Chiral Symmetry Breakdown

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Chiral symmetry breakdown. I. Gauge dependence in constant vertex approximation

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An approximate quark propagator equation in a Landau-like gauge is analyzed and it is shown that there is a critical value of the coupling constant, corresponding to the onset of dynamical chiral symmetry breakdown, provided that (a) there is an infrared cutoff, which can be supplied by an effective gluon mass, and (b) there is an ultraviolet cutoff, which may be engendered by a running coupling constant. Dynamical chiral symmetry breakdown is shown not to occur in other gauges under the same circumstances, thus casting doubt upon the approximations commonly used.

I. INTRODUCTION

The idea that quarks obtain effective (constituent) masses as a result of a dynamical breakdown of chiral symmetry has received a great deal of attention in recent years.[1–10] We propose to examine this attractive hypothesis by a detailed analysis of truncated Dyson–Schwinger equations for the quark propagator. In this paper, we will restrict attention to the approximation in which the gluon–quark vertex is replaced by the bare value, whereas the gluon acquires an effective mass, while its propagator retains the tensor structure of the bare propagator. This resembles the first Johnson–Baker–Willey (JBW) approximation for the electron propagator of QED.[11–13]

In pioneering work over a decade ago, Maskawa and Nakajima[2] studied the truncated Dyson–Schwinger equation in a JBW-like approximation. Their analysis in a "Landau-like" gauge showed that spontaneous chiral symmetry breakdown occurs when a Pauli–Villars ultraviolet cutoff parameter Λ is introduced, *and* that spontaneous breakdown survives in the continuum limit $\Lambda \to \infty$. We obtain similar conclusions in that gauge, but using a smooth ultraviolet cutoff function, the choice being motivated by QCD. Like Maskawa and Nakajima in Ref. 2, and unlike several recent authors,[3–9] we have gone to some care in analyzing coupled Dyson–Schwinger equations for the two functions appearing in the quark propagator. The formalism is described in Sec. II, and the Landau-like gauge is analyzed in Sec. III.

The Landau-like gauge of Ref. 2 leads to Dyson–Schwinger equations which are relatively well behaved in the ultraviolet, whereas in other covariant gauges they become more singular. The case of Feynman gauge with *finite* momentum cutoff parameter Λ has also been analyzed in Ref. 2. We show in Sec. IV that, because of ultraviolet singularities in the continuum limit, $\Lambda \to \infty$, in Feynman gauge the regularized quark propagator corresponds to massless, free quarks. The Dyson–Schwinger equations exhibit spontaneous chiral symmetry breaking at finite cutoff Λ, because the quark mass operator satisfies a homogeneous Fredholm integral equation in that case, but the solution becomes "trivialized" upon renormalization in the continuum limit. Our logarithmic ultraviolet cutoff function reduces the degree of the divergence in the continuum limit, before renormalization, from log Λ (Ref. 2) to log log Λ; but it does not eliminate the need for regularization.

We have established that, in the JBW-like approximation, the quark propagator exhibits a sensitivity to the choice of gauge. This apparent "gauge dependence" of spontaneous chiral symmetry breaking is a consequence of the fact that truncated Dyson–Schwinger equations in the JBW-like scheme have ultraviolet singularities in most gauges. It is our conclusion that such a truncation is inadequate for studying spontaneous chiral symmetry breaking, and we intend in the future to study the problem for truncation schemes in which our choice of vertex function is motivated by the Slavnov–Taylor identities. In addition, asymptotic freedom imposes constraints upon the ultraviolet behavior of the propagator and vertex function.

II. DYSON–SCHWINGER EQUATION

The quark propagator satisfies the integral equation

$$S_F'^{-1}(p) = \not{p} - \frac{i\lambda}{(2\pi)^4} \int d^4p' \, \gamma_\mu S_F'(p') \gamma_\nu D_F'^{\mu\nu}(p'-p), \tag{2.1}$$

where we have approximated the full by the bare vertex. Here λ is the square of the QCD coupling constant, times a color factor, and the bare quark mass is zero. We suppose that the gluon has an effective mass, generated by self-interaction. The correct form for a massive vector propagator in a gauge theory is

$$\frac{1}{k^2 - m^2 + i\epsilon}\left[-g^{\mu\nu} + (1-a)\frac{k^\mu k^\nu}{k^2 - am^2 + i\epsilon}\right]. \tag{2.2}$$

