# Chiral symmetry breakdown

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# Chiral symmetry breakdown. III. Delbourgo's gauge technique

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The quark propagator in massless quantum chromodynamics (QCD) is analyzed using the gauge technique. In both the Feynman and Landau gauges with a Pauli–Villars cutoff, a chirally symmetric solution is found, while a nonsymmetric solution appears at a critical coupling $\lambda_c > 0$. As the cutoff is removed, $\lambda_c$ tends to 0 but the nonsymmetric solution vanishes in the continuum limit, so that chiral symmetry is then restored.

## I. INTRODUCTION

Spontaneous dynamical breakdown of chiral symmetry represents an interesting and potentially important means through which particles become massive at a certain critical coupling strength because of interactions with the quantum field.[1] In QCD, this phenomenon provides a mechanism for quarks to acquire constituent masses in a formally massless theory through nonperturbative effects. The occurrence of dynamical symmetry breaking is investigated by analysis of Dyson–Schwinger equations for the fermion propagator. Such analyses are hampered by a degree of arbitrariness, in that the effect of unavoidable truncation of Dyson–Schwinger equations is difficult to assess.

Some time ago Maskawa and Nakajima showed that, in the context of quantum electrodynamics (QED), when the fermion–fermion–photon vertex function and the photon propagator are replaced by their free values, the solutions of the Dyson–Schwinger equation exhibit spontaneous chiral symmetry breaking.[2] In a recent paper we demonstrated that this spontaneous symmetry breaking occurs when one introduces both an infrared and an ultraviolet cutoff.[3] In a more detailed analysis of the problem, this truncation procedure was found to be unacceptably gauge dependent. We showed that chiral symmetry was indeed broken in the theory with cutoffs present, but that the symmetry was restored in the continuum limit in certain gauges.[4] It was found that chiral symmetry breaking was very sensitive to the infrared and ultraviolet behavior of the gauge-dependent free photon propagator. Furthermore, we showed that this sensitivity to choice of gauge remained even when the truncation procedure was modified to maintain consistency with the "transversality" constraint coming from gauge invariance in the ultraviolet.[5] It is our conclusion that the free vertex approximation cannot provide a reliable indication as to the occurrence of spontaneous chiral symmetry breaking, and that other truncation schemes should be explored.

Truncation of Dyson–Schwinger equations is certainly an *ad hoc* procedure based primarily upon expediency, but one must make use of physically motivated conditions or requirements whenever possible. Several authors have chosen the quark–quark–gluon vertex in QCD to decrease as $1/\log q^2$ at large $q^2$, in correspondence with the ultraviolet behavior imposed by asymptotic freedom.[6–8] We have shown in Ref. 4 that chiral symmetry remains unbroken in certain gauges in the continuum limit in this case as well, because of formal divergence of the loop integral in the Dyson–Schwinger equation. We conclude that this truncation scheme has essentially the same difficulties as the free vertex approximation, and thus that one must look further to obtain a resolution of the ambiguity associated with gauge-dependent results.

The free vertex truncation, as well as its renormalization-group-improved counterpart discussed in Ref. 4, is not consistent with the Slavnov–Taylor identity, and therefore stands in direct conflict with the requirements of gauge invariance. The gauge dependence of the results on chiral symmetry breaking is a manifestation of the gauge dependence of the truncation scheme. By contrast, in the truncation scheme proposed some time ago by Salam and Delbourgo,[9] and studied by a number of people,[10,11] one maintains consistency with the Slavnov–Taylor identity for the vertex function, and thus the effects of gauge dependence should be reduced.

Here we will study chiral symmetry breaking in the Salam–Delbourgo scheme, which is based upon spectral *Ansätze* for the propagator and vertex functions. We replace the gluon propagator by a free massive one, and discuss both Feynmann and Landau gauges in the Dyson–Schwinger equation in Sec. II. The formal divergence of the loop integral is removed by introducing a Pauli–Villars cutoff; and the Dyson–Schwinger equation reduces to a homogeneous linear equation for the spectral density function, which has nontrivial solutions.

We find that when the cutoff is present, only a chirally symmetric solution occurs below a critical coupling $\lambda_c > 0$, whereas above $\lambda_c$ nonsymmetric solutions are also present. In the continuum limit as the cutoff is removed, $\lambda_c$ goes formally to 0. However, in both gauges only the chirally symmetric solution survives. Therefore in the Salam–Delbourgo formalism chiral symmetry is restored in the continuum limit.

