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Nonperturbative confinement in quantum chromodynamics. III. Improved gluon propagator

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An ansatz is introduced for the three-gluon vertex that is consistent with the Slavnov–Taylor identity in Landau gauge. It is shown that the gluon has a confining infrared singularity; but there is also a tachyon, indicating an insufficiency either of quarkless QCD or at least of our approximation to it.

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I. INTRODUCTION

It is an attractive hypothesis that a severe infrared singularity is a signal of confinement in quarkless quantum chromodynamics (QCD).[1,2] Although the gluon self-energy is gauge-dependent, the gauge-invariant Wilson loop can be constructed from it; and it has been shown that a sufficiently singular propagator leads to an area law in leading order.[3] A proper study of such singularities entails a nonperturbative approximation. As an alternative to the lattice approach, with attendant uncertainties concerning the continuum limit, one may study truncated continuum Dyson–Schwinger (DS) equations, using either covariant or axial gauges. The DS equation for the gluon propagator may be truncated through use of Slavnov–Taylor (ST) identities, parametrizing longitudinal parts of vertex-functions.

In an axial gauge, the gluon field decouples from the ghost field, so that the DS equations and the ST identities have a simple form. If one makes the *ad hoc* assumption that the full propagator has the same tensor structure as the bare propagator, then the scalar function that multiplies it may be obtained as the solution of the scalar equation obtained by projection of the DS equation onto the direction of the axial vector n. This equation does not contain the four-gluon vertex. A disadvantage is that the equation involves the unphysical, gauge-dependent parameter $(pn)^2$, where p is the momentum variable. Furthermore, since the ST identity involves projection onto p, rather than n, there is a certain arbitrariness in the projected DS equation.[4]

In the Landau gauge, the DS equation for the gluon propagator involves a single tensor structure, and hence reduces to a scalar equation. However, both the DS equations and the ST identities involve ghost couplings. We do not expect the ghost fields themselves to produce infrared singularities, so we replace the ghost propagator and ghost–ghost–gluon vertex by bare values. The four-gluon vertex does appear in the scalar equation, but we drop it in the interest of simplicity, not expecting cancellations between three- and four-gluon couplings.

The approximation of Mandelstam[5] in Landau gauge involved the replacement of one internal gluon line and of the three-gluon vertex by free values. Such a replacement is motivated by the form of the ST identity, but is not strictly consistent with it. In our analysis of Mandelstam's equation in Refs. 1 and 2, we confirm that the gluon propagator is of order p^{-4} at small spacelike momenta. However, there are first-sheet branch points in the variable p^2, which accumulate at $p^2 = 0$ in the timelike direction. These singularities are presumably unphysical, and in any event they invalidate the Wick rotation to Euclidean momenta.

In this paper we propose an ansatz for the vertex function that has the same tensor structure as the corresponding bare vertex. The multiplicative scalar function can be chosen so that the ST identity in the external leg is automatically satisfied. The resultant scalar equation, which is generally similar to that obtained in Mandelstam's approach, with, however, a somewhat more intricate structure, is derived in Sec. II. We analyze this equation in Sec. III, and show by methods similar to those of Refs. 1 and 2 that there is a solution which is free from complex branch points. This solution has the infrared asymptote p^{-4}, uniformly in the cut p^2-plane, and is therefore suggestive of confinement.

The resultant gluon propagator also has a simple pole at a real, spacelike momentum. Such a pole does not spoil the Wick rotation, but would imply the existence of an unstable tachyon, if it were taken seriously. Recall that in perturbative QED there is a tachyon (Landau ghost), which would not be expected in perturbative QCD because of the opposite sign in the self-energy. Our nonperturbative tachyon might conceivably be an indication of the insufficiency of quarkless QCD; unless it is merely a deficiency of our approximation scheme. It may be that neglect of the four-gluon vertex has produced an instability of the type familiar in a scalar theory with ϕ^3 interaction.