We multiply this by a factor $\omega(-k^2)$ that satisfies $\omega(0) = 1$ and $\omega(-k^2) \sim [\log(-k^2)]^{-1}$ as $-k^2 \to \infty$, in order to allow for the decrease of the running coupling constant in a non-Abelian gauge theory. Thus

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$$D_F'^{\mu\nu}(k) = \omega(-k^2)\left[\frac{-g^{\mu\nu}}{k^2 - m^2 + i\epsilon} + \frac{k^\mu k^\nu}{m^2} \times\left(\frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - am^2 + i\epsilon}\right)\right], \tag{2.3}$$

where we have rearranged the tensor for calculational convenience and where a is the gauge parameter. A suitable form for ω is

$$\omega(-k^2) = \frac{k^2}{k^2 - m^2 + i\epsilon}\left[\log\left(1 - \frac{k^2}{m^2} - i\epsilon\right)\right]^{-1}, \tag{2.4}$$

although the results do not depend on the detailed expression.

The inverse quark propagator has the form

$$S_F'^{-1}(p) = \alpha(-p^2) + \not p\beta(-p^2), \tag{2.5}$$

where α and β are scalar functions, so that

$$S_F'(p) = \frac{\alpha(-p^2) - \not p\beta(-p^2)}{\alpha^2(-p^2) - p^2\beta^2(-p^2)}. \tag{2.6}$$

Upon insertion of these formulas into (2.1), two coupled equations for α and β can be obtained; and, after Wick rotation, one gets (with $x = p^2$, $y = p'^2$),

$$\alpha(x) = \frac{\lambda}{\pi^2}\int_0^\infty dy\, K(x,y)\frac{y\alpha(y)}{\alpha^2(y) + y\beta^2(y)}, \tag{2.7}$$

$$\beta(x) = 1 + \frac{\lambda}{\pi^2}\int_0^\infty dy\, L(x,y)\frac{y\beta(y)}{\alpha^2(y) + y\beta^2(y)}, \tag{2.8}$$

where

$$K(x,y) = \int_0^\pi d\theta \sin^2\theta\cdot\omega(k^2)\left[\frac{3}{k^2 + m^2} + \frac{a}{k^2 + am^2}\right] \tag{2.9}$$

and

$$L(x,y) = \int_0^\pi d\theta \frac{\omega(k^2)}{k^2 + m^2}\left[2(y/x)^{1/2}\cos\theta + (1-a) \times\frac{(y-x)k^2[(y/x)^{1/2}\cos\theta - 1]}{k^2 + am^2}\right]\sin^2\theta, \tag{2.10}$$

with

$$k^2 = (p' - p)^2 = x + y - 2(xy)^{1/2}\cos\theta. \tag{2.11}$$

We require that $\omega(x)$ be a monotonically decreasing function of x for Euclidean momenta. With the Euclidean version of (2.4), namely,

$$\omega(x) = \frac{x}{x + m^2}\left[\log\left(1 + \frac{x}{m^2}\right)\right]^{-1}, \tag{2.12}$$

it is not possible to evaluate the kernels K and L in terms of elementary functions.

A simplification is to replace $\omega(k^2)$ by unity, i.e., the coupling does not run. This has a profound (and nonphysical) effect on the behavior of the equation. The kernels can now be evaluated,

$$K(x,y) = \tfrac{3}{8}k(x,y,m^2) + \tfrac{1}{8}ak(x,y,am^2), \tag{2.13}$$

$$L(x,y) = \tfrac{1}{4}yk^2(x,y,m^2) - (1/16m^2x) \times[[(y-x)^2 + m^2(y+x)]k(x,y,m^2) - [(y-x)^2 + am^2(y+x)]k(x,y,am^2)] \tag{2.14}$$

with

$$k(x,y,m^2) = [x + y + m^2 + [(x+y+m^2)^2 - 4xy]^{1/2}]^{-1}. \tag{2.15}$$

This was essentially the case considered by Maskawa and Nakajima,[2] together with the Pauli–Villars cutoff version.

