; \left(\frac{x-1}{x+1} \right)^2 \right), \\ = (\lambda)_n (n!)^{-1} \left(\frac{x-1}{x+1} \right)^n {}_2F_1 \left(\lambda, -n; 1-\lambda-n; \left(\frac{x+1}{x-1} \right)^2 \right), \end{aligned}$$

we derive the interesting relation,

$$\begin{aligned} \left(\frac{4x}{1-x^2} \right)^{-l} C_{l-n}^{-l} \left(\frac{x^2+1}{x^2-1} \right) \\ = \frac{\Gamma^2(l+1)}{\Gamma(l+n+1)\Gamma(l-n+1)} \left(\frac{1-x}{1+x} \right)^n \\ \times P_l^{(n, -n)} \left(\frac{1}{2}x + \frac{1}{2}x^{-1} \right), \quad |n| \leq l. \end{aligned} \quad (3.3c)$$

By inserting this in (3.3b) we get

$$\begin{aligned} A_l = x^l c_{l\gamma}^{-1} \gamma^2 \sum_{n=0}^l \frac{\epsilon_n}{n^2 + \gamma^2} \frac{\Gamma^2(l+1)}{\Gamma(l+n+1)\Gamma(l-n+1)} \\ \times \left(\frac{1-x}{1+x} \right)^n P_l^{(n, -n)} \left(\frac{1}{2}x + \frac{1}{2}x^{-1} \right). \end{aligned} \quad (3.4a)$$

Further, we have

$$P_l^{(n, -n)}(z) = (l+1)_n (z+1)^{n/2} (z-1)^{-n/2} \mathfrak{P}_l^{-n}(z),$$

where $\mathfrak{P}_l^{-n}$ is Legendre's function of the first kind. Substitution of this expression yields

$$\begin{aligned} A_l = x^l c_{l\gamma}^{-1} \gamma^2 \sum_{n=0}^l \frac{\epsilon_n}{n^2 + \gamma^2} \frac{\Gamma(l+1)}{\Gamma(l-n+1)} \mathfrak{P}_l^{-n} \left(\frac{1}{2}x + \frac{1}{2}x^{-1} \right), \\ 0 < x < 1. \end{aligned} \quad (3.4b)$$

When $x > 1$ the Legendre function here has to be multiplied by $(-)^n$.

From Eq. (3.2a) one can find an expression containing either ${}_2F_1(\dots; \frac{1}{2})$ or ${}_2F_1(\dots; -1)$ or ${}_2F_1(\dots; 2)$. It turns out that the formula with ${}_2F_1(\dots; 2)$ is the more convenient one. We obtain this formula by using the binomial expansion, which yields

$$\begin{aligned} A_l = 4^{-l} i\gamma c_{l\gamma}^{-1} \int_0^1 (1-t)^{i\gamma-l-1} \\ \times \sum_{m=0}^l \binom{l}{m} (2-t)^{2m} (-x^2 t^2)^{l-m}. \end{aligned}$$

By again using the binomial expansion,

$$(2-t)^{2m} = \sum_{n=0}^{2m} \binom{2m}{n} 2^n (-t)^{2m-n},$$

the integration can be performed, with the result,

$$\begin{aligned} \int_0^1 (1-t)^{i\gamma-l-1} t^{2l-n} dt \\ = \Gamma(2l-n+1) \Gamma(i\gamma-l) / \Gamma(i\gamma+l-n+1). \end{aligned}$$

In this way we get

$$\begin{aligned} A_l = 4^{-l} i\gamma c_{l\gamma}^{-1} (2l)! [\Gamma(i\gamma-l) / \Gamma(l+1+i\gamma)] \\ \times \sum_{m=0}^l \binom{l}{m} (-x^2)^{l-m} \sum_{n=0}^{2m} (-2m)_n (-l-i\gamma)_n \\ \times 2^n / [n! (-2l)_n]. \end{aligned}$$

The sum Σ_n is a terminating hypergeometric series for which we write ${}_2F_1(-2m, -l-i\gamma; -2l; 2)$. One should be careful here, since the third parameter, $-2l$, is a nonpositive integer. By using expression (2.11) for $c_{l\gamma}$ we obtain

$$\begin{aligned} A_l = 4^{-l} \binom{2l}{l} \sum_{m=0}^l \binom{l}{m} (-)^m x^{2l-2m} \\ \times {}_2F_1(-2m, -l-i\gamma; -2l; 2). \end{aligned}$$