## II. SALAM–DELBOURGO ANSATZ

The Dyson–Schwinger equation for the quark propagator $S'_F(p)$ can be written

$$\not{p} S'_F(p) - \Sigma(p) S'_F(p) = 1, \tag{2.1}$$

where the self-energy is given by

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$$\Sigma(p)=\frac{ig^2}{(2\pi)^4}\int d^4k\,\gamma_\mu S'_F(p-k)\Gamma_\nu(p-k,p)D'^{\mu\nu}_F(k), \tag{2.2}$$

where $\Gamma_\nu$ is the gluon-quark vertex function and $D'_F$ the gluon propagator. Here $g$ is proportional to the SU(3) coupling constant.

The Lehmann representation for the quark propagator can be written

$$S'_F(p)=\int_{-\infty}^{\infty}dw'\,\frac{\rho(w')}{\not p-w'+i\epsilon\epsilon(w')}, \tag{2.3}$$

where $\epsilon(w')=\mathrm{sgn}(w')$, and $\rho$ is an unknown spectral function. If $\rho(w)$ is even in $w$, $S'_F(p)$ has the form $\not p$ times a function of $p^2$, and so chiral symmetry is then unbroken. The Salam–Delbourgo *Ansatz* assumes that

$$S'_F(p-k)\Gamma_\nu(p-k,p)S'_F(p)=\int_{-\infty}^{\infty}dw'\rho(w')\,\frac{1}{\not p-\not k-w'}\gamma_\nu\frac{1}{\not p-w'}, \tag{2.4}$$

where the $i\epsilon\epsilon(w')$ is to be understood in both denominators. It is easy to see that (2.4) is consistent with the Slavnov–Taylor identity for $\Gamma_\nu$.

With this *Ansatz*, the Dyson–Schwinger equation (2.1) can be rewritten in the form

$$\int_{-\infty}^{\infty}dw'\,\rho(w')[\not p-\Sigma(\not p-w')]\frac{1}{\not p-w'}=1, \tag{2.5}$$

where

$$\Sigma(\not p,w')=\frac{ig^2}{(2\pi)^4}\int d^4k\,\gamma_\nu\frac{1}{\not p-\not k-w'}\gamma_\nu D'^{\mu\nu}_F(k). \tag{2.6}$$

In this paper, we propose to consider the Feynman and the Landau gauges. Moreover, we shall remove the divergence of the loop integral (2.6) by introducing a Pauli–Villars cutoff. We write

$$D'^{\mu\nu}_F(k)=D^{\mu\nu}(k,m)-D^{\mu\nu}(k,\Lambda), \tag{2.7}$$

where the cutoff $\Lambda$ is much greater than $m$, the gluon mass, and where, in the Feynman gauge,

$$D^{\mu\nu}(k,m)=-g^{\mu\nu}/(k^2-m^2+i\epsilon), \tag{2.8a}$$

and in the Landau gauge,

$$D^{\mu\nu}(k,m)=\left[-g^{\mu\nu}+\frac{k^\mu k^\nu}{k^2+i\epsilon}\right]\frac{1}{k^2-m^2+i\epsilon}. \tag{2.8b}$$

The corresponding expressions for $\Sigma$ can be calculated:

$$\Sigma(\not p,w')=\lambda\int_{-\infty}^{\infty}dw\,\frac{\Omega(w,w',m)-\Omega(w,w',\Lambda)}{\not p-w}. \tag{2.9}$$

Here $\lambda=g^2/(16\pi^2)$ and

$$\begin{aligned}\Omega(w,w',m)&\\&=[\epsilon(w)/w^3]\theta(|w|-|w'|-m)\\&\quad\times[w^2-4ww'+w'^2-m^2]\\&\quad\times\{[w^2-(w'+m)^2][w^2-(w'-m)^2]\}^{1/2}\end{aligned} \tag{2.10a}$$

in the Feynman gauge, and

$$\begin{aligned}\Omega(w,w',m)&\\&=\tfrac{3}{2}[\epsilon(w)/w^3]\theta(|w|-|w'|-m)[(w-w')^2-m^2]\\&\quad\times\{[w^2-(w'+m)^2][w^2-(w'-m)^2]\}^{1/2}\end{aligned} \tag{2.10b}$$

in the Landau gauge. The expression (2.9) for $\Sigma(p)$ involves a convergent integral in both gauges, because of the Pauli–Villars cutoff.

