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Chiral symmetry breakdown. II. Ansatz for vertex in ultraviolet

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An earlier analysis of the Dyson–Schwinger equation for the quark propagator is improved by taking the Slavnov–Taylor identity into account in the ultraviolet. It is found that chiral symmetry breaking occurs above a critical coupling in the Landau gauge; but that this result does not hold in other gauges.

I. INTRODUCTION

Much study has been devoted to the tantalizing possibility that the (constituent) masses of quarks arise from the nonperturbative breakdown of chiral symmetry.[1–5] More specifically, it is supposed that the bare quark mass vanishes; and the Dyson–Schwinger equation for the quark propagator is then analyzed for signs of chiral symmetry breaking. The most popular scenario is that in which this breakdown occurs only if the QCD coupling λ is greater than a certain critical value λ_C: this point constitutes then a bifurcation of the mass function from the trivial to a nontrivial solution of the equation.

Some authors confine themseleves to the Landau gauge and assume that the trace of $\not p$ times the inverse quark propagator is p^2. This is only correct if the gluon remains massless. If the gluon acquires an effective mass, as a result of self-interaction, this trace is $p^2\beta(p^2)$, where β is a function that has to be obtained from the Dyson–Schwinger equation. In Refs. 6 and 7, we showed that, in the approximation $\beta(p^2)\equiv 1$, a positive bifurcation point λ_c exists only if both infrared and ultraviolet cutoffs are introduced. In Ref. 8 we elaborated the analysis by treating β properly: in the presence of an infrared cutoff, in the form of an effective gluon mass, and an ultraviolet cutoff, provided naturally by the logarithmic decrease of the running coupling constant, we found again that $\lambda_c>0$ in the Landau gauge. However, in the Feynman gauge (and in other gauges), it turned out that there is no solution of the equation for $\beta(p^2)$, unless a Pauli–Villars cutoff Λ is introduced. As $\Lambda\to\infty$, so $\lambda_c\to 0$, thus indicating an extreme gauge dependence that casts doubt on the credibility of the approach.

The most questionable approximation made in Ref. 8 is the replacement of the full quark–gluon vertex $\Gamma_\nu(p',p)$ by its bare value γ_ν. Since the difficulties in the Feynman gauge are associated with ultraviolet divergences, and since the inverse quark propagator behaves like $\not p\beta(p^2)$ as $p^2\to\infty$, a better approximation for Γ_ν should be γ_ν, multiplied by β, since this is consistent with the Ward–Takahashi identity in the ultraviolet regime. It is true that the correct Slavnov–Taylor identity of a non-Abelian theory contains matrix elements of ghost fields, as Miransky has pointed out[9]; but it might reasonably be hoped that these do not alter the ultraviolet behavior of the quark propagator.

In this paper we undertake a treatment of the quark propagator, with the above-mentioned improvement in the approximation for Γ_ν. We find that the analysis is much easier than that of Ref. 8; but the fundamental conclusions remain unchanged: λ_c is positive in the Landau gauge, and $\lambda_c\to 0$ as $\Lambda\to\infty$ in the Feynman gauge.

Ten years ago, Weinberg[10] suggested that a positive bifurcation point λ_c is not to be expected, since, if it were to exist, it would surely be gauge dependent; and the onset of a phenomenon such as chiral symmetry breaking presumably ought not to depend on the gauge that one chooses. Our conclusions support this conjecture; and, in this connection, a parallel analysis that employs Delbourgo's gauge technique,[11] in which the Ward–Takahashi identity is respected at all momenta, similarly yields a gauge dependence of λ_c.

In Sec. II, we briefly recall the formalism, while the analysis is carried out in Sec. III. An Appendix is devoted to the bifurcation theory that is required in the body of the paper.

In conclusion, although the general result of this work suggests that the existence of a gauge-independent bifurcation point $\lambda_c>0$ is untenable, the hope might reasonably be entertained that our general methods will yield more positive results in other situations. In particular, in finite-temperature field theory, one expects a phase transition to the plasma state above a critical temperature T_c and bifurcation theory should prove a useful tool.

