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On the invertibility of Møller morphisms

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Local perturbations of the dynamics of infinite quantum systems are considered. It is known that, if the Møller morphisms associated to the dynamics and its perturbation are invertible, the perturbed evolution is isomorphic to the unperturbed one, and thereby shares its ergodic properties. It was claimed by V. Ya. Golodets [Theor. Math. Phys. **23**, 525 (1975)] that the above condition holds whenever the observable algebra is asymptotically abelian for the unperturbed evolution, and the perturbed evolution has a KMS state. The present paper contains a counterexample to this statement, and a construction of a spatial representation of the Møller morphisms.

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I. INTRODUCTION

Let us consider a quantum mechanical system that can be described by a C^*-algebra $\mathscr{A}$ and a group $\{\alpha_t\}_{t\in\mathbb{R}}$ of *-automorphisms of $\mathscr{A}$. We interpret $\mathscr{A}$ as the set of (bounded) observables of the system, and $\{\alpha_t\}$ as its dynamics. For $A\in\mathscr{A}$, $t\rightarrow\alpha_t(A)$ is the time evolution of the observable A. In Refs. 1 and 2 it is assumed that $t\rightarrow\alpha_t(A)$ is continuous. This seemingly innocent assumption excludes many important cases from the discussion as, for instance, the free Bose gas. It is, however, not vital for the conclusions to be drawn here, so let us also make the assumption, for the sake of simplicity. Being strongly continuous, the group $\{\alpha_t\}$ has an infinitesimal generator, δ say,

$$\alpha_t = \exp(t\delta). \tag{1}$$

Now, let V be any self-adjoint element of $\mathscr{A}$, and define

$$\widetilde{\alpha}_t = \exp[t(\delta + [iV,\cdot])]. \tag{2}$$

$\{\widetilde{\alpha}_t\}_{t\in\mathbb{R}}$ is another strongly continuous group of *-automorphisms of $\mathscr{A}$, which we shall call "the perturbed dynamics". Now suppose that $\{\alpha_t\}$ has some nice ergodic property. It may be that $\{\alpha_t\}$ is ergodic:

$$\{A\in\mathscr{A}\,|\,\forall_t:\alpha_t(A)=A\} = \mathbb{C}1, \tag{3}$$

i.e., $\{\alpha_t\}$ has no nontrivial fixed points ("constants of the motion"). Or it may be that $\{\mathscr{A},\alpha\}$ is asymptotically abelian, i.e.,

$$\forall_{A,B\in\mathscr{A}}:\|[A,\alpha_t(B)]\| \underset{|t|\rightarrow\infty}{\longrightarrow} 0. \tag{4}$$

In these cases, it is interesting to know whether or not $\{\widetilde{\alpha}_t\}$ shares the ergodic property. In order to answer these, and related questions, it was proposed by Robinson[1] to study the limits

$$\gamma_\pm(A) = \lim_{t\rightarrow\pm\infty} \widetilde{\alpha}_{-t}\circ\alpha_t(A) \tag{5}$$

in the norm topology of $\mathscr{A}$. Suppose these limits exist for all $A\in\mathscr{A}$. (A sufficient condition for this was given in Ref. 1). Then $\gamma_\pm$ are isometric *-morphisms of $\mathscr{A}$, intertwining α and $\widetilde{\alpha}$:

$$\gamma_\pm\circ\alpha_t = \widetilde{\alpha}_t\circ\gamma_\pm\,. \tag{6}$$

Clearly, if γ_+ or γ_- is invertible, $\{\widetilde{\alpha}_t\}$ is similar to $\{\alpha_t\}$, and inherits its ergodic properties.

The maps $\gamma_\pm$ are called the "Møller morphisms", by analogy with the Møller operators in scattering theory. Now, in scattering theory, the nonunitary of the Møller operators is generally thought of as due to to the existence of bound states for the perturbed Hamiltonian. It turns out that, analogously, we may consider the noninvertibility of $\gamma_\pm$ as roughly equivalent to the existence of nontrivial fixed points of $\{\widetilde{\alpha}_t\}$, i.e., constants of the motion for the perturbed evolution. In fact, if $\{\widetilde{\alpha}_t\}$ has a fixed point that is not a fixed point of $\{\alpha_t\}$, then γ_+ are not invertible.