In the limit that the gluon mass m tends to zero, we find

$$k(x,y,0) = \frac{\theta(x-y)}{2x} + \frac{\theta(y-x)}{2y}. \tag{2.16}$$

For $m \neq 0$, the approximation

$$k(x,y,m^2) \approx \frac{\theta(x-y)}{2(x+m^2)} + \frac{\theta(y-x)}{2(y+m^2)} \tag{2.17}$$

is exact in the limits $x \to 0$ and $x \to \infty$, $y \to 0$ and $y \to \infty$, and it is a strict upper bound for all positive x and y. In this paper, we shall use the form (2.17) exclusively, although we propose to consider the exact expression in a future publication.

The above approximation is improved by reinstating the running coupling constant. Unfortunately, the dependence of k^2 on the angle θ in (2.9) and (2.10) makes it impossible to evaluate the integrals in closed form when the ω of (2.12) is present. However, if one sets

$$\omega(k^2) \approx \omega(p^2)\theta(p^2 - p'^2) + \omega(p'^2)\theta(p'^2 - p^2), \tag{2.18}$$

one obtains, instead of the kernels K and L of Eqs. (2.13) and (2.14), respectively,

$$[\omega(x)\theta(x-y) + \omega(y)\theta(y-x)]K(x,y), \tag{2.19}$$

$$[\omega(x)\theta(x-y) + \omega(y)\theta(y-x)]L(x,y). \tag{2.20}$$

The approximation (2.18) for the smooth, monotonic function $\omega(k^2)$ is good when $p^2 \gg p'^2$ or $p^2 \ll p'^2$, but not when p^2 and p'^2 are comparable in magnitude. However, the approximation is not expected to affect either the infrared or the ultraviolet behaviors of the solution.

With the approximations (2.17) and (2.18), the kernels read

$$K(x,y) = \tfrac{1}{16}[\omega(x)\mu(x)\theta(x-y) + \omega(y)\mu(y)\theta(y-x)], \tag{2.21}$$

$$L(x,y) = (y/32)[\omega(x)\nu(x,y)\theta(x-y) + \omega(y)\nu(y,x)\theta(y-x)], \tag{2.22}$$

where

$$\mu(x) = 3/(x + m^2) + a/(x + am^2), \tag{2.23}$$

$$\nu(x,y) = \frac{2}{(x+m^2)^2} - \frac{1}{m^2x}\left[\frac{(y-x)^2 + m^2(y+x)}{x + m^2} - \frac{(y-x)^2 + am^2(y+x)}{x + am^2}\right]. \tag{2.24}$$

The Feynman gauge ($a = 1$) is especially simple,

$$D_F'^{\mu\nu}(k) = [-g^{\mu\nu}/(k^2 - m^2 + i\epsilon)]\omega(-k^2), \tag{2.25}$$

$$\mu(x) = 4/(x+m^2), \tag{2.26}$$

$$\nu(x,y) = 2/(x+m^2)^2. \tag{2.27}$$

It turns out that in this gauge (and others), an ultraviolet cutoff is necessary (see Sec. IV). On the other hand, no such cutoff is required in the Landau gauge ($a = 0$),

$$D_F^{\prime\mu\nu}(k) = \frac{-g^{\mu\nu} + k^\mu k^\nu/(k^2+i\epsilon)}{k^2 - m^2 + i\epsilon}\,\omega(-k^2), \tag{2.28}$$

$$\mu(x) = 3/(x+m^2), \tag{2.29}$$

$$\nu(x,y) = \frac{2}{(x+m^2)^2} + \frac{y-3x}{x^2(x+m^2)}. \tag{2.30}$$

As can be seen from the denominator in (2.30), an infrared singularity has been introduced, a gauge artifact, and this turns out to be a nuisance. To avoid this difficulty, Maskawa and Nakajima[2] introduce what they called the Landau-like gauge, with the gluon propagator

$$D_F^{\prime\mu\nu}(k) = \left[\frac{-g^{\mu\nu}}{k^2 - m^2 + i\epsilon} + \frac{k^\mu k^\nu}{(k^2 - m^2 + i\epsilon)^2}\right]\omega(-k^2), \tag{2.31}$$

for which the kernels K and L, with the approximations (2.17) and (2.18), have the form (2.21) and (2.22), with

$$\mu(x) = 3/(x+m^2) + m^2/(x+m^2)^2, \tag{2.32}$$

$$\nu(x,y) = 2m^2/(x+m^2)^3. \tag{2.33}$$

Here the good ultraviolet properties have been retained, while the artificial infrared divergence has been removed.

