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H. van Haeringen, and L. P. Kok

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The Coulomb unitarity relation and some series of products of three Legendre functions

H. van Haeringen  
Department of Mathematics, University of Technology, Delft, The Netherlands  
L. P. Kok  
Institute for Theoretical Physics, P. O. Box 800, University of Groningen, The Netherlands

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We obtain from the off-shell Coulomb unitarity relation a closed expression for \( \Sigma_{l=0}^{\infty}(2l + 1)P_l(x) \times Q_l(y) \times Q_l^{-1}(z) \), and we consider some related series of products of Legendre functions.

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In this paper we shall consider the Coulomb unitarity relation and derive from this relation a closed expression for an infinite series of products of three Legendre functions, \( P_l, Q_l, \) and \( Q_l^{-1} \) [see Eq. (12)]. By taking the limit \( r \to 0 \) we obtain agreement with an expression for the corresponding series, which exists in the literature. However, our expression has a much simpler form, which means that we have obtained a substantial reduction of the expression given in.\(^5\) After the derivation of our main result, Eq. (12), we shall briefly consider some related series of Legendre functions [see Eqs. (14)-(25)].

The unitarity relation, or generalized optical theorem, or Low equation, in quantum-mechanical scattering theory establishes a simple relation between the imaginary part of the scattering amplitude and the cross section. The left-hand side of Eq. (2) is known in closed form (Ref. 4). We rewrite the right-hand side by inserting (Ref. 4), and using the orthogonality relation

\[
\int \frac{P_l(\hat{k}) \times P_{l'}(\hat{k})}{\hat{k}} \, d\hat{k} = 4\pi(2l + 1)^{-2}P_{l'}(\hat{k}) |_y = r \, d\hat{k}.
\]

In Eq. (3), \(|k + \lambda\rangle\) is the partial-wave Coulomb scattering state. Denoting \( p^2 + k^2/(2\hbar) \) by \( \gamma \) and assuming \( \gamma > k \), we have

\[
\left\langle p | V_c | k + \lambda \right\rangle = 2\gamma(\gamma - 1) e^{i(\gamma - 1)/2} Q_l^{-1}(y),
\]

where \( y \) is Sommerfeld's parameter, which is real \( (k > 0) \). It is important to note that \( Q_l^{-1}(y) \) is not real-analytic: For the complex conjugate of both members of Eq. (5) we obtain

\[
\left\langle p | V_c | k + \lambda \right\rangle = 2\gamma(\gamma - 1) e^{-i(\gamma - 1)/2} Q_l^{-1}(y).
\]

In the above indicated way we obtain from Eqs. (2)-(6),

\[
\sum_{l=0}^{\infty} (2l + 1)P_l(x)Q_l(y)Q_l^{-1}(z)
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\]
(cf. Ref. 5) for \( n, m = 0, 1, 2, 3, x_i \in \mathbb{C}, y_i \in \mathbb{C} \setminus [-1,1] \). When \( \text{Re} \, x_i > 0, \text{Re} \, y_i > 0 \), this series is convergent if
\[
\prod_{i=1}^{n} |x_i + (x_i^3 - 1)^{1/2}| < \prod_{i=1}^{n} |y_i + (y_i^3 - 1)^{1/2}|.
\]
(15)

Let us first consider \( F_{12} \). By taking the limit for \( y \to 0 \) in Eq. (12) we obtain
\[
F_{12}(x,y,z) = \sum_{i=0}^{\infty} (2l + 1)P_l(x)P_l(y)Q_l(z)
\]
\[
= \int_{-1}^{1} W^{-1/2} \ln \frac{yz - x + W^{1/2}}{yz - x - W^{1/2}} \, dx.
\]
(16)

It is interesting to note that
\[
F_{12}(x,y,z) = \int_{-\pi}^{\pi} \sqrt{\frac{z - \beta \cdot \hat{q} - 1}{|z - \beta \cdot \hat{q}|}} \, d\hat{q},
\]
where \( \beta \cdot \hat{q} = x \) and the integration is over the unit sphere. In Ref. 5 an expression has been given for \( F_{12}(x,y,z) \). Our result given by Eq. (16) means a considerable reduction of that expression. Indeed, in the notation of Ref. 5 we have
\[
(12) \text{ gives} \quad \text{Eq. (16) gives}
\]
\[
F_{12}(x,y,z) = \frac{1}{4\pi} \int \left[ \ln \left( \frac{t + d^{1/2}}{t - d^{1/2}} \right) \right] dt,
\]
(17)

