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The Coulomb and Coulomb-like off-shell Jost functions

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The off-shell Jost functions are studied for a potential which is the sum of the Coulomb potential and an arbitrary local short-range central potential. We derive their singular on-shell behavior and their connection with the pure Coulomb off-shell Jost functions. For the latter we derive a large variety of interesting explicit analytic expressions which are useful for various purposes.

1. INTRODUCTION

In this paper we investigate the off-shell Jost functions $f_{\alpha}(k,q)$ for the Coulomb potential and the off-shell Jost functions $f_{l}(k,q)$ for a Coulomb plus short-range potential, $V = V_c + V_s$, where $V_c$ is assumed to be local and central. As is now well known, these off-shell Jost functions are particularly interesting in connection with the transition matrices.

In Sec. 2 we show that $f_{\alpha}(k,q)$ is a basic constituent of $f_{l}(k,q)$. In particular, we prove that $f_{l}(k,q)$ has exactly the same singularity in $q = k$ as $f_{\alpha}(k,q)$. In order to obtain the most convenient formula for $f_{\alpha}(k,q)$, a regrouping of certain hypergeometric function expressions has to be performed. By doing this we supply the supplementary proof of the simple formula for $f_{\alpha}(k,q)$ that we have given before.1 This formula contains Jacobi polynomials and certain polynomials of two variables, $A_{l,\alpha}$.

In Sec. 3 we derive a large number of interesting expressions for these polynomials $A_{l,\alpha}$. Each of these is useful for different purposes, as is clearly illustrated at the end of Sec. 3. We shall use the notation of Ref. 1.

2. THE OFF-SHELL JOST FUNCTIONS

In this section we will express the off-shell Jost function $f_{l}(k,q)$ for a Coulomb-like potential in terms of the Coulomb off-shell Jost function $f_{\alpha}(k,q)$. By using this expression the on-shell behavior at $q = k$ is easily obtained. Further, we shall sketch the derivation of a simple closed formula for $f_{l}(k,q)$.

We start by noting that

$$f_{l}(k,q) = 1 + \frac{1}{\sqrt{2\pi}} \text{sgn}(q/k) \mathcal{F}_l(k) \langle q|l|V_l|kl+\rangle. \quad (2.1)$$

Here $f_{l}(k)$ is the Jost function and $|kl+\rangle$ the “outgoing” scattering state, with energy $k^2$, for the potential $V_l = V_{el} + V_{si}$. We use the Coulomb analog of Eq. (2.1) and apply the two-potential formalism. In this way we get the convenient expression

$$f_{l}(k,q)f_{l}(k,q) = f_{l}(k,q) + \langle kl - |V_{el}G_l|X_l\rangle. \quad (2.2)$$

Here $G_l$ is the partial-wave Green operator for $V_l$, and $|X_l\rangle$ is defined by

$$|X_l\rangle = \frac{1}{\sqrt{2\pi k G_{0l}}} \left[ (q/k)^l + |q|l\rangle_0 - |kl\rangle_0 \right].$$

By inserting

$$\langle p|q|l\rangle_0 = 2\pi (\pi)^{-1} (p/q)^l (p^2 - q^2)^{-1},$$

we obtain a simple expression for $|X_l\rangle$ in the momentum representation,

$$\langle p|X_l\rangle = (p/k)^l (k^2 - q^2)/(p^2 - q^2). \quad (2.3)$$

Equation (2.2) is very interesting, since it clearly shows that $f_{l}(k,q)$ has exactly the same singularity in $q = k$ as $f_{\alpha}(k,q)$. As a matter of fact, by using Eq. (2.3) we have

$$\lim_{q \to k} \text{sgn}(q/k) f_{l}(k,q) = 1, \quad k > 0,$$

and therefore,

$$\lim_{q \to k} \langle q|l|X_l\rangle = 0, \quad k > 0. \quad (2.4)$$

Here

$$\omega \equiv \left( \frac{q - k}{q + k} \right) \text{sgn}\left( \frac{q - k}{q + k} \right) = \frac{f_{\alpha}(k)}{f_{\alpha}(k,q)},$$