In each gauge we substitute (2.9) into (2.5) and partially fractionate, obtaining

$$\begin{aligned}1=\int_{-\infty}^{\infty}dw'\,\rho(w')\frac{\not p}{\not p-w'}-\lambda\int_{-\infty}^{\infty}dw'\,\rho(w')\\\times\int_{-\infty}^{\infty}dw''\,\frac{\Omega(w'',w',m)-\Omega(w'',w',\Lambda)}{w''-w'}\\\times\left[\frac{1}{\not p-w''}-\frac{1}{\not p-w'}\right].\end{aligned} \tag{2.11}$$

The imaginary part of this equation is

$$\phi(w)\rho(w)=\lambda\int_{-\infty}^{\infty}dw'\,\rho(w')\times\frac{\Omega(w,w',m)-\Omega(w,w',\Lambda)}{w-w'}, \tag{2.12}$$

where

$$\phi(w)=w+\lambda\int_{-\infty}^{\infty}dw'\,\frac{\Omega(w',w,m)-\Omega(w',w,\Lambda)}{w'-w}. \tag{2.13}$$

The function $\phi(w)$ may be calculated in closed form for either gauge. Equation (2.12) will be analyzed, in these gauges, in the next section.

## III. ANALYSIS OF THE EQUATION

Equation (2.12) can be written

$$\begin{aligned}\phi(w)\rho(w)&\\&=\lambda\theta(|w|-m)\int_{-|w|+m}^{|w|-m}dw'\,\rho(w')\frac{\Omega(w,w',m)}{w-w'}\\&\quad-\lambda\theta(|w|-\Lambda)\int_{-|w|+\Lambda}^{|w|-\Lambda}dw'\,\rho(w')\frac{\Omega(w,w',\Lambda)}{w-w'}.\end{aligned} \tag{3.1}$$

The $\theta$ functions are already implied by those in (2.10), they are included explicitly above to emphasize that the right-hand side of (3.1) vanishes when $|w|<m$.

First note that the denominator $w-w'$ never vanishes, and that the kernel in continuous. The function $\phi(w)$ is also bounded and continuous, and we may write

$$\phi(w)/w=1-\lambda\,[\psi(w,m)-\psi(w,\Lambda)], \tag{3.2}$$

where

$$\psi(w,m)=2\int_{|w|+m}^{M}\frac{dw'}{w'^3}P(w'^2,w^2,m^2)\times\{[w'^2-(w+m)^2][w'^2-(w-m)^2]\}^{1/2} \tag{3.3}$$

with

$$P(w'^2,w^2,m^2)=3+(2w^2+m^2)/(w'^2-w^2) \tag{3.4a}$$

in the Feynman gauge, and

$$P(w'^2,w^2,m^2) = \tfrac{3}{2}(1 + m^2/(w'^2 - w^2)) \tag{3.4b}$$

in the Landau gauge. It is possible to express $\psi(w,m)$ in terms of elementary functions in both gauges. We have introduced a cutoff parameter $M$ in the integral (3.3), which must be taken to infinity in (3.2). The integral in (3.3) diverges logarithmically as $M \to \infty$, but the function $\phi(w)$ in (3.2) is well defined in this limit. In particular,

$$\phi(w)/w \to 1 - b\lambda \log(\Lambda/m) \tag{3.5}$$

as $w \to 0$, where $b = 6$ in Feynman gauge and $b = 3$ in Landau gauge. In both gauges, $\phi(w)/w$ is monotonically increasing for $w > 0$ and

$$\phi(w)/w \to 1 \tag{3.6}$$

as $w \to \pm\infty$.

The odd function $\phi(w)$ is zero at $w = 0$. In addition, under the condition

$$\lambda > \lambda_c(\Lambda) = [b \log(\Lambda/m)]^{-1}, \tag{3.7}$$

it has a pair of zeros at $w = \pm w_0$, where $w_0$ is positive by convention. The function $\phi(w)$ has no positive zero for $\lambda < \lambda_c$. Note that the critical coupling $\lambda_c(\Lambda)$ goes to 0 as the Pauli–Villars cutoff parameter tends to infinity.