It follows from a result of Araki[3] that, whenever there exists an $\{\alpha,\beta\}$-KMS state ω on $\mathscr{A}$ for some $\beta>0$ (i.e., a state, satisfying the Kubo–Martin–Schwinger condition[4] at inverse temperature β w.r.t. $\{\alpha_t\}$), there also is an $\{\widetilde{\alpha},\beta\}$-KMS state, and it is quasiequivalent to ω. This holds regardless of the existence of fixed points for $\{\widetilde{\alpha}_t\}$.

In view of the above remarks, Theorem 3 of Ref. 2 is surprising. Indeed, we shall see that it is not valid.

II. A COUNTEREXAMPLE

Let us assume that

(I) $\{\mathscr{A},\alpha\}$ is asymptotically abelian [i.e., (4) holds],
(II) the limits $\gamma_\pm(A)$ in (5) exists for all $A\in\mathscr{A}$,
(III) for some $\beta>0$ there is an $\{\widetilde{\alpha},\beta\}$-KMS state $\widetilde{\omega}$ on $\mathscr{A}$.

Let $\widetilde{\pi}$ be the representation of $\mathscr{A}$ determined by $\widetilde{\omega}$ according to the Gel'fand–Naimark–Segal (GNS) construction. It is the content of Theorem 3 of Ref. 2 that, under these assumptions, there exist *-automorphisms $\bar{\gamma}_\pm$ of $\widetilde{\pi}(\mathscr{A})''$, such that for all $A\in\mathscr{A}$

$$\bar{\gamma}_\pm\circ\widetilde{\pi}(A) = \widetilde{\pi}\circ\gamma_\pm(A). \tag{7}$$

The following example shows that this cannot be true.

Let H be the self-adjoint operator $-\partial^2/\partial x^2$ on $L^2(\mathbb{R})$. For $g\in L^2(\mathbb{R})$, let P_g denote the orthogonal projection on g. If $g\in L^1\cap L^2(\mathbb{R})$ is such that $\int g\,dx\neq 0$, the operator $\widetilde{H}:=H-P_g$ has an eigenvector $h\neq 0$.

Let $\mathscr{B}$ be the C^*-algebra, embodying the canonical anticommutation relations (CAR) over $L^2(\mathbb{R})$, and let $\mathscr{A}$ be its even subalgebra. Then the groups $\{\alpha_t\}$ and $\{\widetilde{\alpha}_t\}$ of automorphisms of $\mathscr{B}$, defined by

$$\alpha_t(a(f)) = a(e^{itH}f) \quad\text{and}\quad \widetilde{\alpha}_t(a(f)) = a(e^{it\widetilde{H}}f), \tag{8}$$

are related by (1) and (2), with $V = -a(g)^*a(g)/\|g\|^2$. $\{\alpha_t\}$ and $\{\tilde{\alpha}_t\}$ both leave $\mathscr{A}$ invariant, and V is in $\mathscr{A}$.

The system $\{\mathscr{A},\alpha\}$ is asymptotically abelian because $\exp(itH)$ tends to zero weakly as $|t|\to\infty$. Furthermore, by Kato's theorem[5] on perturbations of rank one, the strong limits

$$W_{\pm} = \lim_{t\to\pm\infty} e^{-it\tilde{H}}e^{itH} \tag{9}$$

exist, and $W_{\pm}$ are isometries onto the absolutely continuous spectral subspace of $\tilde{H}$. Because h is an eigenvector of $\tilde{H}$, it follows that

$$h\perp \text{Range}(W_{\pm}). \tag{10}$$

Now, define the *-morphisms $\gamma_{\pm}:\mathscr{B}\to\mathscr{B}$ by

$$\gamma_{\pm}(a(f)) = a(W_{\pm}f).$$

Then $\gamma_{\pm}(A)$ are indeed the norm limits of $\tilde{\alpha}_{-t}\circ\alpha_t(A)$ as $t\to\pm\infty$ because of (8) and (9) and the continuity of $a(f)$ in f. Moreover, for any $\beta>0$ there is an $\{\tilde{\alpha},\beta\}$–KMS state on $\mathscr{A}$; namely the gauge invariant quasifree state $\tilde{\omega}$ with two-point function:

