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Gardner's deformations of the graded Korteweg–de Vries equations revisited

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We solve the Gardner deformation problem for the $N = 2$ supersymmetric $a = 4$ Korteweg–de Vries equation [P. Mathieu, "Supersymmetric extension of the Korteweg–de Vries equation," J. Math. Phys. **29**(11), 2499–2506 (1988)]. We show that a known zero-curvature representation for this super-equation yields the system of new nonlocal variables such that their derivatives contain the Gardner deformation for the classical KdV equation. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4754288]

I. INTRODUCTION

The classical problem of construction of the Gardner deformation[1] for an infinite-dimensional completely integrable system of evolutionary partial differential equations essentially amounts to finding a recurrence relation between the integrals of motion. For the $N = 2$ supersymmetric generalizations of the Korteweg–de Vries (KdV) equation,[2,3] the deformation problem was posed when the integrable triplet of such super-systems was discovered. Various attempts to solve it were undertaken since then (e.g., see Ref. 2) but the progress was limited. In particular, in Ref. 4 we prove the "no-go" theorem stating that a classical polynomial Gardner deformation for the $N = 2$ supersymmetric $a = 4$ KdV equation does *not* exist within the superfield formalism (but that in principle, the deformation may exist whenever the superfields split in components), cf. Ref. 2. This is in contrast with the $N = 1$ sKdV case when the two approaches yield the supersymmetry-invariant deformation.[3] The first solution for the $N = 2$, $a = 4$ SKdV in the triplet $a \in \{-2, 1, 4\}$ was achieved in Ref. 4. We then presented the two-step solution of the deformation problem: We obtained the Gardner deformation for the Kaup–Boussinesq equation, which is the bosonic limit of the super-equation that precedes the $N = 2$, $a = 4$ super-KdV in its hierarchy. We thus derived the recurrence relation between the Hamiltonians of the bosonic limit hierarchy and then we showed how each conserved density is extended to the super-density for the $N = 2$ super-system. In other words, we deformed the bosonic subsystem of the super-equation at hand within the frames of the classical scheme,[1] whence we recovered the full $N = 2$ supersymmetry-invariance.

In this paper we re-address, from a basically different viewpoint, the Gardner deformation problem for a vast class of (not necessarily supersymmetric) KdV-like systems. Namely, in Ref. 4 we emphasized the geometric similarity of the Gardner deformations and zero-curvature representations, each of them manifesting the integrability of nonlinear systems (cf. Refs. 5 and 6). Indeed, both constructions generate infinite sequences of nontrivial integrals of motion. However, the standard Lax approach relies on the calculus of pseudodifferential operators whereas the Gardner technique is more geometric and favourable from a computational viewpoint. We stress that, in general, the two constructions are *not* equivalent, although they provide coinciding results. It is precisely this correspondence which we study in this paper for the graded KdV systems.

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Developing further the approach of Ref. 7, we reformulate the Gardner deformation problem for the graded extensions of the KdV equation in terms of constructing parameter-dependent families of new bosonic and fermionic variables. We require that the "nonlocalities" possess two defining properties:[8,9] on one hand, they should reproduce the classical Gardner deformation from Ref. 1 under the shrinking of the $N = 2$ super-equation back to the KdV equation. On the other hand, we consider the nonlocalities that encode the parameter-dependent zero-curvature representations for the super-systems at hand. In this reformulation, we solve Mathieu's open problem 2 of Ref. 3 for the $N = 2$ supersymmetric $a = 4$-KdV equation. However, our approach is applicable to a much wider class of completely integrable (super-)systems.

This paper is structured as follows. We first fix some notation and compare the notions of Gardner's deformations and zero-curvature representations by using their unifying description in terms of nonlocal structures for partial differential equations (PDE). In Sec. III we proceed with this correspondence for $\mathbb{Z}_2$-graded systems of evolutionary PDE and solve the Gardner deformation problem for the $N = 2$, $a = 4$ SKdV (29). The nature of the new solution is geometric and it presents an alternative to the analytic two-step algorithm that works for graded systems and which we reported earlier in Ref. 4.

II. PRELIMINARIES

In this section we briefly recall some basic notions and facts and fix the notation, which follows Refs. 10–13; we refer to the book[13] for further details.