Now we are going to summarily derive explicit expressions for $f_{l}(k,q)$ [cf. Eqs. (4) and (7) of Ref. 1]. In order to evaluate $\langle q|l|V_{el}|kl+\rangle$, which occurs in

$$f_{l}(k,q) = 1 + \frac{1}{\sqrt{2\pi}} \text{sgn}(q/k) f_{l}(k) \langle q|l|V_{el}|kl+\rangle,$$

we use the well-known expressions,

$$\langle q|l|\rangle = (2\pi)^{-i/2} \left( \Gamma(1 + q) \right) f_{l}(k,q).$$

We obtain

$$\langle q|l|\rangle = (-1)^{(q+1)/2} e^{-iqr/2} (\pi)^{-1/2} e^{-iqr} F_{l}(l + 1 - i\gamma;2l + 2;ikr).$$

By using Ref. 3, p. 278, one obtains

$$\langle q|l|V_{el}|kl+\rangle = 2i\pi \left[ \pi G_{0l} \langle q|l|V_{el}|kl+\rangle \right]^{-1} \times F_{l}(l + 1 - i\gamma;2l + 2;ikr).$$

where $z = 2k/(q + k)$. The important step now is to separate off that part which contains the branch-point singularity in $q = k$. To this end we apply two transformations to the hypergeometric function $F_{l}$ on the right-hand side of Eq. (2.5) and find (Ref. 3, p. 47),

$$F_{l}(l + 1 - i\gamma;2l + 2;ikr) = (1 - z)^{m - i\gamma} \left( \Gamma(2l + 2) \Gamma(i\gamma - m) \right) \left( \Gamma(l + 1 - m) \Gamma(l + 1 + i\gamma) \right) z^{2l + 1},$$

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The hypergeometric series for the \( F_i \)'s on the right-hand side break off. Therefore, these \( F_i \)'s can be rewritten in terms of Jacobi polynomials. One has, with \( z = 2/(1 + x) \),

\[
P_{\nu}^{(\gamma - m, - \nu - m)}(x) = \frac{(1 + m) P_{\nu}^{(\gamma - m, - \nu - m)}(z)}{(l + m)!} F_i(- m - l, i \gamma - l; 1 + i \gamma - m; z - 2).
\]

and so

\[
P_{\nu}^{(\gamma - m, + \nu - m)}(x) = \frac{(l - m) P_{\nu}^{(- l - m, + \nu - m)}(z)}{(l - m)!} F_i(- m - l, - i \gamma - l; 1 - i \gamma + m; z - 2).
\]

When we insert all this in Eq. (2.5) we get a complicated expression. In order to simplify this expression we introduce the polynomials \( A_i \),

\[
A_i(q'^2/k^2, z'^2) = \sum_{m=0}^{\infty} \left( \frac{l + m}{l} \right) \left( \frac{q^2 - k^2}{4kq} \right)^m P_{\nu}^{(\gamma - m, - \nu - m)}(x).
\]

Furthermore, we shall now prove that

\[
\frac{1}{x} \int_0^x (1 + t)^{- \nu - 1} (1 - t)^{- \nu - 1} \, dt
\]

For this proof we use

\[
P_n^{(a, b)}(\xi) = \binom{n + a}{n} F_n(- n, n + 1 + a + b, 1 + \alpha \xi + \frac{b - \alpha}{2} \xi),
\]

and the well-known integral representation

\[
\nu F_i(a, b; c; \xi) = \frac{\Gamma(c) \Gamma(l + 1)}{\Gamma(b) \Gamma(c - b)} \left( \frac{\Gamma(c - b)}{\Gamma(c)} \right) \nu \sum_{m=0}^{\infty} \left( \frac{l + m}{l} \right) \left( \frac{\nu}{4kq} \right)^m dt.
\]

The left-hand side of Eq. (2.7) then becomes

\[
\Gamma(l + 1 - i \gamma) \Gamma(l - i \gamma - 1) \Gamma(l + 1) \nu \frac{(1 - t)^{i \gamma - 1} (1 + t)^{i \gamma - 1} \nu}{(1 - t)^{i \gamma - 1} (1 + t)^{i \gamma - 1} \nu}\nu \sum_{m=0}^{\infty} \left( \frac{l + m}{l} \right) \left( 1 - \frac{\nu}{4kq} \right)^m dt.
\]

By performing the summation and using again Eqs. (2.8) we obtain the desired expression, i.e.,

\[
(1 - i \gamma) F_i(- l, l + 1; 1 - i \gamma, - (q - k)^2) = \frac{\nu}{4kq} \nu \sum_{m=0}^{\infty} \left( \frac{l + m}{l} \right) \left( 1 - \frac{\nu}{4kq} \right)^m dt.
\]

This completes the proof of Eq. (2.7).