$$\tilde{\omega}(a(f_1)^*a(f_2)) = \langle f_2,F(\tilde{H})f_1\rangle,$$

where

$$F(x) = (1+e^{\beta x})^{-1}.$$

Now, because $h\perp$ Range$(W_{\pm})$, $a(h)^*a(h)$ commutes with any element of the range of $\gamma_{\pm}$, a fact that contradicts (7). Indeed, let $\tilde{\pi}$ be the representation determined by $\tilde{\omega}$, and assume that automorphisms $\bar{\gamma}_{\pm}$ satisfying (7) exist. Then $\tilde{\pi}(\gamma_{\pm}(\mathscr{A}))'' = \tilde{\pi}(\mathscr{A})''$, and this would lead to the conclusion that $\tilde{\pi}(a(h)^*a(h))$ commutes with $\tilde{\pi}(\mathscr{A})''$, a contradiction.

Remark: The above example describes a noninteracting one-dimensional Fermi gas in a rank one "potential" P_g. The perturbed one-particle evolution has a bound state h, and consequently there is a constant of the motion for the perturbed evolution of the gas, namely, the observable $a(h)^*a(h)$, counting the particles in the bound state. As the unperturbed evolution is ergodic (i.e., has no constants of the motion), the two evolutions are not isomorphic, and $\gamma_{\pm}$ cannot be invertible. If the claim to be disproved had been that $\gamma_{\pm}$ are automorphisms of $\mathscr{A}$, our argument could stop here. However, only the existence and invertibility of $\bar{\gamma}_{\pm}$ is actually asserted, and it could be that $\bar{\gamma}_{\pm}^{-1}$, mapping $\tilde{\pi}(\mathscr{A})''$ into itself, did not leave $\tilde{\pi}(\mathscr{A})$ invariant. Therefore we need a slightly different argument, the one presented above, based on the fact that any fixed point of $\tilde{\alpha}$ commutes with $\gamma_{\pm}(\mathscr{A})$ if $\mathscr{A}$ is asymptotically abelian for α.

III. A PRELIMINARY RESULT

In what follows, we will have a closer look at the action of $\tilde{\pi}(\gamma_{\pm}(\mathscr{A}))$ on the Hilbert space $\tilde{H}$. The following result, taken from Ref. 2, will enable us to do this:

Lemma 1: Suppose conditions (I), (II), and (III) hold. Let $\{\tilde{H},\tilde{\pi},\tilde{\xi}\}$ be the GNS-triple associated to $\{\mathscr{A},\tilde{\omega}\}$. Then there is $\xi\in\tilde{H}$, cyclic and separating for $\tilde{\pi}(\mathscr{A})''$, such that the state ω, defined by

$$\omega(A) = \langle\xi,\tilde{\pi}(A)\xi\rangle, \tag{11}$$

is an $\{\alpha,\beta\}$–KMS state, and

$$\omega = \tilde{\omega}\circ\gamma_{\pm}. \tag{12}$$

The vector ξ can be chosen to lie in the positive cone[3] $\mathscr{P}_{\tilde{\xi}}$ of $\tilde{\xi}$.

Remark: The reverse is also true: If there is an $\{\alpha,\beta\}$–KMS state ω on $\mathscr{A}$, then there is a $\tilde{\xi}$ in the GNS-space of $\{\mathscr{A},\omega\}$ that implements an $\{\tilde{\alpha},\beta\}$–KMS state $\tilde{\omega}$, satisfying (12). In fact, this reversed statement is the more useful one. In examples where α is "simple", the existence of ω is easier to establish than that of $\tilde{\omega}$. I choose to state the less useful version in order to agree with Ref. 2. Let me emphasize on the other hand that it would certainly not be advisable to entirely interchange α and $\tilde{\alpha}$, and to replace condition (II) of the existence of $\gamma_{\pm}$ by a condition ($\tilde{\text{II}}$), the existence of

$$\tilde{\gamma}_{\pm}(A) = \lim_{t\to\pm\infty} \alpha_{-t}\circ\tilde{\alpha}_t(A).$$

In examples where α is "simple", ($\tilde{\text{II}}$) is much harder to test than (II).