Let Σ^n be an n-dimensional manifold, $1 \le n < \infty$, let $\pi : E^{m+n} \to \Sigma^n$ be a vector bundle over Σ^n of fiber dimension m. In what follows we let $n = 2$ so that $x^1 = x$ and $x^2 = t$ are the independent variables; we have that $m = 1$ for the Korteweg–de Vries equation, $m = (1|1)$ for the $\mathbb{Z}_2$-graded $N = 1$ supersymmetric KdV equation, and we let $m = (2|2)$ for the $N = 2$ SKdV, see Refs. 2, 3, and 14. Consider the jet space $J^\infty(\pi)$ of sections of the vector bundle π. The local coordinates on $J^\infty(\pi)$ are composed by the coordinates x^i on Σ^n, coordinates u^j along the fibers of π, and coordinates u^j_σ along the fibers of $J^\infty(\pi) \to \Sigma$; here σ is a multi-index that labels the partial derivatives of a field u^j, so that $u^j_\varnothing \equiv u^j$. The commuting vector fields $\frac{\mathrm{d}}{\mathrm{d}x^i} = \partial/\partial x_i + \sum_{k,\sigma} u^k_{\sigma i}\partial/\partial u^k_\sigma$ on $J^\infty(\pi)$ are called *the total derivatives*. The operators $\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}x^\sigma}$ that act on the space of smooth functions on $J^\infty(\pi)$ are defined inductively by the formula $\frac{\mathrm{d}^{|\sigma i|}}{\mathrm{d}x^{\sigma i}} = \frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}x^\sigma} \circ \frac{\mathrm{d}}{\mathrm{d}x^i}$. For example, set $\sigma = xt$ whence $\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}x^\sigma} = \frac{\mathrm{d}}{\mathrm{d}x} \circ \frac{\mathrm{d}}{\mathrm{d}t}$; we use the standard notation $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 = \frac{\mathrm{d}^3}{\mathrm{d}x^3}$, etc.

Consider a system $\mathcal{E}$ of r partial differential equations,

$$\mathcal{E} = \left\{ F^\ell(x^i, u^j, \ldots, u^k_I, \ldots) = 0, \quad \ell = 1, \ldots, r \right\}.$$

The system $\{F^\ell = 0\}$ and all its differential consequences $\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}x^\sigma} F^\ell = 0$, $|\sigma| \ge 1$ generate the infinite prolongation of $\mathcal{E}$, which we denote by $\mathcal{E}^\infty$. The restrictions of $\frac{\mathrm{d}}{\mathrm{d}x^i}$ on $\mathcal{E}^\infty$ determine the Cartan distribution $\mathcal{C}$ on the tangent space $T\mathcal{E}^\infty$. Here and in what follows, the notation $\frac{\mathrm{d}}{\mathrm{d}x^i}$ stands for the restrictions of the total derivatives onto $\mathcal{E}^\infty$. At each point $\theta \in \mathcal{E}^\infty$, there is the decomposition of the tangent space $T_\theta\mathcal{E}^\infty$ to the direct sum of the horizontal (spanned by the Cartan distribution) and the vertical vector spaces, $T_\theta\mathcal{E}^\infty = \mathcal{C}_\theta \oplus V_\theta\mathcal{E}^\infty$. Let $\Lambda^{1,0}(\mathcal{E}^\infty) = \operatorname{Ann}\mathcal{C}$ and $\Lambda^{0,1}(\mathcal{E}^\infty) = \operatorname{Ann} V\mathcal{E}^\infty$ be the $C^\infty(\mathcal{E}^\infty)$-modules of contact 1-forms and horizontal 1-forms vanishing on $\mathcal{C}$ and $V\mathcal{E}^\infty$, respectively. Let $\Lambda^r(\mathcal{E}^\infty)$ denote the C^∞-module of r-forms on $\mathcal{E}^\infty$. We have the decomposition $\Lambda^r(\mathcal{E}^\infty) = \bigoplus_{q+p=r} \Lambda^{p,q}(\mathcal{E}^\infty)$, where $\Lambda^{p,q}(\mathcal{E}^\infty) = \bigwedge^p \Lambda^{1,0}(\mathcal{E}^\infty) \wedge \bigwedge^q \Lambda^{0,1}(\mathcal{E}^\infty)$. According to this decomposition, the exterior differential splits to the sum $d = \bar{d} + d_\mathcal{C}$ of the horizontal differential $\bar{d} : \Lambda^{p,q}(\mathcal{E}^\infty) \to \Lambda^{p,q+1}(\mathcal{E}^\infty)$ and the vertical differential $d_\mathcal{C} : \Lambda^{p,q}(\mathcal{E}^\infty) \to \Lambda^{p+1,q}(\mathcal{E}^\infty)$.