II. DYSON–SCHWINGER EQUATION AND SLAVNOV–TAYLOR IDENTITY

The Dyson–Schwinger equation for the quark propagator may be written in Euclidean space in the form

$$S_F'^{-1}(p)=\not p+\frac{\lambda}{(2\pi)^4}\int d^4p'\,\gamma_\mu S_F'(p')\Gamma_\nu(p',p)\times D'_{F\mu\nu}(p'-p), \tag{2.1}$$

where λ is the square of the QCD coupling constant, times a color factor. Here D_F' is the gluon propagator, and we shall equip it with a mass and a running coupling,

$$D'_{F\mu\nu}(k)=\omega(k^2)D_{F\mu\nu}(k). \tag{2.2}$$

Here D_F is the bare propagator for a massive vector field, and $\omega(x)$ is a given function with the following properties:

$$\omega(0)=1,\quad \omega(x)\sim(\log x)^{-1}\quad \text{as } x\to\infty,\quad \frac{d\omega(x)}{dx}\leqslant 0. \tag{2.3}$$

The Slavnov–Taylor identity, with ghosts neglected, is

$$(p'-p)_\nu\Gamma_\nu(p'-p)=S_F'^{-1}(p)-S_F'^{-1}(p'), \tag{2.4}$$

and this relates the longitudinal part of the quark–gluon vertex to the inverse of the quark propagator. If we set

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$$S_F'^{-1}(p) = \alpha(p^2) - \not{p}\beta(p^2), \tag{2.5}$$

then we expect that, as $p \to \infty$ at fixed p', (2.4) will read asymptotically

$$p_\nu \Gamma_\nu(p',p) \sim \not{p}\beta(p^2) \tag{2.6}$$

and similarly for $p' \to \infty$ at fixed p. This motivates the ansatz

$$\Gamma_\nu(p',p) \approx \gamma_\nu \beta(p_>^2), \tag{2.7}$$

where $p_>^2 = \max(p^2, p'^2)$, which should respect the ultraviolet behavior of the theory better than does the constant vertex approximation of Ref. 8.

As in Ref. 8, we approximate the running coupling function ω by

$$\omega(k^2) = \omega((p'-p)^2) \approx \omega(p_>^2), \tag{2.8}$$

and we evaluate the angular integrals in (2.1). This results in the coupled integral equations

$$\alpha(x) = \frac{\lambda}{\pi^2}\int_0^\infty dy\, K(x,y)\, \frac{y\alpha(y)\beta(x_>)}{\alpha^2(y) + y\beta^2(y)}, \tag{2.9}$$

$$\beta(x) = 1 + \frac{\lambda}{\pi^2}\int_0^\infty dy\, L(x,y)\, \frac{y\beta(y)\beta(x_>)}{\alpha^2(y) + y\beta^2(y)}. \tag{2.10}$$

The kernels K and L were given explicitly in Ref. 8, and we do not reproduce them here, nor shall we repeat the discussion of their further approximation.

III. BIFURCATION EQUATIONS

As in I, we shall consider the Feynman gauge, and a modification of the Landau gauge, the so-called Landau-like gauge of Maskawa and Nakajima,[1] for technical convenience. Upon differentiating (2.9) functionally with respect to α, and setting $\alpha = 0$, we obtain the following equations:

$$\delta\alpha(x) = \frac{\lambda}{16\pi^2}\int_0^\infty dy\, \rho(x_>)\beta(x_>)\frac{\delta\alpha(y)}{\beta^2(y)}, \tag{3.1}$$

and

$$\beta(x) = 1 + \frac{\lambda}{16\pi^2}\int_0^\infty dy\, \sigma(x_>)\beta(x_>)\frac{y}{\beta(y)}, \tag{3.2}$$

where $x_> = \max(x,y)$, and where

$$\rho(x) = [4/(x+m^2)]\omega(x), \tag{3.3}$$

$$\sigma(x) = [1/(x+m^2)^2]\omega(x), \tag{3.4}$$

in the Feynman gauge, and

$$\rho(x) = [3/(x+m^2) + m^2/(x+m^2)^2]\omega(x), \tag{3.5}$$

$$\sigma(x) = [m^2/(x+m^2)^3]\omega(x), \tag{3.6}$$

in the Landau-like gauge. Here m is the effective gluon mass, which is assumed to arise from gluon–gluon interaction.