We note that A_l is a function of γ^2 rather than of γ , as can be seen from Eq. (3.3b). So we have, by replacing m by $l-n$,

$$\begin{aligned} A_l = 4^{-l} \binom{2l}{l} \sum_{n=0}^l \binom{l}{n} (-)^{l-n} x^{2n} \\ \times {}_2F_1(2n-2l, -l \pm i\gamma; -2l; 2). \end{aligned} \quad (3.5a)$$

The hypergeometric function ${}_2F_1(\dots; 2)$ can be expressed in terms of a Jacobi polynomial with argument 0. By using Ref. 3, p. 212, we have

$$\begin{aligned} A_l = \frac{(-)^l}{l!} \sum_{n=0}^l \frac{(2n)!}{n!} \frac{(2l-2n)!}{(l-n)!} (-\frac{1}{4}x^2)^n \\ \times P_{2l-2n}^{(2n-l+i\gamma, 2n-l-i\gamma)}(0), \end{aligned} \quad (3.5b)$$

or

$$A_l = \frac{1}{l!} \left(\frac{1}{4}x^2\right)^l \sum_{n=0}^l \frac{(2n)!}{n!} \frac{(2l-2n)!}{(l-n)!} \left(-\frac{1}{4}x^2\right)^{-n} \times P_{2n}^{(l-2n+i\gamma, l-2n-i\gamma)}(0). \quad (3.5c)$$

Now we come to the derivation of the most elegant formula for A_l , i.e., a generalized hypergeometric function ${}_3F_2$ with argument $1-x^2$. From Eq. (3.2a) we have

$$A_l = i\gamma c_{l\gamma}^{-1} \int_0^1 (1-t)^{i\gamma-l-1} [1-t+\frac{1}{4}(1-x^2)t^2]^l dt.$$

After substitution of

$$[1-t+\frac{1}{4}(1-x^2)t^2]^l = \sum_{n=0}^l \binom{l}{n} (1-t)^{l-n} t^{2n} 2^{-2n} \times (1-x^2)^n,$$

we can perform the integration, the result being

$$\int_0^1 (1-t)^{i\gamma-l-1-n} t^{2n} dt = \Gamma(i\gamma-n)\Gamma(2n+1)/\Gamma(i\gamma+n+1).$$

In this way we obtain

$$A_l = c_{l\gamma}^{-1} \sum_{n=0}^l \frac{(2n)!}{n!} \frac{(-l)_n}{(1+i\gamma)_n(1-i\gamma)_n} 2^{-2n} (1-x^2)^n. \quad (3.6a)$$

By using the doubling formula for the gamma function we have

$$(2n)! = (\frac{1}{2})_n 2^{2n} n!,$$

and so

$$A_l = c_{l\gamma}^{-1} {}_3F_2(-l, 1, \frac{1}{2}; 1+i\gamma, 1-i\gamma; 1-x^2). \quad (3.6b)$$

An alternative expression is

$$A_l = \sum_{n=0}^l \frac{\Gamma(l+1+i\gamma)}{\Gamma(n+1+i\gamma)} \frac{\Gamma(l+1-i\gamma)}{\Gamma(n+1-i\gamma)} \times \frac{(\frac{1}{2})_n (x^2-1)^n}{\Gamma(l+1)\Gamma(l+1-n)}, \quad (3.6c)$$

where we have inserted Eq. (2.11). Furthermore, we have the terminating hypergeometric series,

$$A_l = \frac{(\frac{1}{2})_l}{l!} (x^2-1)^l \sum_{n=0}^l \frac{(i\gamma-l)_n (-i\gamma-l)_n}{(\frac{1}{2}-l)_n} \frac{(1-x^2)^{-n}}{n!}. \quad (3.6d)$$

From Eq. (3.6c) one can derive an expression involving a ${}_3F_2$ with argument 1. By inserting

$$(x^2-1)^n = \sum_{m=0}^n \binom{n}{m} x^{2m} (-)^{n-m}$$

in (3.6c) and introducing the new summation variable $v = n - m$, we have

$$\sum_{n=0}^l \sum_{m=0}^n \dots = \sum_{m=0}^l \sum_{v=0}^{l-m} \dots$$

It turns out that the sum Σ_v is a ${}_3F_2(\dots; 1)$, and thus we obtain

$$A_l = \sum_{n=0}^l \frac{x^{2n} (\frac{1}{2})_n}{\Gamma(l+1)\Gamma(l+1-n)} \times \frac{\Gamma(l+1+i\gamma)\Gamma(l+1-i\gamma)}{\Gamma(n+1+i\gamma)\Gamma(n+1-i\gamma)} \times {}_3F_2(n-l, n+1, n+\frac{1}{2}; n+1+i\gamma, n+1-i\gamma; 1). \quad (3.7)$$