II. ANSATZ FOR VERTEX FUNCTION

We shall use a consistent notation, in which primes distinguish full from bare propagators and vertex functions. We suppress all color indices, since they only yield a trivial multiplicative factor in the final equation. The full propagator is

$$D'_{\mu\nu}(q) = F(-q^2)D_{\mu\nu}(q) = -q^{-2}F(-q^2)\Delta_{\mu\nu}(q)\,, \qquad (2.1)$$

where

$$\Delta_{\mu\nu}(q) = g_{\mu\nu} - q_\mu q_\nu/q^2\,. \qquad (2.2)$$

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The object is to obtain an equation for the scalar function $F(-q^2)$.

The Slavnov–Taylor identity relating the full three-gluon vertex, Γ', to the propagator[6] is

$$p^\lambda \Gamma'_{\lambda\mu\nu}(p,q,r) D'^{\mu\sigma}(q) D'^{\nu\tau}(r) = G(-p^2) D'^{\mu\sigma}(q) \Delta^{\nu\tau}(r) \widetilde{\Gamma}'_{\mu\nu}(p,r{:}q) - G(-p^2) \Delta^{\mu\sigma}(q) D'^{\nu\tau}(r) \widetilde{\Gamma}'_{\mu\nu}(p,q{:}r) . \tag{2.3}$$

Here $\widetilde{\Gamma}'_{\mu\nu}$ is the full ghost–ghost–gluon vertex, and the ghost propagator is $-p^{-2}G(-p^2)$. As discussed in the Introduction, we replace the ghost functions by their bare values, $\widetilde{\Gamma}'_{\mu\nu} \to g_{\mu\nu}$ and $G \to 1$. With use of (2.1), we then obtain

$$p^\lambda \Gamma'_{\lambda\mu\nu}(p,q,r) D'^{\mu\sigma}(q) D'^{\nu\tau}(r) = [F(-r^2)/r^2 - F(-q^2)/q^2]\, \Delta^{\mu\sigma}(q) \Delta^{\nu\tau}(r) g_{\mu\nu} . \tag{2.4}$$

Our basic ansatz[7] is to suppose that

$$\Gamma'_{\lambda\mu\nu}(p,q,r) D'^{\mu\sigma}(q) D'^{\nu\tau}(r) = f(p,q,r) \Gamma_{\lambda\mu\nu}(p,q,r) D^{\mu\sigma}(q) D^{\nu\tau}(r) , \tag{2.5}$$

where f is some scalar function that is to be related to F by requiring that the Slavnov–Taylor identity (2.4) is satisfied. Since the bare version of (2.4) is obtained by removing the primes and setting F equal to unity, we find, by contracting both sides of (2.5) against p^λ,

$$\frac{F(-r^2)}{r^2} - \frac{F(-q^2)}{q^2} = f(p,q,r)\left(\frac{1}{r^2} - \frac{1}{q^2}\right), \tag{2.6}$$

so that

$$f(p,q,r) = [r^2 F(-q^2) - q^2 F(-r^2)]/(r^2 - q^2) , \tag{2.7}$$

which can be inserted into (2.5). For comparison, the Mandelstam ansatz, in which the three-gluon vertex and one, but not both, of the gluon propagators are replaced by their bare values, is of the form (2.5), with the function $f(p,q,r)$ equal to $F(-q^2)$. It must be emphasized that this form, unlike (2.7), in inconsistent with the Slavnov–Taylor identity (2.4).

The Dyson–Schwinger equation for the gluon propagator, in which only the three-gluon vertex contribution is retained, with the ansatz (2.5), takes the form

$$D'_{\lambda\rho}(p) = D_{\lambda\rho}(p) = D_{\lambda\mu}(p) \Pi^{\mu\nu}(p) D'_{\nu\rho}(p) , \tag{2.8}$$

where the self-energy is

$$\Pi^{\mu\nu}(p) = \frac{ig^2}{(2\pi)^4} \int d^4q\, f(p,q,r) \Gamma^{\mu\sigma\tau}(p,q,r) \times \Gamma^{\nu\rho\omega}(p,q,r) D_{\sigma\rho}(q) D_{\tau\omega}(r) . \tag{2.9}$$