The differential $\bar{d}$ can be expressed in coordinates by its actions on the elements $\phi \in C^\infty(\mathcal{E}^\infty) = \Lambda^{0,0}(\mathcal{E}^\infty)$, whence

$$\bar{d}\phi = \sum_i \frac{\mathrm{d}}{\mathrm{d}x^i}(\phi)\, dx^i.$$

This be our working formula.

A. Zero curvature representations and coverings

Consider the tensor product over the ring of $\mathbb{C}$-valued smooth function on $\mathcal{E}^\infty$ of the exterior algebra $\bar{\Lambda}(\mathcal{E}^\infty) = \bigwedge^* \Lambda^{i,0}(\mathcal{E}^\infty)$ with a finite-dimensional complex Lie algebra $\mathfrak{g}$. The product is endowed with the bracket $[A\mu, B\nu] = [A, B]\mu\wedge\nu$ for $\mu, \nu \in \bar{\Lambda}(\mathcal{E}^\infty)$ and $A, B \in \mathfrak{g}$. The tensor product $\bar{\Lambda}(\mathcal{E}^\infty) \otimes \mathfrak{g}(\mathbb{C})$ is a differential graded associative algebra with respect to the multiplication $A\mu \cdot B\nu = (A \cdot B)\mu\wedge\nu$ induced by the ordinary matrix multiplication so that

$$\bar{d}(\rho \cdot \sigma) = \bar{d}\rho \cdot \sigma + (-1)^r \rho \cdot \bar{d}\sigma$$

for $\rho \in \bar{\Lambda}^r(\mathcal{E}^\infty) \otimes \mathfrak{g}(\mathbb{C})$ and $\sigma \in \bar{\Lambda}^s(\mathcal{E}^\infty) \otimes \mathfrak{g}(\mathbb{C})$, whereas

$$[\rho, \sigma] = \rho \cdot \sigma - (-1)^{rs} \rho \cdot \sigma.$$

The elements of $C^\infty(\mathcal{E}^\infty) \otimes \mathfrak{g}$ will be called the $\mathfrak{g}$-*matrices*.[10]

Definition 1 (Ref. 10): A horizontal 1-form $\alpha \in \bar{\Lambda}^1(\mathcal{E}^\infty) \otimes \mathfrak{g}$ is called the $\mathfrak{g}$-valued *zero-curvature representation* for the equation $\mathcal{E}$ if the Maurer–Cartan condition holds:

$$\bar{d}\alpha = \frac{1}{2}[\alpha, \alpha]. \tag{1}$$

If $\alpha = \sum_{i=1}^n A_i\, \mathrm{d}x^i$, where $A_i \in \mathfrak{g}$, is a $\mathfrak{g}$-valued zero-curvature representation for $\mathcal{E}$, then we have

$$\bar{d}\alpha - \frac{1}{2}[\alpha, \alpha] = \sum_{i<j} (\tfrac{\mathrm{d}}{\mathrm{d}x^i} A_j - \tfrac{\mathrm{d}}{\mathrm{d}x^j} A_i) dx^i \wedge dx^j - \sum_{i<j} [A_i, A_j] dx^i \wedge dx^j.$$

Therefore, Eq. (1) is equivalent to the following set of conditions upon the $\mathfrak{g}$-matrices A_i:

$$\tfrac{\mathrm{d}}{\mathrm{d}x^j} A_i - \tfrac{\mathrm{d}}{\mathrm{d}x^i} A_j + [A_i, A_j] = 0, \quad \forall i, j = 1, \ldots, m : i \neq j. \tag{2}$$

The most interesting zero-curvature representations for $\mathcal{E}$ are those which contain a non-removable spectral parameter; in this case the system $\mathcal{E}$ is integrable. (The parameter is *removable* if one obtains gauge-equivalent zero-curvature representations, see Sec. II C for details, at different values of the parameter; otherwise, the parameter is non-removable).