By inserting the above expressions in Eq. (2.5) and using Eqs. (2.6) and (2.7) we obtain

\[
\sum_{l=0}^{\infty} |x^{l+1} F_i(l+1; l+1; 1; \xi)|^2 = 2c_{\nu}(q')^{-\nu} f_{\nu}(k, q')^{-\nu} f_{\nu}(k, q')^{-\nu} f_{\nu}(k, q')^{-\nu},
\]

for the polynomials \( A_i \).

In Sec. 3, we will derive a large number of useful expressions for the polynomials \( A_i \).

3. THE TWO-VARIABLE POLYNOMIALS \( A_i \)

In this section, we shall derive a number of interesting explicit expressions for the polynomials \( A_i \), that occur in the Coulomb off-shell Jost functions \( f_{\nu}(k, q') \).

To start with, we have

\[
A_i = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(l + n)}{l} \left( \frac{q^2 - k^2}{4kq} \right)^m P_{\nu}^{(l + n, - \nu - n)}(x)
\]

Substitution of

\[
P_{\nu}^{(l + n, - \nu - n)}(x) = \frac{\Gamma(l + 1 + i \gamma) \Gamma(l + n + 1) \Gamma(l - n + 1) \Gamma(i \gamma - 1)}{\Gamma(l + 1 + i \gamma) \Gamma(l + i \gamma) \Gamma(l - n - 1) \Gamma(i \gamma - l)} - 1
\]

yields

\[
A_i = \Gamma(l + 1 + i \gamma) \Gamma(l + n + 1) \Gamma(l - n + 1) \Gamma(i \gamma - 1) - 1
\]

The sum is easily carried out. We then get

\[
A_i = 4 \nu c_{\nu} \nu \sum_{n=0}^{\infty} \nu \sum_{m=0}^{\infty} \frac{(l + n)}{l} \left( \frac{q^2 - k^2}{4kq} \right)^m dt.
\]

\[
\sum_{n=0}^{\infty} \nu \sum_{m=0}^{\infty} \frac{(l + n)}{l} \left( \frac{q^2 - k^2}{4kq} \right)^m dt.
\]

It is well known that

\[
(1 - 2\tau^2 + \tau^4)^{\nu} = \sum_{n=0}^{\infty} \nu \sum_{m=0}^{\infty} \frac{(l + n)}{l} \left( \frac{q^2 - k^2}{4kq} \right)^m dt.
\]

Therefore, these polynomials can be rewritten in terms of Gegenbauer polynomials \( C_\nu^{(l + n, - \nu - n)} \).
Because of
\[ C_n^{(a)} \equiv 0, \quad n = 2l + 1, 2l + 2, \ldots, \]
we can apply the above expansion to Eq. (3.2b), the result being,
\[ A_l = \left(1 - \frac{x^2}{4}\right)^{\frac{1}{
\mu + i
}} \sum_{l=0}^{2l} \frac{1}{n - l + i
} C_n \left(\frac{x^2 + 1}{x^2 - 1}\right). \quad (3.3a) \]
By using
\[ C_i^{(a)}(\xi) \equiv C_i^{(a)}(\xi), \quad l \leq n < l, \]
we recast the above sum in the more convenient form,
\[ \sum_{n=0}^{2l} \frac{1}{n - l + i
} C_n \left(\frac{x^2 + 1}{x^2 - 1}\right) = -i
\sum_{n=0}^{\infty} \frac{\epsilon_n}{x^2 + \gamma^2} C_i^{(a)}(\frac{x^2 + 1}{x^2 - 1}). \]
Here \( \epsilon_n \) is the Neumann symbol,
\[ \epsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n = 1, 2, 3, \ldots \end{cases} \]
In this way we obtain from Eq. (3.3a),
\[ A_l = \left(1 - \frac{x^2}{4}\right)^{\frac{1}{
\mu + i
}} \sum_{n=0}^{l} \frac{\epsilon_n}{x^2 + \gamma^2} C_i^{(a)}(\frac{x^2 + 1}{x^2 - 1}). \quad (3.3b) \]
This expression can be rewritten in terms of the Jacobi polynomials \( P^{(a)}(\mu - n) \). By using
\[ C_i^{(a)}(\xi) = (\lambda)_n^a(\xi)^a \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(x^2 + 1)^n}{x^2 - 1}, \]
we derive the interesting relation,
\[ \left(\frac{4\lambda - 1}{x^2 - 1}\right) \Gamma^{-\mu}(\frac{x^2 + 1}{x^2 - 1}) = P_i^{(a)}(\mu - n)(\frac{1}{x^2 + \gamma^2}) \Gamma^{-\mu}(\frac{x^2 + 1}{x^2 - 1}). \quad (3.3c) \]
When \( x > 1 \) the Legendre function here has to be multiplied by \((-1)^n\).