Consider first Eq. (3.2), which can be written

$$\beta(x) = 1 + \frac{\lambda}{16\pi^2}\int_0^x y\, dy\, \sigma(x)\frac{\beta(x)}{\beta(y)} + \frac{\lambda}{16\pi^2}\int_x^\infty y\, dy\, \sigma(y). \tag{3.7}$$

The last integral here is convergent in the Landau-like gauge; but it is log log divergent in the Feynman gauge. Convergence can be achieved in this case by the imposition of a Pauli–Villars cutoff, which has the effect of replacing (3.4) by

$$\sigma(x) = [1/(x+m^2)^2 - 1/(x+\Lambda^2)^2]\omega(x). \tag{3.8}$$

Divide (3.7) throughout by $\beta(x)$ and define $\gamma(x) = [\beta(x)]^{-1}$, thus obtaining

$$\frac{\gamma(x)}{f(x)} = 1 - \frac{\lambda}{16\pi^2}\sigma(x)\int_0^x y\, dy\, \gamma(y), \tag{3.9}$$

where

$$[f(x)]^{-1} = 1 + \frac{\lambda}{16\pi^2}\int_x^\infty y\, dy\, \sigma(y). \tag{3.10}$$

Note that (3.9) is a linear Volterra equation that can be converted into a linear differential equation for $\gamma(x)$. The unique solution of the Volterra equation is

$$\gamma(x) = f(x) - \frac{\lambda}{16\pi^2}\sigma(x)\int_0^x y\, dy f^2(y). \tag{3.11}$$

From (3.10) we see that $f(x) \to 1$ as $x \to \infty$, whether we take σ to be given by (3.6), the Landau-like gauge, or by (3.8), the Feynman gauge with Pauli–Villars cutoff. Hence, from (3.11), $\gamma(x) \to 1$ as $x \to \infty$.

Further,

$$\gamma(0) = f(0) = \left[1 + \frac{\lambda}{16\pi^2}\int_0^\infty y\, dy\, \sigma(y)\right]^{-1} > 0; \tag{3.12}$$

and moreover, it is easy to check from (3.11) that

$$\gamma'(x) = -\frac{\lambda}{16\pi^2}\sigma'(x)\int_0^x dy f^2(y), \tag{3.13}$$

which is positive, since $\sigma'(x)$ is negative. Hence, as x increases from zero to infinity, $\gamma(x)$ increases monotonically from $f(0)$ to unity, and $\beta(x)$ decreases monotonically from $[f(0)]^{-1}$ to unity.

We turn now to (3.1), which we rewrite

$$\delta\alpha(x) = \frac{\lambda}{16\pi^2}\int_0^\infty dy\, F(x,y)\delta\alpha(y), \tag{3.14}$$

where

$$F(x,y) = \frac{\rho(x)\beta(x)}{\beta^2(y)}\theta(x-y) + \frac{\rho(y)}{\beta(y)}\theta(y-x). \tag{3.15}$$

The kernel F is square integrable,

$$\begin{aligned} \|F\|^2 &= \int_0^\infty dx \int_0^x dy\, \frac{\rho^2(x)\beta^2(x)}{\beta^4(y)} \\ &\quad + \int_0^\infty dx \int_x^\infty dy\, \frac{\rho^2(y)}{\beta^2(y)} \\ &\leqslant \left[\frac{1}{[f(0)]^2} + 1\right]\int_0^\infty dx\, x\rho^2(x). \end{aligned} \tag{3.16}$$

In the Landau-like gauge,

$$\begin{aligned} \rho(x) &= [3/(x+m^2) + m^2/(x+m^2)^2]\omega(x) \\ &\leqslant [4/(x+m^2)]\omega(x), \end{aligned} \tag{3.17}$$

and the last expression is just ρ in the Feynman gauge. So in both gauges

$$\|F\|^2 < 16\left[\frac{1}{[f(0)]^2} + 1\right]\int_0^\infty dx\, \frac{x}{(x+m^2)^2}\omega^2(x), \tag{3.18}$$

which is convergent, since $\omega^2(x) \sim (\log x)^{-2}$ as $x \to \infty$. Notice that the running coupling function ω is essential for this convergence. Since (3.14) is a homogeneous Fredholm equation, it only has a nontrivial L^2 solution $\delta\alpha$ for λ on a point set. The smallest positive point in this set, say λ_c, which necessarily satisfies

$$\lambda_c \geqslant 16\pi^2/\|F\|, \tag{3.19}$$

corresponds to the bifurcation of a nontrivial L^2 solution $\alpha(x)$ of Eqs. (2.9) and (2.10) away from the trivial solution (see the Appendix).