whereas in Ref. 5 the following result is given,
\[
F_{12}(y,z,x) = \frac{1}{4\pi d^{1/2}} \int \left[ \ln \left( \frac{t + 1}{t - d^{1/2}} \right) \right] dt.
\]
To demonstrate the equivalence of this result and that in Eq. (17) is not completely trivial. It can be done by dividing out the common factor \( \ln \left[ \left( 1 + t^2 d^{-1/2} \right)^{1/2} \left( \frac{z^2 - 1}{x} \right)^{-1} \right] \) from the numerator and the denominator of the fraction which forms the argument of the logarithm. By this procedure Eq. (17) is retrieved.

We shall briefly consider some other interesting particular cases of the general function \( F_{mn} \). By taking \( x = 1 \) in Eq. (16) we obtain the well-known result
\[
F_{00}(y,z) = \frac{\ln (y - z)}{y + z} - \frac{1}{2} \ln \frac{y - 1}{y + 1}.
\]
(18)

Other well-known formulas are\textsuperscript{6}
\[
F_{02}(x,y,z) = W^{-1/2},
\]
(19)
\[
F_{10}(y,z) = (z - y)^{-1},
\]
(20)
\[
F_{01}(z) = (z - 1)^{-1}.
\]
(21)

Eq. (20) is called Heine's formula.
When \( n = 0 \) we shall restrict \( x, y, z \) in \( F_{mn} \) to the interval \([-1,1]\). According to Ref. 6, p. 307 we have
\[
F_{00}(x,y,z) = \begin{cases} 0 & \text{if } W > 0 \\ 2\pi^{-1/2} & \text{if } W < 0 \end{cases}
\]
(22)

Furthermore, we have [cf. Eq. (4)]
\[
F_{00}(x,y) = 2\delta(x - y),
\]
(23)
\[
F_{00}(x) = 2\delta(1 - x),
\]
(24)
where \( \delta \) is Dirac's delta distribution.

Finally we shall briefly consider \( F_{03} \). In virtue of Eq. (16) we have
\[
F_{00}(y,z,x) = \sum_{l=0}^{\infty} (2l + 1)Q_l(w)Q_l(v)Q_l(z)
\]
\[
= \frac{1}{2} \int_{-1}^{1} dx \left| \sum_{l=0}^{\infty} (2l + 1)p_l(x)Q_l(w)Q_l(z) \right|
\]
\[
= \frac{1}{4} \int_{-1}^{1} dx \left| \frac{W^{1/2}}{y - v + W^{1/2}} \right| \ln \frac{v - x + W^{1/2}}{y - v - W^{1/2}}.
\]
(25)

Putting \( a = (y^3 - 1)^{1/2} \), \( v = \arccosh(\sqrt{y - x}/a) \), \( v_+ = \arccosh(\sqrt{y + x}/a) \), we get \( W^{1/2} = a \sinh \theta \) and
\[
F_{00}(y,z,x) = (1/2) \int_{-1}^{1} \sinh \theta \left| \frac{v_+}{v - v_+} \right| \cos v \, dv.
\]
According to formula 2.478.7 of Ref. 7 we have
\[
\int \frac{x \, dx}{\cosh 2x - \cos 2t} = \frac{1}{4 \sin 2t} \left[ (u - t) - (u + t) + 2L(t) \right],
\]
(26)

where \( u = \arctan(\tanh x \cos t) \) and \( L(t) \) is Lobachevski's function, defined by
\[
L(t) = - \int_{0}^{\infty} \ln(\cos t) \, dt.
\]
(27)

This implies that \( F_{03} \) cannot be expressed in terms of elementary functions.

By using the series representation
\[
L(t) = - \int_{0}^{\pi} \frac{\sin^2 t}{\sin^2 t + \cos^2 t} \, dt = \sum_{n=1}^{\infty} \frac{(-1)^n \sin 2nt}{n},
\]
(28)

the right member of Eq. (26) can be rewritten as
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{4 \sin 2t} \left| \frac{v_+}{v - v_+} \right| \cos 2nt.
\]
(29)

We point out that on p. 377 of Ref. 6, Eq. (30.2.1), a closed formula is given for the series
\[
\sum_{l=0}^{\infty} (2l + 1)p_l(x)P_l(y)P_l(z),
\]
(30)

where \( m \in \mathbb{N} \) and \( x,y,z \in [-1,1] \).

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