We transform this ${}_3F_2$ into a ${}_3F_2$ with different parameters by applying a generalization of Dixon's theorem, see Slater (Ref. 4, p. 52),

$${}_3F_2(n-l, n+1, n+\frac{1}{2}; n+1+i\gamma, n+1-i\gamma; 1) = \Gamma \left[\begin{matrix} l-n+\frac{1}{2}, & n+1+i\gamma, & n+1-i\gamma \\ \frac{1}{2}, & l+1, & n+1 \end{matrix} \right] \times {}_3F_2(i\gamma, -i\gamma, l-n+\frac{1}{2}; \frac{1}{2}, l+1; 1).$$

Then we have from (3.7),

$$A_l = \frac{\Gamma(l+1+i\gamma)\Gamma(l+1-i\gamma)}{\Gamma^2(l+1)} \times \sum_{n=0}^l \frac{x^{2n}}{n!} \frac{(\frac{1}{2})_n \Gamma(l-n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(l-n+1)} \times {}_3F_2(i\gamma, -i\gamma, l-n+\frac{1}{2}; \frac{1}{2}, l+1; 1). \quad (3.8a)$$

Note that the hypergeometric series for this ${}_3F_2$ breaks off when $i\gamma = 0, -1, -2, \dots$. The case $i\gamma = 0$ corresponds to no Coulomb interaction at all. On the other hand, $i\gamma = -1, -2, -3, \dots$ occurs for the Coulomb bound states.

It is not difficult to derive from Eq. (3.8a) the corresponding series with *decreasing* powers of x . This expression has almost exactly the same form as (3.8a), namely,

$$A_l = \frac{\Gamma(l+1+i\gamma)\Gamma(l+1-i\gamma)}{\Gamma^2(l+1)} \sum_{n=0}^l \frac{x^{2l-2n}}{n!} \times \frac{(\frac{1}{2})_n \Gamma(l-n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(l-n+1)} \times {}_3F_2(i\gamma, -i\gamma, n+\frac{1}{2}; \frac{1}{2}, l+1; 1). \quad (3.8b)$$

By comparing this expression with Eq. (3.5c) we get the interesting equality

$${}_3F_2(i\gamma, -i\gamma, n+\frac{1}{2}; \frac{1}{2}, l+1; 1) = \frac{(-4)^n (l-n)! l! n!}{\Gamma(l+1+i\gamma)\Gamma(l+1-i\gamma)} P_{2n}^{(l-2n+i\gamma, l-2n-i\gamma)}(0).$$

In the particular case when $l = 2n$ this expression can be simplified. By using (e.g., Ref. 3, p. 167)

$$\Gamma(1+\mu)P_{\nu}^{-\mu}(0) = {}_2F_1(-\nu, \nu+1; \mu+1; \frac{1}{2})$$

$$= \Gamma(1+\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\mu) \times [\Gamma(1+\frac{1}{2}\mu+\frac{1}{2}\nu)\Gamma(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu)]^{-1},$$

we get

$$\begin{aligned} P_{2n}^{(i\gamma, -i\gamma)}(0) &= 2^{2n}\Gamma(\frac{1}{2}+\frac{1}{2}i\gamma+n)[\Gamma(\frac{1}{2}+\frac{1}{2}i\gamma-n)\Gamma(2n+1)]^{-1} \\ &= \frac{(-)^n n!}{(\frac{1}{2})_n} \binom{n-\frac{1}{2}+\frac{1}{2}i\gamma}{n} \binom{n-\frac{1}{2}-\frac{1}{2}i\gamma}{n}, \end{aligned} \quad (3.8c)$$

and so

$$\begin{aligned} {}_3F_2(i\gamma, -i\gamma, n+\frac{1}{2}; \frac{1}{2}, 2n+1; 1) \\ = \frac{\pi\Gamma^2(n+1)}{\Gamma(\frac{1}{2}+\frac{1}{2}i\gamma)\Gamma(\frac{1}{2}-\frac{1}{2}i\gamma)\Gamma(n+1+\frac{1}{2}i\gamma)\Gamma(n+1-\frac{1}{2}i\gamma)}, \end{aligned}$$

cf. Eqs. (2.3) and (3.13) of Ref. 4.