Equation (3.14) is equivalent to the differential equation,

$$\frac{d}{dx}\left[\frac{(d/dx)\delta\alpha(x)}{(d/dx)[\rho(x)\beta(x)]}\right] = \frac{\lambda}{16\pi^2}\frac{\delta\alpha(x)}{\beta^2(x)} \tag{3.20}$$

with the boundary condition,

$$\frac{d}{dx}\delta\alpha(x) \underset{x\to 0}{\to} 0. \tag{3.21}$$

According to the general theory of linear, second-order, ordinary differential equations, Eq. (3.20) has two independent solutions, say f_R and f_I, and the general solution of (3.14) is

$$\delta\alpha(x) = Af_R(x) + Bf_I(x); \tag{3.22}$$

and the ratio of A to B is determined by the boundary condition (3.21). The solution is thus unique, up to a normalization.

The ultraviolet behaviors of the regular and irregular solutions follow from the fact that $\beta(x)$ tends to unity as $x\to\infty$, that $\rho(x)$ is given by (3.3) or (3.5), and that $\omega(x)$ satisfies (2.3). We find

$$f_R(x) \sim x^{-1}(\log x)^{-1+b}, \tag{3.23}$$

$$f_I(x) \sim (\log x)^{-b}, \tag{3.24}$$

as $x\to\infty$, where $b=\lambda/4\pi^2$ in the Feynman gauge and $b=3\lambda/16\pi^2$ in the Landau-like gauge. The solution (3.22) is square integrable only if $B=0$, and the smallest value of λ for which this happens is precisely λ_c, the bifurcation point.

The whole analysis is applicable to the Landau-like gauge without cutoff, or the Feynman gauge with cutoff. As $\Lambda\to\infty$ in the latter case, however, $\beta(0)\sim\log\log\Lambda$. Subtract $\beta(0)$ from (3.7),

$$\beta(x) = \beta(0) + \frac{\lambda}{16\pi^2}\int_0^x y\,dy\left[\sigma(x)\frac{\beta(x)}{\beta(y)} - \sigma(y)\right], \tag{3.25}$$

and define a renormalized $\tilde\beta(x) = Z_2\beta(x)$, where $Z_2 = [\beta(0)]^{-1}$. The renormalized version of (3.25) is

$$\tilde\beta(x) = 1 + \frac{\lambda Z_2}{16\pi^2}\int_0^x y\,dy\left[\sigma(x)\frac{\tilde\beta(x)}{\tilde\beta(y)} - \sigma(y)\right]. \tag{3.26}$$

As $\Lambda\to\infty$, $Z_2\to 0$ and $\tilde\beta(x)\to 1$. The renormalization constant Z_2 may *not* be absorbed into a redefinition of the coupling $\tilde\lambda = \lambda Z_2$ for the coupling should be renormalized by the gluon renormalization constant Z_3 which we have effectively approximated by unity. In the usual perturbative renormalization, one would expand the integral in (3.26) to order λ^n, and Z_2 to order λ^{n-1}, allowing the infinities to cancel in the usual way. However, the present nonperturbative approach, if it is to be viable, must deal with all divergences in one fell swoop. The renormalized $\delta\tilde\alpha(x) = Z_2\delta\alpha(x)$ satisfies

$$\delta\tilde\alpha(x) = \frac{\lambda Z_2}{16\pi^2}\int_0^\infty dy\,\rho(x_>)\tilde\beta(x_>)\frac{\delta\tilde\alpha(y)}{\tilde\beta^2(y)}, \tag{3.27}$$

from which we see that $\delta\tilde\alpha(x)\to 0$ as $\Lambda\to\infty$. Hence, as the cutoff is removed, the quark propagator tends to the bare form, $(\not p)^{-1}$. Thus we have demonstrated a gauge dependence of a most extreme kind: chiral symmetry breakdown in the Landau-like gauge and none in the Feynman gauge—a most absurd result.