In Sec. III, we consider this Landau-like gauge, without ultraviolet cutoff; while the Feynman gauge is treated in Sec. IV. In the latter case, a Pauli–Villars cutoff has to be introduced.

The major purpose is to find out conditions under which there is a critical $\lambda_c > 0$, such that, for $0 < \lambda < \lambda_c$, Eqs. (2.7) and (2.8) only have the chiral solution $\alpha \equiv 0$, while for $\lambda > \lambda_c$, there is also a nontrivial solution, $\alpha \neq 0$. To investigate such a bifurcation point λ_c we differentiate the equations functionally w.r.t. α, and set $\alpha = 0$,

$$\delta\alpha(x) = \frac{\lambda}{\pi^2}\int_0^\infty dy\, K(x,y)\,\frac{\delta\alpha(y)}{\beta^2(y)}, \tag{2.34}$$

$$\beta(x) = 1 + \frac{\lambda}{\pi^2}\int_0^\infty dy\, L(x,y)\,\frac{1}{\beta(y)}. \tag{2.35}$$

III. LANDAU-LIKE GAUGE

In the case (2.31)–(2.33), we can write the bifurcation equations (2.34) and (2.35) in the form

$$\delta\alpha(x) = \frac{\lambda}{16\pi^2}\left\{\int_0^x dy\,\rho(x) + \int_x^\infty dy\,\rho(y)\right\}\frac{\delta\alpha(y)}{\beta^2(y)}, \tag{3.1}$$

$$\beta(x) = 1 + \frac{\lambda}{16\pi^2}\left\{\int_0^x dy\,\sigma(x) + \int_x^\infty dy\,\sigma(y)\right\}\frac{y}{\beta(y)}, \tag{3.2}$$

where

$$\rho(x) = [3/(x+m^2) + m^2/(x+m^2)^2]\,\omega(x), \tag{3.3}$$

$$\sigma(x) = [m^2/(x+m^2)^3]\,\omega(x). \tag{3.4}$$

We study first Eq. (3.2). To this end, consider the mapping

$$\bar\beta(x) = P(\beta;x), \tag{3.5}$$

where $P(\beta;x)$ is just the right-hand side of (3.2). Let B be the Banach space of real, continuous functions $f(x)$ with supremum norm, for which the following inequalities hold:

$$0 < \beta_m \leqslant f(x) \leqslant \beta_M < \infty. \tag{3.6}$$

We shall specify β_m and β_M in a moment.

Next, define the function

$$P(x) = \frac{1}{16\pi^2}\left\{\int_0^x y\,dy\,\sigma(x) + \int_x^\infty y\,dy\,\sigma(y)\right\}. \tag{3.7}$$

It is easy to see that $P(x)$ is non-negative and monotonically decreasing in $0 < x < \infty$. Thus

$$0 \leqslant P(x) \leqslant P(0) < \infty. \tag{3.8}$$

A computer estimate gives

$$P(0) = \frac{1}{16\pi^2}\int_0^\infty d\omega\,\frac{\omega^2}{(1+\omega)^4\log(1+\omega)} \approx 0.001\,82. \tag{3.9}$$

Because of the positivity of $\sigma(x)$, we see from (3.2) and (3.5) that

$$\bar\beta(x) \geqslant 1 \tag{3.10}$$

and

$$\bar\beta(x) \leqslant 1 + \lambda P(0), \tag{3.11}$$

so that, if we define

$$\beta_m = 1 \tag{3.12}$$

and

$$\beta_M = 1 + \lambda P(0), \tag{3.13}$$

we see that the space B is mapped into itself by the nonlinear operator, P. Indeed, the image of B is actually compact in norm, since

$$\frac{d}{dx}\bar\beta(x) = \frac{\lambda}{16\pi^2}\left[\frac{d}{dx}\sigma(x)\right]\int_0^x \frac{y\,dy}{\beta(y)}. \tag{3.14}$$

Now $d\sigma/dx$ is negative, so $d\bar\beta/dx$ is also negative, and

$$-\frac{d}{dx}\bar\beta(x) \leqslant -\frac{\lambda}{32\pi^2}x^2\left[\frac{d}{dx}\sigma(x)\right] \leqslant \text{const}, \tag{3.15}$$

i.e., $|d\bar\beta/dx|$ has a bound that is independent of β.