Let us consider Eq. (3.1). We know that any solution $\rho(w)$ vanishes for $|w| < m$. Furthermore, for $m \leqslant |w| < 2m$, the values of $w'$ in the integrals on the right-hand side of (3.1) satisfy $|w'| < m$, for which $\rho(w')$ is 0. Consequently, $\rho(w)$ actually vanishes for $|w| < 2m$. In fact, one may iterate this procedure to show that $\rho(w)$ vanishes for all $w$. The conclusion is unavoidable if one requires the spectral density $\rho(w)$ to be a bounded continuous function. However, it is also possible for $\rho(w)$ to have delta distributions at those values of $w$ for which $\phi(w)$ is 0; $\phi(w)\rho(w)$ and the integrals in (3.1) would then be ordinary functions.[12] Therefore, location of the zeros of $\rho(w)$ is a crucial ingredient in solving (3.1).

Let us set the mass scale by taking $m = 1$, and define the function

$$\lambda_0(w_0,\Lambda) = [\psi(w_0,1) - \psi(w_0,\Lambda)]^{-1}. \tag{3.8}$$

For coupling strength $\lambda = \lambda_0(w_0,\Lambda)$ with cutoff $\Lambda$, the function $\phi(\pm w_0)$ is 0, provided that $\lambda_0$ is greater than the critical coupling $\lambda_c(\Lambda)$; cf. Eq. (3.2). We find that $\lambda_0(w_0,\Lambda)$ is monotonically decreasing in $w_0$ for fixed $\Lambda$. The function $\lambda_0(w_0,\Lambda)$ is plotted against $w_0$ in Figs. 1(a) (Feynman gauge) and 1(b) (Landau gauge). Note that $w_0$ goes to 0 as $\lambda$ approaches the critical coupling $\lambda_c(\Lambda)$ from above.

In Figs. 2(a) (Feynman gauge) and 2(b) (Landau gauge), $w_0$ is plotted against the cutoff parameter $\Lambda$ for various choices of coupling strength $\lambda_0$. There is an approximate linear relation between $\Lambda$ and $w_0$, which can be understood by examining the integral (3.3) in the parameter regime $M^2 \gg \Lambda^2 \gg w_0^2 \gg m^2 = 1$, to obtain

$$\psi(w_0,1) - \psi(w_0,\Lambda) \simeq 6 \log(\Lambda/w_0) + 2 \tag{3.9a}$$

in Feynman gauge, and

$$\psi(w_0,1) - \psi(w_0,\Lambda) \simeq 3 \log(\Lambda/w_0) - \tfrac{3}{4} \tag{3.9b}$$