We recall that n is the dimension of the base Σ^n for the vector bundle π. From now on, we consider mainly $(1 + 1)$-dimensional systems, i.e., we let $n = 2$ and interpret one independent variable as the time t and the other as the spatial coordinate x. With the conventions $n = 2$, $x^1 = x$, $x^2 = t$, $A_1 = A$, and $A_2 = B$, the Maurer–Cartan equations (1) and (2) read

$$\tfrac{\mathrm{d}}{\mathrm{d}t} A - \tfrac{\mathrm{d}}{\mathrm{d}x} B + [A, B] = 0. \tag{2$'$}$$

This is the compatibility condition for the auxiliary linear system

$$\Psi_x = A\Psi, \quad \Psi_t = B\Psi,$$

where Ψ is the wave function and the matrices A and B belong to the tensor product of a matrix Lie algebra $\mathfrak{g}$ and the ring $C^\infty(\mathcal{E}^\infty)$ of smooth functions on the prolongation $\mathcal{E}^\infty$. If the matrices A and B depend on the non-removable spectral parameter, then the equation $\mathcal{E}$ is integrable.[15]

The construction of Gardner's deformations[1] is another way to prove the integrability of evolution equations $\mathcal{E}$.

Definition 2 (Gardner's deformation (Ref. 1)): Let $\mathcal{E} = \{u_t = f(x, u, u_x, \ldots, u_k)\}$ be a system of evolutionary partial differential equations upon the unknowns $u(x, t)$ in two variables. Let $\mathcal{E}_\varepsilon = \{\tilde{u}_t = f_\varepsilon(x, \tilde{u}, \tilde{u}_x, \ldots, \tilde{u}_{\tilde{k}}; \varepsilon) \mid f_\varepsilon \in \operatorname{im} \tfrac{\mathrm{d}}{\mathrm{d}x}\}$ be a deformation of $\mathcal{E}$ such that at each point $\varepsilon \in \mathcal{I} \subseteq \mathbb{R}$ there exists the Miura contraction $\mathfrak{m}_\varepsilon = \{u = u(\tilde{u}, \tilde{u}_x, \ldots; \varepsilon)\} : \mathcal{E}_\varepsilon \to \mathcal{E}$. Then the pair $(\mathcal{E}_\varepsilon, \mathfrak{m}_\varepsilon)$ is the *Gardner deformation* for the system $\mathcal{E}$.

One obtains the recurrence relations between the conserved densities $\tilde{u}_n(x, u, u_x, \ldots)$ for $\mathcal{E}$ using the contraction $\mathfrak{m}_\varepsilon$ and the expansion $\tilde{u} = \sum_{n=0}^{+\infty} \tilde{u}_n \varepsilon^n$ of the deformed unknowns $\tilde{u}$ in ε.

In the recent paper[9] we understood Gardner's deformations in the extended sense, namely, in terms of coverings over PDE and diagrams of coverings. The zero-curvature representations and

Gardner's deformations can be considered as such geometric structures[16] that obey some extra conditions.

Definition 3 (Ref. 12): Let $\mathcal{E}$ be a differential equation that admits the nonempty infinite prolongation $\mathcal{E}^\infty$. A *covering* over the equation $\mathcal{E}$ is another (usually, larger) system of partial differential equations $\tilde{\mathcal{E}}$ endowed with the n-dimensional Cartan distribution $\tilde{\mathcal{C}}$ and such that there is a mapping $\tau : \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ for which, at each point $\theta \in \tilde{\mathcal{E}}$ the tangent map $\tau_{*,\theta}$ is an isomorphism of the plane $\tilde{\mathcal{C}}_\theta$ to the Cartan plane $\mathcal{C}_{\tau(\theta)}$ at the point $\tau(\theta)$ in $\mathcal{E}^\infty$.

The construction of a covering over $\mathcal{E}$ means the introduction of new variables such that their compatibility conditions lie inside the initial system $\mathcal{E}^\infty$. In practice (see Ref. 13), it is the rules to differentiate the new variable which are specified in a consistent way; this implies that those new variables acquire the nature of nonlocalities if their derivatives are local but the variables themselves are not (e.g., consider the potential $\mathfrak{v} = \int u\,dx$ satisfying $\mathfrak{v}_x = u$ and $\mathfrak{v}_t = -u_{xx} - 3u^2$ for the KdV equation $u_t + u_{xxx} + 6uu_x = 0$). Whenever the covering is indeed realized as the fibre bundle $\tau : \tilde{\mathcal{E}} \to \mathcal{E}$, the forgetful map τ discards the nonlocalities.