One can see from Eq. (3.6c) in particular that the degree of the polynomial $A_l \equiv A_l(x^2; \gamma^2)$ is l , both in x^2 and in γ^2 ,

$$A_l = \sum_{n=0}^l x^{2l-2n} D_n^{(l)}(\gamma^2), \quad (3.9a)$$

$$A_l = \sum_{n=0}^l \gamma^{2l-2n} F_n^{(l)}(x^2). \quad (3.9b)$$

Here $D_n^{(l)}$ and $F_n^{(l)}$ are certain polynomials of degree n . It turns out that Eq. (3.9b) is less suitable for practical applications, so we shall mainly restrict ourselves to the expansion in the $D_n^{(l)}$'s. One can also write A_l as

$$A_l = \sum_{n=0}^l \sum_{m=0}^n x^{2l-2n} \gamma^{2m} a_{n,m}^{(l)}. \quad (3.10)$$

Here the coefficients $a_{n,m}^{(l)}$ are *real positive* numbers, as can be proven with the help of Eq. (3.8).

It is of interest to discuss a number of special cases. In the first place we consider the zero-energy case, $k=0$. Recalling $x \equiv q/k$ and $\gamma \equiv -s/k$, we have from Eq. (3.6c),

$$A_l \simeq \gamma^{2l} (l!)^{-2} {}_3F_0(-l, l, \frac{1}{2}; -x^2/\gamma^2), \quad k \rightarrow 0,$$

and so

$$\lim_{k \rightarrow 0} \gamma^{-2l} A_l = (l!)^{-2} {}_3F_0(-l, l, \frac{1}{2}; -q^2/s^2). \quad (3.11)$$

On the other hand, for $k \rightarrow \infty$ we have $x \rightarrow 0$ and $\gamma \rightarrow 0$. In this case we get from Eq. (3.8),

$$A_l(0; 0) = a_{l,0}^{(l)} = \frac{(\frac{1}{2})_l}{l!} = 4^{-l} \binom{2l}{l}. \quad (3.12)$$

For $x=1$ one easily derives from Eq. (3.6b)

$$A_l(1; \gamma^2) = c_{i\gamma}^{-1} = \binom{l+i\gamma}{l} \binom{l-i\gamma}{l}. \quad (3.13)$$

The numbers $a_{n,m}^{(l)}$ ($n, m = 0, 1, \dots, l$) can be considered as a matrix, which is triangular because of

$$a_{n,m}^{(l)} = 0, \quad n < m.$$

The matrix elements on the principal axis are given by

$$a_{n,n}^{(l)} = \frac{4^{n-l} (2l-2n)!}{l! n! (l-n)!}. \quad (3.14)$$

In particular for $n=l$ one has

$$a_{l,l}^{(l)} = F_0^{(l)} = (l!)^{-2}. \quad (3.15)$$

Equation (3.14) is obtained by considering

$$D_n^{(l)}(\gamma^2) = \sum_{m=0}^n \gamma^{2m} a_{n,m}^{(l)}$$

and

$$\begin{aligned} D_n^{(l)}(\gamma^2) &= (-)^n 4^{n-l} \frac{(2n)!(2l-2n)!}{l! n! (l-n)!} P_{2n}^{(l-2n+i\gamma, l-2n-i\gamma)}(0) \\ &= (-)^n 4^{-l} \binom{2l}{l} \binom{l}{n} {}_2F_1(-2n, -l \pm i\gamma; -2l; 2). \end{aligned} \quad (3.16)$$

It is interesting to note the connection of $D_n^{(l)}$ with certain known polynomials, namely Krawtchouk's polynomials $k_n(z)$, which depend in addition on a positive variable $p < 1$ and a positive integer N . These polynomials are associated with the binomial distribution in probability theory. According to Refs. 5 and 6 one has, with $p = \frac{1}{2}$ and $N = 2l$,

$$\begin{aligned} k_{2n}(i\gamma + l) &= 4^{-n} \binom{i\gamma + l}{2n} {}_2F_1(-2n, i\gamma - l; 1 + l + i\gamma - 2n; -1) \\ &= P_{2n}^{(l-2n \pm i\gamma, l-2n \mp i\gamma)}(0). \end{aligned} \quad (3.17)$$

Since $k_n(z)$ is defined for an integer variable z only, $D_n^{(l)}$ may be considered as a generalization of k_{2n} .