From Eq. (3.2a) one can find an expression containing either \( \mu_{F(n)} \) or \( \mu_{F(n)} \) or \( \mu_{F(n)} \). It turns out that the formula with \( \mu_{F(n)} \) is the more convenient one. We obtain this formula by using the binomial expansion, which yields
\[ A_l = 4^{\frac{1}{n}} \sum_{m=0}^{l} (l - m)^{2m}(\frac{x^2 + 1}{x^2 - 1}). \]
By again using the binomial expansion,
\[ (2 - t)^{2m} = \sum_{n=0}^{\infty} \frac{(-2)^m}{n!} (-1)^n t^{n+1}, \]
the integration can be performed, with the result,
\[ \int_0^1 (1 - t)^{\mu - l} t^{2l - n} dt = \frac{\Gamma(2 - n + 1)(\Gamma(\mu - l))/\Gamma(\mu + l + m + 1)}{\Gamma(2 + m - n + 1)} \]
In this way we get
\[ A_l = 4^{\frac{1}{n}} \sum_{m=0}^{l} \left(\frac{1}{n!}\right) (-1)^n x^{2l - 2m} \times P_i^{(a)}(\mu - n)(-1 - iy)^n \]
The sum \( \Sigma_d \) is a terminating hypergeometric series for which we write \( \mu_{F(2 - 2m - l - iy)} \). One should be careful here, since the third parameter, \( -2l \), is a nonpositive integer. By using expression (2.11) for \( c_{r_i} \) we obtain
\[ A_l = 4^{\frac{1}{n}} \sum_{m=0}^{l} \left(\frac{1}{n!}\right) (-1)^n x^{2l - 2m} \times P_i^{(a)}(2,l - l + iy; - 2l/2). \]
We note that \( A_l \) is a function of \( \gamma^2 \) rather than of \( \gamma \), as can be seen from Eq. (3.3b). So we have, by replacing \( m \) by \( l - n \),
\[ A_l = 4^{\frac{1}{n}} \sum_{m=0}^{l} \left(\frac{1}{n!}\right) (-1)^n x^{2l - 2m} \times P_i^{(a)}(2 + m, - l + iy; - 2l/2). \quad (3.5a) \]
The hypergeometric function \( \mu_{F(2 - 2m - l - iy)} \) can be expressed in terms of a Jacobi polynomial with argument 0. By using Ref. 3, p. 212, we have
\[ A_l = 4^{\frac{1}{n}} \sum_{m=0}^{l} \left(\frac{1}{n!}\right) (-1)^n x^{2l - 2m} \times P_i^{(a)}(2 + m, - l + iy; - 2l/2). \]
Further, we have
\[ P_i^{(a)}(\mu - n)(\xi) = (\mu + 1)_{\mu}(\mu + 1)_{\mu}(\xi - 1)^{\mu}/(\mu + n)_{\mu}(\xi - 1)^{\mu} \times P_i^{(a)}(\mu + 1)_{\mu}(\xi - 1)^{\mu}, \quad 0 < x < 1. \quad (3.4b) \]
or
\[ A_t = \frac{1}{l!} (\frac{4}{x^2})^l \sum_{n=0}^l (2n)! (2l-2n)! (\frac{4}{x^2})^{-n} \times P_{\frac{2n}{2n}}^{(l-2n+iy, l-2n-iy)} (0). \] (3.5c)