Proof: Let $\mathscr{Z}$ be the center of $\tilde{\pi}(\mathscr{A})''$, i.e., $\mathscr{Z} = \tilde{\pi}(\mathscr{A})''\cap\tilde{\pi}(\mathscr{A})'$. By the perturbation theory of KMS states,[3] there exists $\eta\in\mathscr{P}_{\tilde{\xi}}$, cyclic and separating for $\tilde{\pi}(\mathscr{A})''$, such that $A\to\langle\eta,\tilde{\pi}(A)\eta\rangle$ is $\{\alpha,\beta\}$–KMS. Now consider the states $Z\to\langle\eta,Z\eta\rangle$ and $Z\to\langle\tilde{\xi},Z\tilde{\xi}\rangle$. They are both faithful normal states on $\mathscr{Z}$. It follows that there is a vector $\xi\in\overline{\mathscr{Z}^+\eta}$ such that

$$\forall_{Z\in\mathscr{Z}}:\langle\xi,Z\xi\rangle = \langle\tilde{\xi},Z\tilde{\xi}\rangle. \tag{13}$$

This vector ξ is also in the cone $\mathscr{P}_{\tilde{\xi}}$, and is cyclic and separating for $\mathscr{Z}$. Let ω be given by ξ as in (11). It is not hard to show that, because $\xi\in\overline{\mathscr{Z}\eta}$, ω is $\{\alpha,\beta\}$–KMS. And then ξ must be cyclic and separating for the whole of $\tilde{\pi}(\mathscr{A})''$.

Let us now prove that

$$\forall_{A\in\mathscr{A}}: \lim_{t\to\pm\infty} \tilde{\omega}\circ\alpha_t(A) = \omega(A). \tag{14}$$

Suppose the contrary. Then there are $A\in\mathscr{A}$, $\epsilon>0$, and a sequence $\{t_n\}$ of times, such that $|t_n|\to\infty$ and

$$|\tilde{\omega}\circ\alpha_{t_n}(A) - \omega(A)|\geqslant\epsilon, \tag{15}$$

Now, the sequence $\{\tilde{\pi}(\alpha_{t_n}(A))\}_{n\in\mathbf{N}}$ must have a w^*-converging subnet, because it remains inside the w^*-compact set $\{X\in\tilde{\pi}(\mathscr{A})''\,|\,\|X\|\leqslant\|A\|\}$. So let $\{n(\sigma)\}$ be a net in $\mathbf{N}$, such that $\lim_\sigma n(\sigma) = \infty$ and $w^* - \lim_\sigma\tilde{\pi}(\alpha_{t_{n(\sigma)}}(A)) = Z\in\tilde{\pi}(\mathscr{A})''$. Then for all $B\in\mathscr{A}$

$$\begin{aligned}[\tilde{\pi}(B),Z] &= w^* - \lim_\sigma\left[\tilde{\pi}(B),\tilde{\pi}(\alpha_{t_{n(\sigma)}}(A))\right]\\ &= w^* - \lim_\sigma\tilde{\pi}(\left[B,\alpha_{t_{n(\sigma)}}(A)\right]) = 0,\end{aligned}$$

because $\{\mathscr{A},\alpha\}$ is asymptotically abelian. So $Z\in\mathscr{Z}$, and we can apply (13). But then

$$\begin{aligned}&\lim_\sigma\tilde{\omega}(\alpha_{t_{n(\sigma)}}(A))\\ &\quad= \lim_\sigma\langle\tilde{\xi},\tilde{\pi}(\alpha_{t_{n(\sigma)}}(A))\tilde{\xi}\rangle = \langle\tilde{\xi},Z\tilde{\xi}\rangle = \langle\xi,Z\xi\rangle\\ &\quad= \lim_\sigma\langle\xi,\tilde{\pi}(\alpha_{t_{n(\sigma)}}(A))\xi\rangle = \lim_\sigma\omega(\alpha_{t_{n(\sigma)}}(A)) = \omega(A),\end{aligned}$$

because ω is α-invariant. This contradicts (15) and we conclude that (14) holds. Finally, note that for all $A\in\mathscr{A}$,

$$\tilde{\omega}\circ\gamma_{\pm}(A) = \lim_{t\to\pm\infty} \tilde{\omega}\circ\tilde{\alpha}_{-t}\circ\alpha_t(A)$$

$$= \lim_{t\to\pm\infty} \tilde{\omega}\circ\alpha_t(A) = \omega(A). \blacksquare$$

IV. INVERTIBILITY OF γ AND EXISTENCE OF $\bar{\gamma}$

For ease of notation, let us from now on identify $A\in\mathscr{A}$ with the operator $\tilde{\pi}(A)$ on $H=\tilde{H}$, so that $\mathscr{A}$ becomes a C^*-algebra of bounded operators on a Hilbert space. Moreover, let us focus our attention on only one of the Møller morphisms: γ_+, and call it γ.