Since P is a completely continuous operator that maps B into itself, we can use the Schauder theorem to assert that there is at least one fixed point, $\bar\beta = \beta$, in B, i.e., at least one solution of (3.2). To show that the solution is unique in B, we subtract $\beta(0)$,

$$\beta(x) = \beta(0) + \frac{\lambda}{16\pi^2}\int_0^x dy[\sigma(x) - \sigma(y)]\,\frac{y}{\beta(y)}. \tag{3.16}$$

Any solution of (3.2) is also a solution of (3.16), on the condition that $\beta(0)$ has the correct value. We first show that no two different solutions in B can have the same $\beta(0)$. For suppose that β_1 and β_2 both satisfy (3.16), and that $\beta_1(0) = \beta_2(0)$. Then

$$\beta_1(x)-\beta_2(x)=\frac{\lambda}{16\pi^2}\int_0^x y\,dy[\sigma(y)-\sigma(x)] \times\frac{\beta_1(y)-\beta_2(y)}{\beta_1(y)\beta_2(y)}. \tag{3.17}$$

Hence

$$|\beta_1(x)-\beta_2(x)|<\frac{\lambda}{16\pi^2\beta_m^2}\sum(x)\sup_{0<y<x}|\beta_1(y)-\beta_2(y)|, \tag{3.18}$$

where

$$\sum(x)=\int_0^x y\,dy[\sigma(y)-\sigma(x)]. \tag{3.19}$$

Let us take $X>0$ to be so small that, for any $x\in[0,X]$,

$$\sum(x)<\frac{16\pi^2\beta_m^2}{\lambda}K \tag{3.20}$$

with $K<1$. Then

$$\sup_{0<x<X}|\beta_1(x)-\beta_2(x)|<K\sup_{0<x<X}|\beta_1(x)-\beta_2(x)|, \tag{3.21}$$

which is only possible if $\beta_1(x)=\beta_2(x)$ for $0<x<X$. Since $\beta(x)$ satisfies the differential equation

$$\left[\frac{\beta'(x)}{\sigma'(x)}\right]'=\frac{\lambda}{16\pi^2}\frac{x}{\beta(x)}, \tag{3.22}$$

it is easy to extend this identity to all x values.

Next consider the case that $\beta_1(0)\neq\beta_2(0)$. For definiteness, we set $\beta_1(0)>\beta_2(0)$. Instead of (3.17) we have

$$\beta_1(x)-\beta_2(x) =\beta_1(0)-\beta_2(0)+\frac{\lambda}{16\pi^2}\int_0^x y\,dy[\sigma(y)-\sigma(x)] \times\frac{\beta_1(y)-\beta_2(y)}{\beta_1(y)\beta_2(y)}. \tag{3.23}$$

This has the structure of a linear Volterra equation for $\beta_1-\beta_2$, given $\beta_1\beta_2$; and the Neumann series is guaranteed to have an infinite radius of convergence. Since $\sigma(y)\geqslant\sigma(x)$ for $y\leqslant x$, each term in the series is non-negative, and so, for all x

$$\beta_1(x)-\beta_2(x)\geqslant\beta_1(0)-\beta_2(0). \tag{3.24}$$

Now it follows from (3.2) that $\beta(\infty)=1$, so by taking $x=\infty$ in (3.24) we find $\beta_2(0)\geqslant\beta_1(0)$, which contradicts $\beta_1(0)>\beta_2(0)$. Hence $\beta_1(0)=\beta_2(0)$ and, as we have seen this implies $\beta_1(x)\equiv\beta_2(x)$.