in Landau gauge. Correspondingly, we get

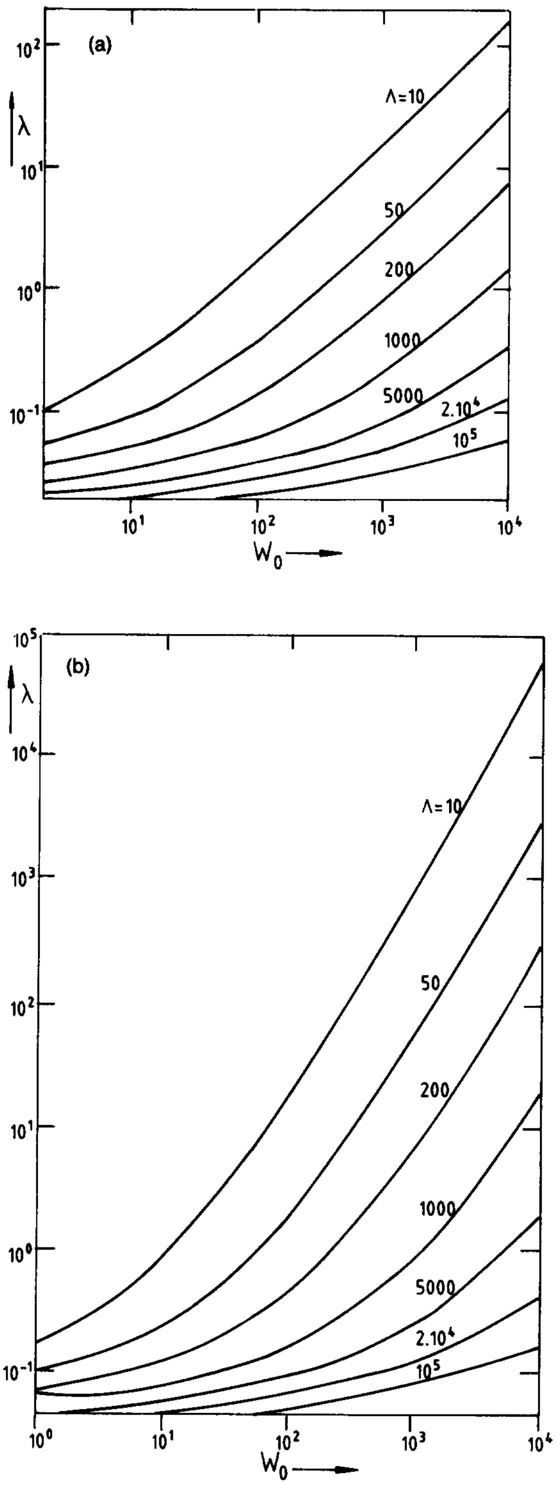


FIG. 1. $\lambda_0$ vs $w_0$ for various values of the cutoff $\Lambda$: (a) Feynman gauge, (b) Landau gauge.

$$\Lambda \simeq w_0 \exp\left[\frac{1}{b\lambda} + \frac{9-2b}{12}\right], \tag{3.10}$$

where $b$ is 6 in Feynman gauge and 3 in Landau gauge, as in Eq. (3.5) above. These relations are valid to within a few percent for both gauges. Note, in particular, that for fixed $\lambda$, $w_0$ tends to infinity with $\Lambda$.

Let us look for solutions of Eq. (3.1) of the form

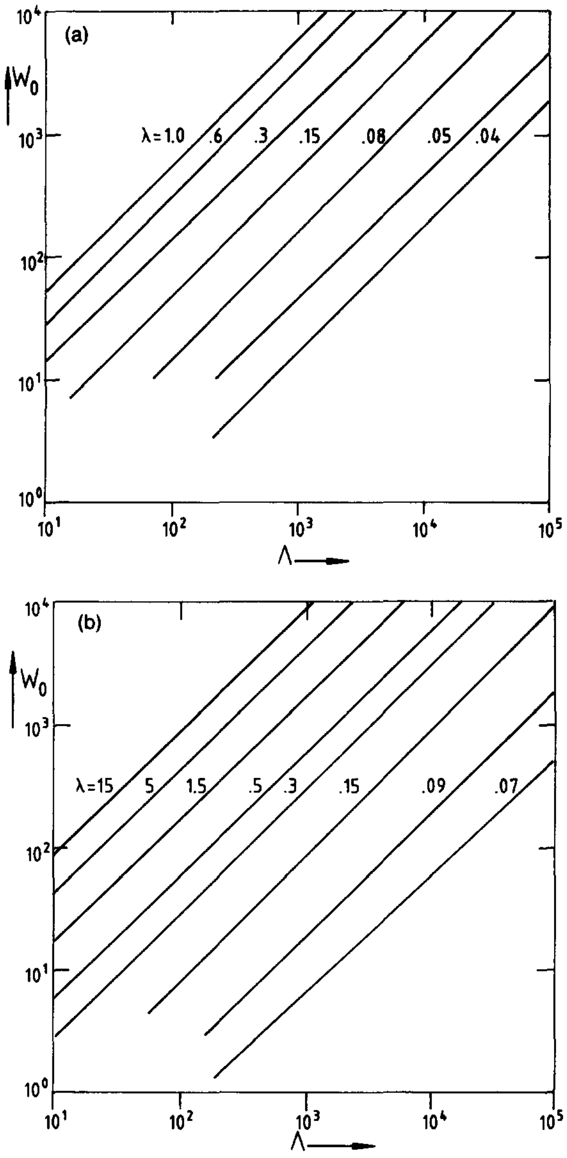


FIG. 2. $w_0$ versus cutoff parameter $\Lambda$ for various values of coupling strength: (a) Feynman gauge, (b) Landau gauge.

$$\rho(w) = A\delta(w - w_0) + B\delta(w + w_0) + \sigma(w), \quad (3.11)$$

for constants $A$ and $B$ chosen arbitrarily, with the function $\sigma(w)$ to be determined. We obtain

$$\begin{aligned}\phi(w)\sigma(w) = {} & \lambda\theta(|w| - m - w_0)\left[A\,\frac{\Omega(w,w_0,m)}{w - w_0} + B\,\frac{\Omega(w,-w_0,m)}{w + w_0}\right] \\ & + \lambda\theta(|w| - m)\int_{-|w|+m}^{|w|-m} dw' \\ & \times\sigma(w')\frac{\Omega(w,w',m)}{w - w'} - [m \to \Lambda].\end{aligned} \quad (3.12)$$