In Ref. 2 a counterpart θ to the map γ: $\mathscr{A}\to\mathscr{A}$ is introduced. θ acts on the commutant $\mathscr{A}'$, which is a von Neumann algebra, unlike $\mathscr{A}$ itself. I shall give a direct construction of θ below.

Lemma 2: Suppose conditions (I), (II), and (III) hold. Then there is an isometry Ω: $H\to H$, such that for all $A\in\mathscr{A}$,

$$\gamma(A)\Omega = \Omega A.$$

Proof: Define Ω_0: $\mathscr{A}\xi\to\gamma(\mathscr{A})\tilde{\xi}$ by

$$\Omega_0 A\xi = \gamma(A)\tilde{\xi}.$$

Then for all $A\in\mathscr{A}$, $\|\Omega_0 A\xi\|^2 = \|\gamma(A)\tilde{\xi}\|^2 = \langle\tilde{\xi},\gamma(A)^*\gamma(A)\tilde{\xi}\rangle = \langle\tilde{\xi},\gamma(A^*A)\tilde{\xi}\rangle = \tilde{\omega}\circ\gamma(A^*A) = \omega(A^*A) = \langle\xi,A^*A\xi\rangle = \|A\xi\|^2$. As $\overline{\mathscr{A}\xi} = H$, Ω_0 extends continuously to an isometry Ω:$H\to H$ with range, $\gamma(\mathscr{A})\tilde{\xi}$. Now for all $A,B\in\mathscr{A}$,

$$\gamma(A)\Omega B\xi = \gamma(A)\gamma(B)\tilde{\xi} = \gamma(AB)\tilde{\xi} = \Omega AB\xi,$$

and the statement follows from the cyclicity of ξ for $\mathscr{A}$.$\blacksquare$

Lemma 3: Suppose (I), (II), and (III) hold. Let ξ be given by Lemma 1 and Ω by Lemma 2. Let J be the modular conjugation $H\to H$, associated with $\{\mathscr{A}'',\xi\}$. Then

$$J\Omega = \Omega J.$$

Proof: Let Δ and $\tilde{\Delta}$ be the modular operators associated with $\{\mathscr{A}'',\xi\}$ and $\{\mathscr{A}'',\tilde{\xi}\}$ according to the Tomita–Takesaki theory[6] and J and $\tilde{J}$ the corresponding modular conjugations.

Let $A\in\mathscr{A}$ be analytic for $\{\alpha_t\}$. Then by the intertwining property (6) of γ, $\gamma(A)$ is analytic for $\{\tilde{\alpha}_t\}$ and

$$\begin{aligned}\tilde{J}\Omega\alpha_{i\beta/2}(A)\xi &= \tilde{J}\gamma(\alpha_{i\beta/2}(A))\tilde{\xi}\\ &= \tilde{J}\tilde{\alpha}_{i\beta/2}(\gamma(A))\tilde{\xi} = \tilde{J}\tilde{\Delta}^{1/2}\gamma(A)\tilde{\xi}\\ &= \gamma(A)^*\tilde{\xi} = \gamma(A^*)\tilde{\xi} = \Omega A^*\xi\\ &= \Omega J\Delta^{1/2}A\xi = \Omega J\alpha_{i\beta/2}(A)\xi.\end{aligned}$$

Now, the linear space $\{\alpha_{i\beta/2}(A)\xi \,|A\in\mathscr{A}$ analytic for $\alpha\}$ is dense in H. Therefore $\tilde{J}\Omega = \Omega J$. And because ξ and $\tilde{\xi}$ are in the same positive cone,[3] J and $\tilde{J}$ coincide, and the statement follows.$\blacksquare$