Now we come to the derivation of the most elegant formula for \( A_t \), i.e., a generalized hypergeometric function \( F \) with argument \( l - x^2 \). From Eq. (3.2a) we have
\[ A_t = \frac{i\gamma}{\sqrt{\pi}} e^{-\frac{1}{2}} \int_0^1 (1-t)^{\gamma-2} [1-t + \frac{1}{2} (1-x^2)t^2] dt. \]

After substitution of
\[ (1 - t + \frac{1}{2} (1-x^2)t^2) = \sum_{n=0}^l \binom{l}{n} (1-t)^n t^{2n} - 2n \times (1-x^2)^n, \]
we can perform the integration, the result being
\[ \int_0^1 (1-t)^{\gamma-1} - n t^{2n} dt = \Gamma (\gamma - n) \Gamma (2n + 1)/ \Gamma (\gamma + n + 1). \]

In this way we obtain
\[ A_t = c_n \frac{1}{l!} \sum_{n=0}^l \frac{(2n)!}{n!} \frac{(1 + i\gamma, n - 1 - i\gamma)}{2^{2n}(1-x^2)^n}. \] (3.6a)

By using the doubling formula for the gamma function we have
\[ (2n)! = (\frac{1}{2})_n 2^{2n} n!, \]
and so
\[ A_t = c_n \frac{1}{l!} F_2 (-l, 1; \frac{1}{2} l + i\gamma, 1 - i\gamma; 1 - x^2). \] (3.6b)

An alternative expression is
\[ A_t = \sum_{n=0}^l \frac{\Gamma (l + 1 + i\gamma) \Gamma (l + 1 - i\gamma)}{\Gamma (n + 1 + i\gamma) \Gamma (n + 1 - i\gamma)} \times \frac{1}{\Gamma (l + 1) \Gamma (l + 1 - n)} \times \frac{1}{\Gamma (\gamma)} \frac{\Gamma (l - n + \frac{1}{2})}{\Gamma (l - n + 1)} \times F_2 (i\gamma, -i\gamma l - n + \frac{1}{2} l l + 1; 1). \] (3.6c)

where we have inserted Eq. (2.11). Furthermore, we have the terminating hypergeometric series,
\[ A_t = \frac{1}{l!} (x^2 - 1)^l \sum_{n=0}^l \frac{(\gamma - l) \Gamma (l - 1) \Gamma (l - \gamma - 1)}{(\frac{1}{2} - l)_n \Gamma (l + 1) \Gamma (l + 1 - l)} n! \times \frac{1}{\Gamma (l + 1) \Gamma (l + 1 - l)} \times \frac{1}{\Gamma (\gamma)} \frac{\Gamma (l - n + \frac{1}{2})}{\Gamma (l - n + 1)} \times F_2 (i\gamma, -i\gamma n + \frac{1}{2} l l + 1; 1). \] (3.6d)

From Eq. (3.6c) one can derive an expression involving a \( F \) with argument 1. By inserting
\[ (x^2 - 1)^n = \sum_{m=0}^n \binom{n}{m} x^{2m} (-y)^{-m} \]
in (3.6c) and introducing the new summation variable \( v = n - m \), we have
\[ \sum_{n=0}^l \sum_{m=0}^l \ldots = \sum_{n=0}^l \sum_{m=0}^l \ldots. \]

It turns out that the sum \( \Sigma_v \) is a \( F \), and thus we obtain
\[ A_t = \frac{1}{l!} \sum_{n=0}^l \frac{x^{2n} (\frac{1}{2})_n \Gamma (l + 1 + i\gamma) \Gamma (l + 1 - i\gamma)}{\Gamma (l + 1) \Gamma (l + 1 - n)} \times \frac{1}{\Gamma (\gamma)} \frac{\Gamma (l - n + \frac{1}{2})}{\Gamma (l - n + 1)} \times F_2 (i\gamma, -i\gamma l - n + \frac{1}{2} l l + 1; 1). \] (3.7)