Having shown that (3.2) has a unique solution in B, we turn to (3.1). Let us write it in the form

$$\delta\alpha(x)=\frac{\lambda}{16\pi^2}\int_0^\infty dy\,H(x,y)\delta\alpha(y), \tag{3.25}$$

where

$$H(x,y)=[\rho(x)\theta(x-y)+\rho(y)\theta(y-x)]\beta^{-2}(y). \tag{3.26}$$

Thanks to the fact that $\beta(y)$ is bounded from below, we can show that H is a positive L^2 kernel,

$$\begin{aligned}\int_0^\infty\int_0^\infty dx\,dy\,H^2(x,y) &=\int_0^\infty dx\,\rho^2(x)\int_0^x\frac{dy}{\beta^4(y)}+\int_0^\infty dy\frac{\rho^2(y)}{\beta^4(y)}dx\\ &<\frac{2}{\beta_m^4}\int_0^\infty\frac{d\omega}{\log^2(1+\omega)}\frac{\omega^3}{(1+\omega)^4}\times\left(3+\frac{1}{1+\omega}\right)^2<\infty.\end{aligned} \tag{3.27}$$

Hence (3.25) is a classic Fredholm equation, and thus, if we require $\delta\alpha(x)$ to belong to L^2, the spectrum is discrete; in particular, there is a smallest value $\lambda_c>0$ such that (3.25) has only the trivial solution, $\delta\alpha(x)\equiv0$, if $0<\lambda<\lambda_c$, while it has a nontrivial solution if $\lambda=\lambda_c$.

The existence of a critical point λ_c is crucially dependent on limiting $\delta\alpha$ to L^2. Equation (3.1) is equivalent to the differential equation

$$\frac{d}{dx}\left[\frac{(d/dx)\delta\alpha(x)}{(d/dx)\rho(x)}\right]=\frac{\lambda}{16\pi^2}\frac{\delta\alpha(x)}{\beta^2(x)}, \tag{3.28}$$

with the boundary condition

$$\frac{d}{dx}\delta\alpha(x)\underset{x\to0}{\to}0. \tag{3.29}$$

The asymptotic behavior of (3.28) for large x is

$$\frac{d}{dx}\left[x^2\log\frac{x}{m^2}\frac{d}{dx}\delta\alpha(x)\right]+\frac{3\lambda}{16\pi^2}\delta\alpha(x)\sim0, \tag{3.30}$$

where we have used the fact that $\beta(\infty)=1$. This admits two solutions, which have the asymptotic behaviors

$$f_R(x)\sim(1/x)[\log(x/m^2)]^{-1+3\lambda/16\pi^2}, \tag{3.31}$$

$$f_I(x)\sim[\log(x/m^2)]^{-3\lambda/16\pi^2}. \tag{3.32}$$

The general solution of (3.28) is

$$\delta\alpha(x)=Af_R(x)+Bf_I(x), \tag{3.33}$$

but in order for this to solve the integral equation (3.1), the boundary condition (3.29) needs to be imposed. This fixes the ratio B/A, the remaining constant being a trivial normalization. It should be noted that a solution of the form (3.33) exists for any λ, but that it is not in L^2 in general. The smallest value of λ for which $B=0$ is precisely λ_c, and $\delta\alpha$ is then the regular solution f_R, which is in L^2.

In conclusion, we have seen that the bifurcation equations, in the Landau-like gauge, yield a critical point λ_c only if some information external to the Dyson–Schwinger system is used, in order to exclude the irregular solution $f_I(x)$.[3,8,9]

IV. FEYNMAN GAUGE

In Feynman gauge the bifurcation equation (2.35) for β has the specific form

$$\beta(x)=1+\frac{\lambda}{16\pi^2}\int_0^\infty dy\,\sigma(x_{\max})\frac{y}{\beta(y)}, \tag{4.1}$$

where $x_{\max}=\max(x,y)$, and where by hypothesis the function

$$\sigma(x)=\omega(x)/(x+m^2)^2 \tag{4.2}$$

is positive and monotonically decreasing. Notice that the function σ has one inverse power of $(x+m^2)$ less than that of Sec. III. We shall in fact show that (4.1) has *no* solutions.