Lemma 4: Suppose (I), (II), and (III) hold. Let J, Ω be as defined before, and let

$$\theta:\mathscr{A}'\to\mathscr{L}(H):A\to\Omega^*A\Omega,$$

$$j:\mathscr{A}''\to\mathscr{A}':A\to JAJ.$$

Then

$$\theta\circ j\circ\gamma = j\restriction\mathscr{A}.$$

Moreover,

$$\theta(\mathscr{A}')\subset\mathscr{A}'.$$

Proof: First we show that $\theta(\mathscr{A}')\subset\mathscr{A}'$. Let $B\in\mathscr{A}'$, $A\in\mathscr{A}$. Then, by Lemma 2,

$$\begin{aligned}[\theta(B),A] &= [\Omega^*B\Omega,A] = \Omega^*B\Omega A - A\Omega^*B\Omega\\ &= \Omega^*B\gamma(A)\Omega - \Omega^*\gamma(A)B\Omega\\ &= \Omega^*[B,\gamma(A)]\Omega = 0.\end{aligned}$$

So $\theta(B)\in\mathscr{A}'$ for all $B\in\mathscr{A}'$. Furthermore, it follows from Lemmas 2 and 3 that, if $A\in\mathscr{A}$,

$$\begin{aligned}\theta\circ j\circ\gamma(A) &= \Omega^*J\gamma(A)J\Omega = \Omega^*J\gamma(A)\Omega J\\ &= \Omega^*J\Omega AJ = \Omega^*\Omega JAJ = JAJ = j(A). \quad\blacksquare\end{aligned}$$

Lemma 5: Again suppose that (I), (II), (III) hold. Then θ, defined in Lemma 4, is the unique map $\mathscr{A}'\to\mathscr{A}'$ satisfying

$$\forall_{A\in\mathscr{A}}\forall_{B\in\mathscr{A}'}:\langle\tilde{\xi},B\gamma(A)\tilde{\xi}\rangle = \langle\xi,\theta(B)A\xi\rangle. \tag{16}$$

Moreover, θ is linear, $*$-preserving, w^*-continuous, and surjective.

Proof: Let $A\in\mathscr{A}$, $B\in\mathscr{A}'$. Then, by Lemma 2, and because $\tilde{\xi} = \Omega\xi$,

$$\begin{aligned}\langle\tilde{\xi},B\gamma(A)\tilde{\xi}\rangle &= \langle\Omega\xi,B\gamma(A)\Omega\xi\rangle\\ &= \langle\xi,\Omega^*B\Omega A\xi\rangle = \langle\xi,\theta(B)A\xi\rangle.\end{aligned}$$

Uniqueness of θ follows from the cyclicity of ξ for $\mathscr{A}$. Clearly, θ is linear, $*$-preserving, and w^*-continuous. It remains to prove surjectivity. So let $B\in\mathscr{A}'$; $\|B\| = 1$, say. We look for an $X\in\mathscr{A}'$ such that $B=\theta(X)$. Now, because j is a bijection $\mathscr{A}''\to\mathscr{A}'$, $j^{-1}(B)$ is a well-defined element of $\mathscr{A}''$; $\|j^{-1}(B)\| = 1$. By Kaplanski's density theorem the unit sphere in $\mathscr{A}$ is dense in the unit sphere in $\mathscr{A}''$. So there is a net $\{B_\sigma\}$ in $\mathscr{A}$ with $\|B_\sigma\|\leqslant 1$ and $w^*\text{-lim}\, B_\sigma = j^{-1}(B)$. Now consider the net $\{j\circ\gamma(B_\sigma)\}$. Being included in the w^*-compact unit ball of $\mathscr{A}'$, it must have a w^*-converging subnet $\{B_{\sigma(\tau)}\}$,

$$w^*\text{-}\lim_\tau j\circ\gamma(B_{\sigma(\tau)}) = X\in\mathscr{A}'.$$

But then it follows from Lemma 4 that

$$\theta(X) = w^*\text{-}\lim_\tau \theta\circ j\circ\gamma(B_{\sigma(\tau)})$$

$$= w^*\text{-}\lim_\tau j(B_{\sigma(\tau)}) = j(j^{-1}(B)) = B,$$

because both θ and j are w^*-continuous.$\blacksquare$

Remark: In Ref. 2, (16) is the defining property of θ. The w^*-continuity and surjectivity of θ are also proved there. But, in addition, it is claimed that θ has the morphism property

$$\forall_{A,B\in\mathscr{A}'}:\theta(AB) = \theta(A)\theta(B), \tag{17}$$

which is now easily seen not to hold if Ω is not unitary, i.e., if $\gamma(\mathscr{A})\tilde{\xi}$ is not dense in H. And, indeed, a close look at the proof of (17) in Ref. 2 reveals that the w^*-density of $\gamma(\mathscr{A})$ in $\mathscr{A}''$ is implicitly assumed there. Once accepting (17), Golodets can prove that $\bar{\gamma}$ exists as an automorphism of $\mathscr{A}''$ by turning the argument around that has proved the existence of θ as an automorphism of $\mathscr{A}'$. Actually, the existence of the automorphism $\bar{\gamma}$ and (17) are equivalent:

Theorem 6: Suppose that the conditions (I), (II), (III) hold, and let Ω be given by Lemma 2, and θ be as defined in Lemma 4. Then the following statements are equivalent:

(i) There is a *-automorphism $\bar{\gamma}$ of $\mathscr{A}''$, such that $\bar{\gamma} \restriction \mathscr{A} = \gamma$,

(ii) $\gamma(\mathscr{A})' \subset \mathscr{A}'$,

(iii) $\Omega H = H$,

(iv) For all $A,B \in \mathscr{A}' : \theta(AB) = \theta(A)\theta(B)$.

Proof: (i)⟹(ii): Suppose (i) holds. Then $\bar{\gamma}$ is w^*-continuous, and therefore $\bar{\gamma}(\mathscr{A})'' \supset \bar{\gamma}(\mathscr{A}'') = \mathscr{A}''$. It follows that $\gamma(\mathscr{A})' = \bar{\gamma}(\mathscr{A})' = (\bar{\gamma}(\mathscr{A})'')' \subset (\mathscr{A}'')' = \mathscr{A}'$.

(ii)⟹(iii): Suppose (ii) holds. Let $P = \Omega\Omega^*$. P is the orthogonal projection on $\overline{\gamma(\mathscr{A})\tilde{\xi}}$, so $P \in \gamma(\mathscr{A})'$, and therefore $P \in \mathscr{A}'$ by (ii). Now $P\tilde{\xi} = \tilde{\xi}$, so $(P - \mathbf{1})\tilde{\xi} = 0$, and because $\tilde{\xi}$ is separating for $\mathscr{A}'$, $P = \mathbf{1}$. It follows that $\Omega H = H$.

(iii)⟹(i): Suppose Ω is unitary; define $\bar{\gamma}(A) = \Omega A \Omega^*$ for all $A \in \mathscr{A}''$. Then for $A \in \mathscr{A}$ we have $\bar{\gamma}(A) = \Omega A \Omega^* = \gamma(A)\Omega\Omega^* = \gamma(A)$, so $\bar{\gamma} \restriction \mathscr{A} = \gamma$. $\bar{\gamma}$ is clearly a *-morphism, and we have to show that it is onto. Let $A \in \mathscr{A}''$ and let $B = \Omega^* A \Omega$. As J commutes with Ω, $B = J\Omega^* JAJ\Omega J = j^{-1}\circ\theta\circ j(A)$; so $B \in \mathscr{A}''$ by Lemma 4. Moreover, $\bar{\gamma}(B) = \Omega B \Omega^* = \Omega\Omega^* A \Omega\Omega^* = A$. We conclude that any $A \in \mathscr{A}''$ is of the form $\bar{\gamma}(B)$, $B \in \mathscr{A}''$.

(iii)⟹(iv): If Ω is unitary then, for all $A,B \in \mathscr{A}'$, $\theta(A)\theta(B) = \Omega^* A \Omega\Omega^* B \Omega = \Omega^* AB \Omega = \theta(AB)$.

(iv)⟹(iii): Suppose (iv) holds. Then for all $A \in \mathscr{A}'$
$\|\Omega^* A\tilde{\xi}\|^2 = \langle\tilde{\xi}, A^*\Omega\Omega^* A\tilde{\xi}\rangle = \langle\xi, \Omega^* A^*\Omega\Omega^* A\Omega\xi\rangle = \langle\xi, \theta(A^*)\theta(A)\xi\rangle = \langle\xi, \theta(A^*A)\xi\rangle = \langle\xi, \Omega^* A^* A\Omega\xi\rangle = \langle\tilde{\xi}, A^*A\tilde{\xi}\rangle = \|A\tilde{\xi}\|^2$, and because $\tilde{\xi}$ is cyclic for $\mathscr{A}'$, Ω^* is an isometry. Hence Ω is unitary, and $\Omega H = H$. ■

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