We transform this \( F \) into a \( F \) with different parameters by applying a generalization of Dixon's theorem, see Slater (Ref. 4, p. 52),
\[ F_2 (n - l, n + 1, n + \frac{1}{2} n + 1 + i\gamma; 1 - i\gamma; 1) = \Gamma [l + n + \frac{1}{2}, n + 1 + i\gamma, 1 + n - i\gamma] \frac{1}{\gamma} \frac{1}{l + 1} \times F_2 (i\gamma, -i\gamma l - n + \frac{1}{2} l l + 1; 1). \]

Then we have from (3.7),
\[ A_t = \frac{\Gamma (l + 1 + i\gamma) \Gamma (l + 1 - i\gamma)}{\Gamma (l + 1)} \times \frac{1}{\Gamma (\gamma)} \frac{\Gamma (l - n + \frac{1}{2})}{\Gamma (l - n + 1)} \times \frac{1}{\Gamma (l + 1) \Gamma (l + 1 - l)} \times F_2 (i\gamma, -i\gamma n + \frac{1}{2} l l + 1; 1). \] (3.8a)

Note that the hypergeometric series for this \( F \) breaks off when \( i\gamma = 0, -1, -2, \ldots \). The case \( i\gamma = 0 \) corresponds to no Coulomb interaction at all. On the other hand, \( i\gamma = -1, -2, -3, \ldots \) occurs for the Coulomb bound states.

It is not difficult to derive from Eq. (3.8a) the corresponding series with decreasing powers of \( x \). This expression has almost exactly the same form as (3.8a), namely,
\[ A_t = \frac{\Gamma (l + 1 + i\gamma) \Gamma (l + 1 - i\gamma)}{\Gamma (l + 1)} \times \frac{1}{\Gamma (\gamma)} \frac{\Gamma (l - n + \frac{1}{2})}{\Gamma (l - n + 1)} \times \frac{1}{\Gamma (l + 1) \Gamma (l + 1 - l)} \times F_2 (i\gamma, -i\gamma n + \frac{1}{2} l l + 1; 1). \] (3.8b)

By comparing this expression with Eq. (3.5c) we get the interesting equality
\[ F_2 (i\gamma, -i\gamma n + \frac{1}{2} l l + 1; 1) = \frac{(-4)^l (l - n)! \Gamma (l + 1 + i\gamma) \Gamma (l + 1 - i\gamma)}{\Gamma (l + 1 + i\gamma) \Gamma (l + 1 - i\gamma)} \times \frac{1}{\Gamma (l + 1 + i\gamma) \Gamma (l + 1 - i\gamma)} \times \times \frac{1}{\Gamma (l + 1) \Gamma (l + 1 - l)} \times \times F_2 (i\gamma, -i\gamma n + \frac{1}{2} l l + 1; 1). \]

In the particular case when \( l = 2n \) this expression can be simplified. By using (e.g., Ref. 3, p. 167)
\[ \Gamma(1 + \mu)P_{-\mu}(0) = \mathcal{F}(c, d, a + b + 1, 1) \]

\[ = \Gamma(1 + \frac{1}{2} \mu) \times [\Gamma(1 + \frac{1}{2} \mu + \frac{1}{2} \nu) / \Gamma(\frac{1}{2} + \frac{1}{2} \mu + \frac{1}{2} \nu)]^2, \]

we get

\[ P^{(2n)\mu}(0) = 2^{2n} \Gamma(\frac{1}{2} + \frac{1}{2} \mu + n) / [\Gamma(\frac{1}{2} + \frac{1}{2} \mu - n) \Gamma(2n + 1)]^{-1} \]

\[ = (-)^n n! \binom{n}{\frac{1}{2} + \frac{1}{2} \mu} \binom{n}{\frac{1}{2} - \frac{1}{2} \mu}, \]

(3.8c)

and so

\[ J_{\nu}(\mu, -\mu, n + \frac{1}{2}, 2n + 1, 1) \]

\[ = \frac{\pi \Gamma(n + 1)}{(\frac{1}{2})^n} \]

cf. Eqs. (2.3) and (3.13) of Ref. 4.