Equation (4.1) is equivalent to the integrodifferential equation

$$\beta'(x) = \frac{\lambda}{16\pi^2}\sigma'(x)\int_0^x dy\,\frac{y}{\beta(y)}, \tag{4.3}$$

along with the boundary condition

$$\beta(\infty)=1. \tag{4.4}$$

Let us consider the case in which $\beta(0)>0$, and define the domain $\mathfrak{D}$ on which β remains positive,

$$\mathfrak{D}=\{x|\beta(y)>0 \text{ for } y\in[0,x]\}. \tag{4.5}$$

It follows from (4.3) that β is monotonically decreasing on $\mathfrak{D}$. As a consequence

$$\beta'(x)\leqslant[\lambda/32\pi^2\beta(0)]x^2\sigma'(x) \tag{4.6}$$

for $x\in\mathfrak{D}$. Integrating, we obtain

$$\beta(0)-\beta(x)\geqslant\frac{\lambda}{16\pi^2\beta(0)}\int_0^x dy\,y\sigma(y). \tag{4.7}$$

It follows from (4.2) and the definition (2.4) of ω that, at large y,

$$\sigma(y)\sim 1/y^2\log y, \tag{4.8}$$

so that the integral in (4.7) approaches log log x asymptotically at large x. Because of this divergence, the function $\beta(x)$ must approach zero at a finite point x_0 on the positive real x axis. In the vicinity of such a point, the solution to the differential equation (4.3) has the behavior

$$\beta(x)\sim(x_0-x)[(\lambda x_0/8\pi^2)\sigma'(x_0)\log(x_0-x)]^{1/2}. \tag{4.9}$$

The solution to Eq. (4.3) consequently has a branch point at $x=x_0$, with the real-analytic continuation having a branch cut for $x>x_0$. Furthermore, this solution of (4.3) has the asymptotic form

$$\beta(x)\sim\pm[(-\lambda/8\pi^2)\log\log x]^{1/2} \tag{4.10}$$

as x becomes large within the cut plane. Such solutions are not consistent with the boundary condition (4.4), so that they do not satisfy the integral equation (4.1), even if x is allowed to be complex.

We have shown that there are no solutions of (4.1) for $\beta(x)$ positive. Since $-\beta(x)$ satisfies Eq. (4.3) if $\beta(x)$ is a solution, there are also no solutions of (4.1) for $\beta(0)$ negative. For $\beta(0)=0$, the solution to (4.3) has the following asymptotic behavior at small x:

$$\beta(x)\sim\pm[(\lambda/12\pi^2)\sigma'(0)x^3]^{1/2}, \tag{4.11}$$

where $\sigma'(0)<0$. In this case the real-analytic solution has a branch cut for $x>0$, and it also has asymptotic behavior (4.10) at large x. Therefore there are no solutions to the integral equation for this case either.

The integral equation (4.1), considered for any positive, strictly decreasing weight functions $\omega(x)$, has no solutions whenever

$$\lim_{x\to\infty}\int_0^x dy\,y\sigma(y)=\infty. \tag{4.12}$$

If the weight function $\omega(x)$ were chosen to decrease slightly faster—say, $O[(\log x)^{-1-\epsilon}]$ for $\epsilon>0$—the integral (4.12) would converge and the integral equation would have a solution. This might well be affected by modifying the approximation for the quark–gluon vertex function—a matter we propose to take up in the future—but for the present we shall discuss the more standard Pauli–Villars cutoff procedure.

In the Pauli–Villars approach, we replace the function $\sigma(x_{\max})$ in the nonlinear integral equation (4.1) by the function $\tau(x_{\max})$,

$$\tau(x)=\omega(x)\,[1/(x+m^2)^2-1/(x+\Lambda^2)^2]; \tag{4.13}$$

with $\Lambda\gg m$. Equivalently, the function $\beta(x)$ will satisfy the nonlinear Volterra equation

$$\beta(x)=\beta(0)+\frac{\lambda}{16\pi^2}\int_0^x dy\,\frac{y}{\beta(y)}[\tau(x)-\tau(y)], \tag{4.14}$$

along with the boundary condition

$$\beta(\infty)=1. \tag{4.15}$$

Let us consider the solution of Eq. (4.14), starting from a given initial value $\beta(0)>0$. We define $\mathfrak{D}$ as the domain over which β remains positive; vide Eq. (4.5). For x in $\mathfrak{D}$, the Volterra equation has a unique monotonically decreasing solution $\beta(x)$. Furthermore, the value of β at fixed x is monotonically increasing as a function of the initial value $\beta(0)$. On the domain $\mathfrak{D}$, $\beta(x)$ satisfies the bound

$$\beta(x)\geqslant\beta(0)-I/\beta(x), \tag{4.16}$$

where

$$I=\frac{\lambda}{16\pi^2}\int_0^\infty dy\,y\tau(y). \tag{4.17}$$

If we choose

$$\beta(0)>[4I]^{1/2}, \tag{4.18}$$

it follows from (4.16) that $\beta(x)$ is positive for all $x\geqslant0$.