Contracting both sides of (2.8) by $g^{\lambda\rho}$ and dividing throughout by $F(-p^2)$, we find

$$1/F(-p^2) = 1 + (1/p^2) \Pi(-p^2) , \tag{2.10}$$

where

$$\Pi(-p^2) = \Delta_{\mu\nu}(p) \Pi^{\mu\nu}(p) = \frac{ig^2}{(2\pi)^4} \int \frac{d^4q}{q^2 r^2} f(p,q,r) \Omega(p,q,r) , \tag{2.11}$$

with

$$\Omega(p,q,r) = \Gamma^{\mu\sigma\tau}(p,q,r) \Delta_{\mu\nu}(p) \Delta_{\sigma\rho}(q) \times \Delta_{\tau\omega}(r) \Gamma^{\nu\rho\omega}(p,q,r) . \tag{2.12}$$

We next make a Wick rotation to Euclidean space: $-p^2 \to p^2_{\rm eucl} \equiv x$. The angular integrations can be performed and we obtain

$$\begin{aligned} \frac{1}{F(x)} = 1 &- \frac{25}{4} \int_0^\infty \frac{dy}{y} F(y) - 9 \int_0^\infty \frac{dy}{x} F(y) \\ &- \frac{x^2}{8} \int_x^\infty \frac{dy}{y^3} F(y) + \int_0^x \frac{dy}{x} \left[\frac{13}{2}\left(1 - \frac{x}{y}\right) \right. \\ &\left. + \frac{7}{4}\left(\frac{x^2}{y^2} - \frac{y}{x}\right) - \frac{1}{4}\frac{y^2}{x^2} + \frac{1}{8}\frac{x^3}{y^3} \right] F(y) \\ &+ \int_0^{x/4} \frac{dy}{x} \left(6 + \frac{17}{2}\frac{x}{y} - 2\frac{x^2}{y^2} - \frac{1}{8}\frac{x^3}{y^3} \right) \\ &\times \left(1 - 4\frac{y}{x}\right)^{1/2} F(y) . \end{aligned} \tag{2.13}$$

Here the coupling constant and other multiplicative factors have been scaled away, as in Ref. 1. This form is similar in its general shape to the Mandelstam equation, except for the last term, involving the square root. This comes from principal-value integrals that are engendered by the quotient (2.7).

The infrared analysis of (2.13) is a little more involved than that in Refs. 1 and 2. We replace the unknown function F by ϕ according to

$$F(x) = A/x + Bx + x^3 \phi(x) . \tag{2.14}$$

The coefficient of the incipient pole on the rhs of (2.13) is zero if

$$A = -0.3456 \int_0^\infty dy\, F(y) , \tag{2.15}$$

while the constant term vanishes if

$$1 = 6.25 \int_0^\infty \frac{dy}{y} F(y) . \tag{2.16}$$

Finally, in order to match the linear term on the lhs of (2.14), x/A, we require

$$B = \gamma/A , \tag{2.17}$$

where

$$\gamma = 0.616\,97 . \tag{2.18}$$

These requirements are formal, in the sense that the integrals (2.15) and (2.16) are actually divergent. However, it is worth stressing that the absence of logarithms when the first two terms of (2.14) are inserted into (2.13) is a result of delicate cancellations.

After these manipulations, (2.13) can be cast into the form

$$\begin{aligned} \frac{\gamma + x^2\phi(x)}{1 + \gamma x^2 + x^4 \phi(x)} &= -\frac{1}{4} \int_0^x \frac{dy}{x} \left(1 - \frac{y}{x}\right)^3 \phi(y) P\left(\frac{y}{x}\right) \\ &+ \frac{1}{8} \int_0^{x/4} \frac{dy}{x} \left(1 - \frac{4y}{x}\right)^{3/2} \phi(y) Q\left(\frac{y}{x}\right), \end{aligned} \tag{2.19}$$

where

$$P(z) = 1 + 10z + z^2 , \tag{2.20}$$

$$Q(z) = 1 + 20z + 12z^2 . \tag{2.21}$$

No divergences are left, and we propose to study this nonlinear equation for the unknown function $\phi(x)$.