In these terms, the zero-curvature representations and Gardner's deformations are coverings of special kinds (see Examples 2 and 4 below). In this paper we use the geometric similarity of the two notions and construct new Gardner's deformations from known zero-curvature representations (but this is *not always* possible).

Example 1 (A zero-curvature representation for the KdV equation): Consider the Korteweg–de Vries equation[1]

$$\mathcal{E}_{\mathrm{KdV}} = \{u_t + u_{xxx} + 6uu_x = 0\} \tag{3}$$

and its Lax representation[1,11,15]

$$\mathcal{L}_t = [\mathcal{L}, \mathcal{A}],$$

where

$$\mathcal{L} = \tfrac{\mathrm{d}^2}{\mathrm{d}x^2} + u, \quad \mathcal{A} = -4\tfrac{\mathrm{d}^3}{\mathrm{d}x^3} - 6u\tfrac{\mathrm{d}}{\mathrm{d}x} - 3u_x. \tag{4}$$

The linear auxiliary problem[17] is

$$\psi_{xx} + u\psi = \lambda\psi,$$

$$-4\psi_{xxx} - 6u\psi_x - 3u_x\psi = \psi_t,$$

By definition, put $\psi_0 = \psi$ and $\psi_1 = \psi_x$. We obtain

$$\psi_{0;x} = \psi_1,$$

$$\psi_{1;x} = (\lambda - u)\psi_0,$$

$$\psi_{0;t} = -4\tfrac{\mathrm{d}}{\mathrm{d}x}((\lambda - u)\psi_0) - 6u\psi_1 - 3u_x\psi_0 = u_x\psi_0 + (-4\lambda - 2u)\psi_!,$$

$$\psi_{1;t} = (-4\lambda^2 + 2u\lambda + 2u^2 + u_x)\psi_0 + (-u_x)\psi_1.$$

We finally rewrite this system as two matrix equations,[17]

$$\underbrace{\begin{pmatrix}\psi_{0;x}\\ \psi_{1;x}\end{pmatrix}}_{\psi_x} = \underbrace{\begin{pmatrix}0 & 1\\ \lambda - u & 0\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}}_{\psi} \quad \underbrace{\begin{pmatrix}\psi_{0;t}\\ \psi_{1;t}\end{pmatrix}}_{\psi_t} = \underbrace{\begin{pmatrix}u_x & -4\lambda - 2u\\ -4\lambda^2 + 2u\lambda + 2u^2 + u_{xx} & -u_x\end{pmatrix}}_{B}\underbrace{\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}}_{\psi}.$$

This yields an $\mathfrak{sl}_2(\mathbb{C})$-valued zero-curvature representation $\alpha^{\mathrm{KdV}} = A\,dx + B\,dt$ for the KdV equation (3). The representation α^{KdV} was rediscovered in Ref. 18.

Example 2 (Zero-curvature representations as coverings): Let $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ as above. We introduce the standard basis e, h, f in $\mathfrak{g}$ such that

$$[e,h] = -2e, \quad [e,f] = h, \quad [f,h] = 2f.$$

We consider, simultaneously, the matrix representation

$$\rho:\ \mathfrak{sl}_2(\mathbb{C}) \to \{M \in \mathrm{Mat}(2,2) \mid \mathrm{tr}\, M = 0\}$$

of $\mathfrak{g}$ and its representation ϱ in the space of vector fields with polynomial coefficients on the complex line with the coordinate w:

$$\rho(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$\varrho(e) = 1\cdot\partial/\partial w, \quad \varrho(h) = -2w\cdot\partial/\partial w, \quad \varrho(f) = -w^2\cdot\partial/\partial w.$$

Let us decompose the matrices A_i (which occur in the zero-curvature representation $\alpha = \sum_i A_i dx^i$) with respect to the basis in the space $\rho(\mathfrak{g})$,

$$A_i = a_e^{(i)} \otimes \rho(e) + a_h^{(i)} \otimes \rho(h) + a_f^{(i)} \otimes \rho(f), \tag{5}$$

for $a_j^{(i)} \in C^\infty(\mathcal{E}^\infty)$.