We have shown that, for $\beta(0)$ chosen sufficiently large, the nonlinear integral equation (4.14) has a unique positive solution for $x>0$. For a particular choice of $\beta(0)$, one satisfies condition (4.15). One can show directly from the integral equation that, to meet (4.15), the initial value $\beta(0)$ lies somewhere between the limits

$$I+[I^2+4]^{1/2}/2\leqslant\beta(0)\leqslant I+1. \tag{4.19}$$

Consequently, there is a unique solution to the integral equation corresponding to (4.1), with a Pauli–Villars cutoff inserted.

We have shown the existence of a unique positive solution of the cutoff integral equation, but the question remains as to the limit in which the cutoff parameter Λ becomes large. For our case the integral $I(\Lambda)$, defined in (4.17), has the form

$$I(\Lambda)=\frac{\Lambda}{16\pi^2}\int_0^\infty dy\,y\omega(y)\left[\frac{1}{(y+m^2)^2}-\frac{1}{(y+\Lambda^2)^2}\right]. \tag{4.20}$$

Because of (4.12), the integral $I(\Lambda)$ must diverge in the limit $\Lambda\to\infty$. In fact, one can show that

$$I(\Lambda)\sim(\lambda/16\pi^2)\log\log\Lambda \tag{4.21}$$

at large Λ. Because of this asymptotic behavior, along with the bounds (4.19), it follows that

$$\beta(0,\Lambda) \sim (\lambda/16\pi^2)\log\log\Lambda \tag{4.22}$$

as $\Lambda\to\infty$. In fact, one may show that, for fixed x, the renormalized function

$$\tilde{\beta}(x) \equiv \lim_{\Lambda\to\infty} [\beta(x,\Lambda)/\beta(0,\Lambda)] = 1. \tag{4.23}$$

The integral equation (2.34) for $\delta\alpha(x)$, with $\beta(x,\Lambda)$ inserted, exhibits chiral symmetry breaking, in that for λ less than some critical value $\lambda_c > 0$, the only solution is $\delta\alpha = 0$. The analysis in Feynman gauge is similar to that of Sec. III in Landau gauge. The critical coupling λ_c depends upon Λ, and in fact

$$\lambda_c \sim [\beta(0,\Lambda)]^2. \tag{4.24}$$

In other words, the only consistent solution of (2.34) for fixed coupling λ in the limit as the cutoff Λ becomes large is

$$\delta\alpha(x,\Lambda) = 0. \tag{4.25}$$

The renormalized function $\delta\tilde{\alpha}(x)$ is also zero,

$$\delta\tilde{\alpha}(x) \equiv \lim_{\Lambda\to\infty} \frac{\delta\alpha(x,\Lambda)}{\beta(0,\Lambda)} = 0.$$

We therefore find that in Feynman gauge, the normalized inverse quark propagator $\tilde{S}^{-1}(p)$ corresponds to a massless, free quark,

$$\tilde{S}^{-1}(p) = \lim_{\Lambda\to\infty} \frac{\alpha(x,\Lambda) + \not{p}\beta(x,\Lambda)}{\beta(0,\Lambda)} = \not{p}. \tag{4.26}$$

In summary, we have shown that there is no solution of the bifurcation equation (4.1) in Feynman gauge, because of problems in the ultraviolet. There is a solution to the Dyson–Schwinger equations when a Pauli–Villars cutoff parameter Λ is introduced, but the renormalized propagator corresponds to free, massless quarks in the limits as $\Lambda\to\infty$. One would expect the phenomenon of chiral symmetry breaking to gauge invariant, but our algorithm for truncation of the Dyson–Schwinger equation is explicitly gauge dependent. The difficulty can be plausibly traced to the naive JBW treatment of the full vertex function.

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