III. ANALYSIS OF EQUATION

In the following, we shall for simplicity replace the polynomials P and Q of (2.20) and (2.21) by constants, in such a way that the threshold value ($x \to 0$) of each integral is unchanged. The averaged equation reads

$$\frac{\gamma + x^2\phi(x)}{1 + \gamma x^2 + x^4\phi(x)} = -\frac{23}{30}\int_0^x \frac{dy}{x}\left(1 - \frac{y}{x}\right)^3 \phi(y) + \frac{53}{168}\int_0^{x/4}\frac{dy}{x}\left(1 - \frac{4y}{x}\right)^{3/2}\phi(y). \tag{3.1}$$

Our previous experience[1,2] leads us to expect that this averaging procedure will only have minor quantitative, but not qualitative effects on the solution.

To examine the nature of the infrared singularity of $\phi(x)$, we linearize the lhs of (3.1), retaining the terms $\gamma - \gamma^2x^2 + x^2\phi(x)$ only. Even with this linearization, we have been unable to give a complete analysis; but there are reasons for expecting the second integral on the rhs of (3.1) to be nondominant in the infrared (see the Appendix). Accordingly, we shall study the linear equation

$$\gamma - \gamma^2x^2 + x^2\phi(x) = -\frac{23}{30}\int_0^x \frac{dy}{x}\left(1 - \frac{y}{x}\right)^3 \phi(y). \tag{3.2}$$

The corresponding homogeneous equation

$$x^6\Psi(x) = -\frac{23}{30}\int_0^x dy (x-y)^3 \Psi(y) \tag{3.3}$$

has solutions expressible in terms of Bessel and Neumann functions. The four independent solutions of the corresponding differential equation have the small-x asymptotic behavior

$$\Psi_i(x) \sim x^{-15/4}\exp(73.6x^{-1/2}z_i), \tag{3.4}$$

where z_i are the four fourth roots of -1. These functions may be used to solve the inhomogeneous Eq. (3.2), using variation of parameters on the corresponding differential equation. Each of the homogeneous solutions Ψ_i becomes unbounded in certain sectors of the plane, cut along $-\infty < x < 0$, as x tends to zero. However, we find that there is one (and only one) function $\phi(x)$ which satisfies the integral Eq. (3.2) in the cut x-plane. That function has the following asymptotic behavior as x approaches zero within the cut plane:

$$\phi(x) \sim \gamma x^{-15/4}\sum_{i=1}^{4} z_i \exp(\kappa x^{-1/2} z_i) \times \Gamma(-\tfrac{13}{2}, \kappa x^{-1/2} z_i), \tag{3.5}$$

where κ is a positive number and the incomplete gamma function is given by

$$\Gamma(\alpha,\omega) = \int_\omega^\infty dy\, y^{\alpha-1}e^{-y}. \tag{3.6}$$

It follows from a standard asymptotic expression for $\Gamma(\alpha,\omega)$ that $\phi(x)$ approaches a constant value as x tends to zero throughout the cut plane.

We wish to emphasize that uniform boundedness of ϕ at small x is a significant improvement over the corresponding infrared behavior obtained for solutions of Mandelstam's equation. In the latter case, we obtained asymptotic behavior $x^{-7/2}$ for the corresponding function as x tends to zero on the left-hand cut. In the present case, the linear approximation (3.2) remains under control at small x, even on the left-hand cut. As a consequence, we expect that the solutions of the nonlinear equation will be analytic in the cut plane, at least in the infrared. In the case of Mandelstam's equation, the linear approximation was out of control near the left-hand cut, and manifestation of this was the accumulation of first-sheet branch points of the full nonlinear equation.