To construct the covering $\tilde{\mathcal{E}}$ over $\mathcal{E}^\infty$ with a new fiber variable w over $\mathcal{E}^\infty$ (the "nonlocality"), we switch from the representation ρ to ϱ. We thus obtain the vector fields

$$V_{A_i} = a_e^{(i)} \otimes \varrho(e) + a_h^{(i)} \otimes \varrho(h) + a_f^{(i)} \otimes \varrho(f) \tag{5$'$}$$

such that the prolongations of the total derivatives $\frac{\mathrm{d}^i}{\mathrm{d}x^i}$ to $\tilde{\mathcal{E}}$ are defined by the formula

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}x^i} = \frac{\mathrm{d}}{\mathrm{d}x^i} - V_{A_i}. \tag{6}$$

The extended derivatives act on the nonlocal variable w as follows,

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}x^i} w = dw \lrcorner (-V_{A_i}).$$

Remark 1: The commutativity of the prolonged total derivatives, $\left[\frac{\tilde{\mathrm{d}}}{\mathrm{d}x^i}, \frac{\tilde{\mathrm{d}}}{\mathrm{d}x^j}\right] = 0$ with $i \neq j$, is equivalent to the Maurer–Cartan equation (2): Indeed, we have that

$$0 = [\tfrac{\tilde{\mathrm{d}}}{\mathrm{d}x^i}, \tfrac{\tilde{\mathrm{d}}}{\mathrm{d}x^j}] = [\tfrac{\mathrm{d}}{\mathrm{d}x^i} - V_{A_i}, \tfrac{\mathrm{d}}{\mathrm{d}x^j} - V_{A_j}] = [\tfrac{\mathrm{d}}{\mathrm{d}x^i}, \tfrac{\mathrm{d}}{\mathrm{d}x^j}] - [\tfrac{\mathrm{d}}{\mathrm{d}x^i}, V_{A_j}] - [V_{A_j}, \tfrac{\mathrm{d}}{\mathrm{d}x^j}] + [V_{A_i}, V_{A_j}] =$$

$$= -V_{\frac{\mathrm{d}}{\mathrm{d}x^i}A_i} + V_{\frac{\mathrm{d}}{\mathrm{d}x^j}A_i} + V_{[A_i,A_j]} = V_{\frac{\mathrm{d}}{\mathrm{d}x^j}A_i - \frac{\mathrm{d}}{\mathrm{d}x^i}A_j + [A_i,A_j]} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x^j}A_i - \frac{\mathrm{d}}{\mathrm{d}x^i}A_j + [A_i, A_j] = 0.$$

This motivates the choice of the minus sign in (6).

Example 3 (A one-dimensional covering over the KdV equation): One obtains the covering over the KdV equation from the zero-curvature representation α (see Example 1) by using representation (5$'$) in the space of vector fields. Applying (5$'$) to the matrices A, $B \in \mathfrak{sl}_2(\mathbb{C})$, we construct the following vector fields with the nonlocal variable w:

$$V_A = (1 - (\lambda - u)w^2)\cdot\partial/\partial w,$$

$$V_B = \left[(-4\lambda - 2u) - 2uw - (-4\lambda^2 + 2u\lambda + 2u^2 + u_{xx})w^2\right]\cdot\partial/\partial w.$$

The prolongations of the total derivatives act on w by the rules

$$w_x = -1 + (\lambda - u)w^2, \tag{7a}$$

$$w_t = -\left((-4\lambda - 2u) - 2u_x w - (-4\lambda^2 + 2u\lambda + 2u^2 + u_{xx})w^2\right). \tag{7b}$$

We thus obtain the one-dimensional covering over the KdV equation (3). It depends on the non-removable[12] spectral parameter λ. In what follows we show that this covering is equivalent to the covering (12) which is derived from Gardner's deformation (11) of the KdV equation (3).

B. The projective substitution and nonlinear representations of Lie algebras in the spaces of vector fields

Suppose $\mathfrak{g}$ is a finite-dimensional Lie algebra. We shall use the projective substitution[7] to construct a covering over the equation $\mathcal{E}$ starting from a $\mathfrak{g}$-valued zero-curvature representation for $\mathcal{E}$.