One can see from Eq. (3.6a) in particular that the degree of the polynomial \( A_i \equiv A \{ x; \nu \} \) is \( i \), both in \( x^2 \) and in \( \gamma^2 \).

\[ A_i = \sum_{n=0}^{\infty} x^{2i - 2n} D_{2n}^{(i)}(\gamma^2). \]

(3.9a)

Here \( D_{2n}^{(i)} \) and \( D_{2n}^{(i)} \) are certain polynomials of degree \( n \). It turns out that Eq. (3.9b) is less suitable for practical applications, so we shall mainly restrict ourselves to the expansion in the \( D_{2n}^{(i)} \)'s. One can also write \( A_i \) as

\[ A_i = \sum_{n=0}^{\infty} \sum_{m=0}^{n} x^{2i - 2n} \gamma^{2m} a_{n,m}^{(i)}. \]

(3.10)

Here the coefficients \( a_{n,m}^{(i)} \) are real positive numbers, as can be proven with the help of Eq. (3.8).

It is of interest to discuss a number of special cases. In the first place we consider the zero-energy case, \( k = 0 \). Recalling \( x \equiv \mu/\kappa \) and \( \gamma \equiv -s/\kappa \), we have from Eq. (3.6c),

\[ A_i \gamma^2 (l - 2)^{i} \mathcal{F}(l - 1, 1, 1; -x^2/\gamma^2), \]

\[ \text{for } k \rightarrow 0, \]

and so

\[ \lim_{k \rightarrow 0} \gamma^{2i} A_i = (l - 2)^{i} \mathcal{F}(l - 1, 1, 1; -x^2/\gamma^2). \]

(3.11)

On the other hand, for \( k \rightarrow \infty \) we have \( x \rightarrow 0 \) and \( \gamma \rightarrow 0 \). In this case we get from Eq. (3.8),

\[ A_i(0, 0) = a_{0,0}^{(i)} = \frac{(l - 2)}{l} \mathcal{F}(l - 1, 1, 1; -x^2/\gamma^2). \]

(3.12)

For \( x = 1 \) one easily derives from Eq. (3.6b),

\[ A_i(1; \gamma^2) = c_{\gamma^2} = \frac{(l + i \gamma)^{l} (l - i \gamma)^{l}}{l!}. \]

(3.13)

The numbers \( a_{n,m}^{(i)}(n, m = 0, 1, \ldots, l) \) can be considered as a matrix, which is triangular because of

\[ a_{n,m}^{(i)} = 0, \quad n < m. \]

The matrix elements on the principal axis are given by

\[ a_{n,n}^{(i)} = \frac{4^m \gamma^m (2l - 2n)!}{l! (l - n)!}. \]

(3.14)

In particular for \( n = l \) one has

\[ a_{l,l}^{(i)} = F_{l,l}^{(i)} = (l - 1)^{-1}. \]

Equation (3.14) is obtained by considering

\[ D_{n}^{(i)}(\gamma^2) = \sum_{m=0}^{n} \gamma^{2m} a_{n,m}^{(i)} \]

and

\[ D_{l}^{(i)}(\gamma^2), \]

\[ = (-)^n 4^{n} - l (2n)(2l - 2n)! / l! (l - n)! \]

\[ p_{n}^{(i)}(2l - 2n + i \gamma, l - 2n - i \gamma) (0). \]

(3.15)

\[ = (-)^n 4^{n} - l (2n)(2l - 2n)! / l! (l - n)! \]

\[ p_{n}^{(i)}(2l - 2n + i \gamma, l - 2n - i \gamma) (0). \]

(3.16)

It is interesting to note the connection of \( D_{n}^{(i)} \) with certain known polynomials, namely Krawtchouk’s polynomials \( k_{n}(z) \), which depend in addition on a positive variable \( p < 1 \) and a positive integer \( N \). These polynomials are associated with the binomial distribution in probability theory. According to Refs. 5 and 6 one has, with \( p = \frac{1}{2} \) and \( N = 2l \),

\[ k_{n}(i \gamma - l) \]

\[ = 4^n \frac{(l - n)(2l - 2n)!}{l! (l - n)!} \]

\[ p_{n}^{(i)}(2l - 2n + i \gamma, l - 2n - i \gamma) (0). \]

(3.17)

Since \( k_{n}(z) \) is defined for an integer variable \( z \) only, \( D_{n}^{(i)} \) may be considered as a generalization of \( k_{n}(z) \).