We have done an extensive numerical study of Eq. (3.1), with the second integral omitted. One may obtain an asymptotic power series for $\phi(x)$ at small x directly from the nonlinear integral equation, which is used for computation of ϕ at small real x. The solution is then obtained at larger values of x by Runge–Kutta integration of the differential equation

$$[x^4G(x)]^{iv} = -\tfrac{23}{5}\phi(x), \tag{3.7}$$

where

$$G(x) = [\gamma + x^2\phi(x)]/[1 + \gamma x^2 + x^4\phi(x)]. \tag{3.8}$$

Using the techniques of Refs. 1 and 2, we have continued $\phi(x)$ into the complex plane, and have not found any branch points on the first Riemann sheet, except at the infrared point, $x = 0$. On penetrating the cut along the timelike axis, $-\infty < x < 0$, however, we have picked up two branch points at $-0.3782 - 0.2239i$ and $-0.0241 - 0.0783i$. It is a reasonable guess that more exist, probably accumulating at the origin on the second or higher Riemann sheets. The fact that the complex branch points, which were on the first sheet in the Mandelstam approximation, are now on secondary sheets, where they cause no trouble, is a definite improvement. The function $F(x)$ [cf. Eq. (2.14)] does indeed have the infrared asymptote

$$F(x) \sim A/x + Bx + Cx^3, \tag{3.9}$$

as $x \to 0$ in any direction on the first sheet.

When we make an analytic continuation of ϕ to larger x, we find a pole on the real axis at $x = 2.1853$. The pole in $\phi(x)$ may be located directly as a zero of the function $1 - x^2G(x)$. Fortunately, the pole does not interfere with the Wick rotation to the Euclidean region, in contrast to the complex singularities found in Refs. 1 and 2. The pole occurs some distance from the infrared, where our approximation scheme is no longer necessarily good. Further details of the numerical work may be found in Ref. 8.

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APPENDIX: TECHNICAL DETAIL

We shall justify neglecting the second term on the right side of Eq. (3.1). The first remark is that, in the asymptotic power series expansion of that equation, the contribution from the first term to the coefficient of x^{2n} dominates that of the second, except at the first few values of n. As a consequence, the asymptotic series $\phi(x)$ is, in effect, controlled by the first term.

Let us consider the linearized, homogeneous version of that equation,

$$x^6\Psi(x) = -\frac{23}{30}\int_0^x dy(x-y)^3\Psi(y) + \frac{53}{168}x^3\int_0^{x/4} dy\left(1-\frac{4y}{x}\right)^{3/2}\Psi(y). \tag{A1}$$

We substitute the solutions $\Psi_i(x)$ [Eq. (3.4)] into (A1), and note that, for x small within a sector of boundedness, the second term in (A1) is asymptotically small compared with the first. Consequently, these functions $\Psi_i(x)$ are asymptotic solutions of Eq. (A1) at small x.

Let us consider an extreme situation in which the first term in (A1) is dropped, and let us replace the "fractional $\frac{5}{2}$-power derivative" on the right by $\Psi(z)/z^{5/2}$, to obtain

$$x^2\Psi(x) = \lambda\Psi(x/4). \tag{A2}$$

The solution to the difference Eq. (A2),

$$\Psi(x) = \exp\left[-\frac{1}{2\ln 2}\ln^2 x + \left(\frac{\ln\lambda}{2\ln 2} - 1\right)\ln x\right], \tag{A3}$$

decreases more rapidly than any power of x at small x, although less rapidly than the functions $\Psi_i(x)$ [Eq. (3.4)]. The asymptotic solution of (A1) with the first term dropped is essentially identical to (A3). In particular, it is analytic and uniformly small throughout the cut plane at small x, so that its infrared behavior is less quixotic than that of $\Psi_i(x)$.

In terms of the Mellin transform,

$$\tilde{\Psi}(p) = \int_0^\infty dx\,\Psi(x)x^{p-1}, \tag{A4}$$

Eq. (A.1) becomes an algebraic difference equation

$$\tilde{\Psi}(p+2) = \left[-\frac{23}{30}B(-p+1,4) + \frac{53}{42}4^p B\left(-p+1,\frac{5}{2}\right)\right]\tilde{\Psi}(p), \tag{A5}$$

where $B(\,,\,)$ is the beta function. We have not been able to solve (A5), subject to the requirement that $\tilde{\Psi}(p)$ be analytic in a vertical strip of width at least 2. However, we can prove that solutions exist, and that they behave as $\Psi_i(x)$ at small x.

The above arguments make it clear that the first integral in (A1) dominates the second in the infrared. Away from $x = 0$, however, the latter integral is not necessarily small.

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