For $\gamma=0$ we get from Eqs. (3.4a) and (3.6b),

$$\begin{aligned} A_l(x^2; 0) &= x^l P_l(\frac{1}{2}x + \frac{1}{2}x^{-1}) \\ &= {}_2F_1(-l, \frac{1}{2}; 1; 1-x^2). \end{aligned} \quad (3.18)$$

By using these expressions we obtain

$$a_{n,0}^{(l)} = a_{l-n,0}^{(l)} = D_n^{(l)}(0) = 4^{-l} \binom{2n}{n} \binom{2l-2n}{l-n}. \quad (3.19)$$

Further we derive from Eqs. (3.8c) and (3.15),

$$D_n^{(2n)}(\gamma^2) = \binom{n-\frac{1}{2}+\frac{1}{2}i\gamma}{n} \binom{n-\frac{1}{2}-\frac{1}{2}i\gamma}{n}, \quad (3.20)$$

which again shows the dependence on γ^2 rather than on γ .

For $x=0$ we have from Eq. (3.15),

$$A_l(0; \gamma^2) = D_l^{(l)}(\gamma^2) = (-)^l \binom{2l}{l} P_{2l}^{(i\gamma-l, -i\gamma-l)}(0). \quad (3.21)$$

In order to obtain explicit expressions for $A_0, A_1, \dots$, Eq. (3.6) is very useful. We first recast Eq. (3.6c) in a more explicit form,

$$\begin{aligned} A_l &= \binom{l+i\gamma}{l} \binom{l-i\gamma}{l} \sum_{n=0}^l \frac{(-l)_n (\frac{1}{2})_n}{(1+i\gamma)_n (1-i\gamma)_n} (1-x^2)^n \\ &= (l!)^{-2} \sum_{\nu=0}^l (-x^2)^\nu \sum_{n=\nu}^l \binom{n}{\nu} (-l)_n (\frac{1}{2})_n \prod_{m=n+1}^l (m^2 + \gamma^2). \end{aligned} \quad (3.22)$$

Therefore, we have

$$D_{l-\nu}^{(l)}(\gamma^2) = (l!)^{-1} \sum_{n=\nu}^l \binom{n}{\nu} \frac{(-)^{n+\nu} (\frac{1}{2})_n}{\Gamma(l+1-n)} \prod_{m=n+1}^l (m^2 + \gamma^2). \quad (3.23)$$

In particular,

$$\begin{aligned} D_0^{(l)} &= \frac{\Gamma(l+\frac{1}{2})}{l!\Gamma(\frac{1}{2})}, \\ D_1^{(l)} &= \frac{\Gamma(l-\frac{1}{2})}{l!\Gamma(\frac{1}{2})} (\gamma^2 + \frac{1}{2}l), \\ D_2^{(l)} &= \frac{\Gamma(l-\frac{3}{2})}{l!\Gamma(\frac{1}{2})} \frac{1}{2} [\gamma^4 + \gamma^2(3l-2) + \frac{3}{4}l(l-1)], \\ D_3^{(l)} &= \frac{\Gamma(l-\frac{5}{2})}{l!\Gamma(\frac{1}{2})} \frac{1}{6} \left[\gamma^6 + \gamma^4 \left(\frac{15}{2}l - 10 \right) \right. \\ &\quad \left. + \frac{1}{4}\gamma^2(45l^2 - 105l + 46) + \frac{15}{8}l(l-1)(l-2) \right]. \end{aligned} \quad (3.24)$$

Finally, we give the first four polynomials A_l in explicit form,

$$\begin{aligned} A_0 &= 1, \\ A_1 &= \frac{1}{2}(x^2 + 1 + 2\gamma^2), \\ A_2 &= \frac{1}{8}[3x^4 + 2x^2(1 + \gamma^2) + 3 + 8\gamma^2 + 2\gamma^4], \end{aligned}$$

$$\begin{aligned} A_3 &= \frac{1}{48} [15x^6 + 3x^4(3 + 2\gamma^2) + x^2(9 + 14\gamma^2 + 2\gamma^4) \\ &\quad + \frac{1}{3}(45 + 136\gamma^2 + 50\gamma^4 + 4\gamma^6)]. \end{aligned} \quad (3.25)$$

ACKNOWLEDGMENTS

I would like to thank Dr. T.H. Koornwinder and Dr. N.M. Temme for a number of interesting discussions. This investigation forms a part of the research program of the Foundation for Fundamental Research of Matter (FOM), which is financially supported by the Netherlands Organization for Pure Scientific Research (ZWO).

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