For \( \gamma = 0 \) we get from Eqs. (3.4a) and (3.6b),

\[ A_i(x^2; 0) = x^l P_l(\frac{x}{2} + \frac{x}{2}) \]

\[ = \mathcal{F}(l - 1, 1, 1; -x^2). \]

(3.18)

By using these expressions we obtain

\[ a_{n,0}^{(i)} = a_{1,0}^{(i)} = D_{0}^{(i)}(0) = 4^{-1} \left( \frac{2l}{l} \right)^{l} \left( \frac{2l - 2n}{l - n} \right). \]

(3.19)

Further we derive from Eqs. (3.8c) and (3.15),

\[ D_{n}^{(i)}(\gamma^2) = \left( \frac{n - \frac{1}{2} + \frac{1}{2} i \gamma}{n} \right) \left( \frac{n - \frac{1}{2} - \frac{1}{2} i \gamma}{n} \right) \]

\[ \text{which again shows the dependence on } \gamma \text{ rather than on } \gamma. \]

For \( x = 0 \) we have from Eq. (3.15),

\[ A_i(0; \gamma^2) = D_{l}^{(i)}(\gamma^2) = (-)^n \left( \frac{2l}{l - i \gamma - 1} \right) (0). \]

(3.21)

In order to obtain explicit expressions for \( A_n, A_{n+1} \ldots \), Eq. (3.6) is very useful. We first recast Eq. (3.6c) in a more explicit form,

\[ A_i = \frac{(l + i \gamma)^l (l - i \gamma)^l}{l!} \sum_{n=0}^{l} \frac{(- l)_n (\frac{1}{2})_n}{(1 + i \gamma)_n (1 - i \gamma)_n} (1 - x^2)^n \]

\[ = (l - 2)^{l} \sum_{n=0}^{l} (1 - x^2)^n \sum_{n=0}^{l} \frac{(- l)_n (\frac{1}{2})_n}{(1 + i \gamma)_n (1 - i \gamma)_n} \prod_{m=0}^{n-1} \left( m^2 + \gamma^2 \right). \]

(3.22)

Therefore, we have
\[ D^{(1)}_n (y^2) = (l)^{1} \sum_{n=1}^{l} \binom{n}{r} \frac{(-y)^r (y^2)^{n-r}}{\Gamma (l+1-n)} \prod_{m=n+1}^{l} (m^2 + y^2). \]

In particular,
\[ D^{(0)}_n = \frac{\Gamma (l + \frac{1}{2})}{l! \Gamma (\frac{1}{2})} , \]
\[ D^{(1)}_n = \frac{\Gamma (l - \frac{1}{2})}{l! \Gamma (\frac{1}{2})} (y^2 + \frac{1}{4}) , \]
\[ D^{(2)}_n = \frac{\Gamma (l + \frac{3}{2})}{l! \Gamma (\frac{3}{2})} \frac{1}{4} \{ \gamma^2 + \gamma^2 (3l - 2) + \frac{3}{2} l (l - 1) \} , \]
\[ D^{(3)}_n = \frac{\Gamma (l + \frac{5}{2})}{l! \Gamma (\frac{5}{2})} \frac{1}{6} \{ \gamma^2 + \gamma^2 (\frac{15}{2} l - 10) \}
+ \frac{1}{2} \gamma^2 (45 l^2 - 105 l + 46) + \frac{15}{8} l (l - 1) (l - 2) \].

Finally, we give the first four polynomials \( A_l \) in explicit form,
\[ A_0 = 1 , \]
\[ A_1 = \frac{1}{8} (x^2 + 1 + 2y^2) , \]
\[ A_2 = \frac{1}{8} [3x^4 + 2x^2 (1 + y^2) + 3 + 8y^2 + 2y^4] . \]

\[ A_3 = \frac{1}{48} [15x^4 + 3x^2 (3 + 2y^2) + x^2 (9 + 14y^2 + 2y^4) \]
\[ + \frac{1}{2} (45 + 136y^2 + 50y^4 + 4y^6)] . \]