APPENDIX: BIFURCATION THEORY

We present a theorem in bifurcation theory and apply it to the coupled equations (2.9) and (2.10), in a neighborhood of the trivial solution, $\alpha(x)=0$.

Theorem: Suppose that

$$\alpha(x) = \lambda T(\alpha;x), \tag{A1}$$

where α belongs to some real Hilbert space H, T is a nonlinear operator on H, and λ is a real number. Suppose further that T is thrice Fréchet differentiable, and that

$$T(-\alpha;x) = -T(\alpha,x), \tag{A2}$$

so that $T(0;x)=0$, which implies that (A1) possesses the trivial solution. Let the first Fréchet derivative at the trivial solution $T'(0;x)$ be compact on H, and suppose that λ_c is such that the linear equation

$$\delta\alpha(x) = \lambda_c[T'(0;\cdot)\delta\alpha](x) \tag{A3}$$

has precisely one nontrivial, linearly independent solution [i.e., λ_c^{-1} belongs to the (point) spectrum of $T'(0;\cdot)$, the corresponding null space of $1-\lambda_c T'(0;\cdot)$ being one dimensional].

Then there exist precisely two nontrivial solutions of (A1), differing only in sign, for λ in a half-neighborhood of λ_c (i.e., $\lambda>\lambda_c$ or $\lambda<\lambda_c$). A proof can be found in Pimbley's book.[12]

In Eqs. (2.9) and (2.10), there is the complication that α and β satisfy coupled equations, the trivial solution corresponding to $\alpha(x)\equiv 0$ and

$$\beta(x) = 1 + \frac{\lambda}{\pi^2}\int_0^\infty dy\,L(x,y)\frac{\beta(x_>)}{\beta(y)}. \tag{A4}$$

However, we can treat β as an implicit function of α. On differentiating (2.9) and (2.10) functionally with respect to α, we find

$$\delta\alpha(x) = \frac{\lambda}{\pi^2}\int_0^\infty y\,dy\,K(x,y) \times\left[\frac{\delta\alpha(y)\beta(x_>) + \alpha(y)\delta\beta(x_>)}{\alpha^2(y)+y\beta^2(y)} - \frac{2\alpha(y)\beta(x_>)[\alpha(y)\delta\alpha(y) + y\beta(y)\delta\beta(y)]}{[\alpha^2(y)+y\beta^2(y)]^2}\right], \tag{A5}$$

$$\delta\beta(x) = \frac{\lambda}{\pi^2}\int_0^\infty y\,dy\,L(x,y) \times\left[\frac{\delta\beta(y)\beta(x_>) + \beta(y)\delta\beta(x_>)}{\alpha^2(y) + y\beta^2(y)} - \frac{2\beta(y)\beta(x_>)[\alpha(y)\delta\alpha(y) + y\beta(y)\delta\beta(y)]}{[\alpha^2(y) + y\beta^2(y)]^2}\right]. \tag{A6}$$

These equations reduce, at $\alpha(x) = 0$, to

$$\delta\alpha(x) = \frac{\lambda}{\pi^2}\int_0^\infty dy\,K(x,y)\,\frac{\beta(x_>)}{\beta^2(y)}\,\delta\alpha(y), \tag{A7}$$

$$\delta\beta(x) = \frac{\lambda}{\pi^2}\int_0^\infty dy\,L(x,y)\left[\frac{\delta\beta(x_>)}{\beta(y)} - \frac{\beta(x_>)\delta\beta(y)}{\beta^2(y)}\right]. \tag{A8}$$

The bifurcation equations (A7) and (A4) are, respectively, equivalent to Eqs. (3.1) and (3.2). The possible existence of a nontrivial solution of (A8) is irrelevent to the applicability of the theorem, since Eqs. (A7) and (A8) are decoupled from one another.

We must now check the conditions of the theorem. The space is L^2, and the nonlinear operator T is given in implicit form by Eqs. (2.9) and (2.10). The oddness condition (A2) is clearly satisfied, and it is easy to check that T is thrice Fréchet differentiable. In Sec. III it is shown that (A7) is a classic Fredholm equation, which means that $T'(0;\cdot)$ is compact on L^2. The fact that the null space of $1 - \lambda_c T'(0;\cdot)$ is one dimensional is implied by the analysis of Eq. (3.14) in Sec. III, in which it is shown that the solution is unique, up to a normalization.

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