The terms involving $A$ and $B$ vanish when $|w| < m + w_0$, and so $\sigma(w) = 0$ for $|w| < m + w_0$, by the argument given before. Hence we may replace $\theta(|w| - m)$ in Eq. (3.12) by $\theta(|w| - m - w_0)$. Indeed, it may be sharpened to $\theta(|w| - 2m - w_0)$, since $|w'|$ can exceed $m + w_0$ only when $|w| > 2m + w_0$. Now $\sigma(w)$ may be evaluated in successive steps of width $m$; for $m + w_0 < |w| < 2m + w_0$, it is given by just the first line of Eq. (3.12), for $2m + w_0 |w| < 3m + w_0$ the integral contributes, but it involves only the domain $m + w_0 < |w'| < 2m + w_0$, for which $\sigma(w')$ is already known, and so on. In short, Eq. (3.12) is not a true equation for $\sigma(w)$; it is rather a progressive algorithm for evaluating $\sigma(w)$ to arbitrarily large $w$ values, and the existence of a solution, parametrized by $A$ and $B$, is assured.

The solution implies the following Lehmann representation for the quark propagator:

$$S'_F(p) = \frac{(A + B)\not p + (A - B)w_0}{p^2 - w_0^2} + \int_{\infty -}^{\infty} dw'\,\frac{\theta(|w'| - m - w_0)\sigma(w')}{\not p - w'}. \quad (3.13)$$

There is a pole in the quark propagator at $p^2 = w_0^2$, but chiral symmetry actually remains unbroken when $A = B$. To verify the latter point, note that $\phi(w)$ is odd, the inhomogeneous term in (3.12) is an odd function of $w$ when $A = B$, and the solution $\sigma(w)$ comes out as an even function of $w$. Thus the quark propagator (3.13) is proportional to $\not p$ and chiral symmetry is preserved. The choice $A \neq B$ leads to a propagator that breaks chiral symmetry.

In the continuum limit ($\Lambda \to \infty$), the pole in the quark propagator at $p^2 = w_0^2$ disappears to infinity. However, the parameters $A$ and $B$ can also be taken to depend upon $\Lambda$. In the continuum limit, the spectral density $\sigma$ vanishes, and the renormalized quark propagator is of the form

$$S'_F(p) = a\not p + b. \quad (3.14)$$

The subtraction constants $a$ and $b$, while not determined in the Salam–Delbourgo formalism, must be set to 0 on physical grounds. Consequently, only the trivial solution $S'_F(p) = 0$ survives in the continuum limit.

Finally, we turn to the zero of $\phi(w)$ at $w = 0$. There is another solution of (3.1) of the form

$$\rho(w) = \delta(w) + \sigma(w), \quad (3.15)$$

where $\sigma(w)$ satisfies Eq. (3.12) with replacements $w_0 = 0$ and $A + B = 1$. The inhomogeneous term in (3.12) is proportional to

$$\lambda\theta(|w| - m)\epsilon(w)(1 - m^2/w^2)^2 \quad (3.16)$$

in both gauges, and the solution $\sigma(w)$ is an even function of $w$. Consequently, the quark propagator becomes

$$S'_F(p) = \frac{\not p}{p^2} + 2\not p\int_m^{\infty} dw'\,\frac{\sigma(w')}{p^2 - w'^2}. \quad (3.17)$$

The quark propagator has a pole at $p^2 = 0$, and chiral symmetry remains unbroken at finite $\Lambda$, as well as in the continuum limit.

In summary, we have found two classes of solutions: in one class there may be a breaking of chiral symmetry with a cutoff present, but such solutions trivialize in the continuum limit; whereas the other class corresponds to unbroken chiral symmetry. Chiral symmetry thus remains unbroken in the continuum limit in the Salam–Delbourgo formalism. These conclusions, which are established here in both Feynman and Landau gauges, are shown in Ref. 13 to apply also in the "Landau-like" gauge proposed by Maskawa and Nakajima in Ref. 2 and treated by us in Ref. 4.


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