Let M be an m-dimensional manifold with local coordinates

$$v = (v^1, v^1, \dots, v^m) \in M, \text{ and put } \partial_v = (\partial_{v^1}, \partial_{v^2}, \dots, \partial_{v^m})^{\mathrm{t}}.$$

For any $g \in \mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{C})$, its representation V_g in the space of vector fields on M is given by the formula

$$V_g = vg\partial_v.$$

We note that V_g is linear in v^i. By construction, the representation preserves the commutation relations in the initial Lie algebra $\mathfrak{g}$:

$$[V_g, V_f] = [vg\partial_v, vf\partial_v] = v[g, f]\partial_v = V_{[g,f]}, \quad f, g \in \mathfrak{g}.$$

At all points of M where $v_1 \neq 0$ we consider the projection

$$\pi : v^i \to w^i = \mu v^i / v^1, \quad \mu \in \mathbb{R} \tag{8}$$

and its differential $d\pi : \partial_v \to \partial_w$. The transformation π yields the new coordinates on the open subset of M where $v^1 \neq 0$ and on the corresponding subset of TM:

$$w = (\mu, w^2, \dots, w^m), \quad \partial_w = (-\frac{1}{\mu}\sum_{i=2}^{m} w^i \partial_{w^i}, \partial_{w^2}, \dots, \partial_{w^m})^{\mathrm{t}}.$$

Consider the vector field $W_g = d\pi(V_g)$. In coordinates, we have

$$W_g = wg\partial_w. \tag{9}$$

We note that, generally, W_g is nonlinear with respect to w^i. The commutation relations between the vector fields of such type are also inherited from the relations in the Lie algebra $\mathfrak{g}$:

$$[W_g, W_f] = [d\pi(V_g), d\pi(V_f)] = d\pi([g, f]) = d\pi(V_{[g,f]}) = W_{[g,f]}.$$

Using representation (9) for the matrices A and B that determine the zero-curvature representation $\alpha^{\mathrm{KdV}} = A dx + B dt$ for the KdV equation, we obtain their realizations in terms of the vector fields:

$$W_A = \frac{1}{\mu}(-\lambda w^2 + \mu^2 + u w^2)\,\partial/\partial w,$$

$$W_B = \frac{1}{\mu}(-u_{xx} w^2 - 2u_x \mu w + 4\lambda^2 w^2 - 4\lambda\mu^2 - 2\lambda u w^2 - 2\mu^2 u - 2u^2 w^2)\,\partial/\partial w.$$

Therefore, the prolongations of the total derivatives act on the nonlocality w as follows:

$$w_x = -\frac{1}{\mu}(-\lambda w^2 + \mu^2 + u w^2), \tag{10a}$$

$$w_t = -\frac{1}{\mu}(-u_{xx} w^2 - 2u_x \mu w + 4\lambda^2 w^2 - 4\lambda\mu^2 - 2\lambda u w^2 - 2\mu^2 u - 2u^2 w^2). \tag{10b}$$

The parameter μ is removable by the transformation $w \to \mu w$, which rescales it to unit. Applying this transformation to (10), we reproduce the covering (7).

Example 4 (A covering which is based on Gardner's deformation): Consider the Gardner deformation[1] of the KdV equation (3),

$$\mathcal{E}_\varepsilon = \left\{\tilde{u}_t = -(\tilde{u}_{xx} + 3\tilde{u}^2 - 2\varepsilon^2\tilde{u}^3)_x\right\}, \tag{11a}$$

$$\mathfrak{m}_\varepsilon = \left\{u = \tilde{u} - \varepsilon\tilde{u}_x - \varepsilon^2\tilde{u}^2\right\} : \mathcal{E}_\varepsilon \to \mathcal{E}_0. \tag{11b}$$

Expressing $\tilde{u}_x$ from (11b) and substituting it in (11a), we obtain the one-dimensional covering over the KdV equation,

$$\tilde{u}_x = \frac{1}{\varepsilon}(\tilde{u} - u) - \varepsilon\tilde{u}^2, \tag{12a}$$

$$\tilde{u}_t = \frac{1}{\varepsilon}(u_{xx} + 2u^2) + \frac{1}{\varepsilon^2}u_x + \frac{1}{\varepsilon^3}u + \left(-2u_x - \frac{2}{\varepsilon}u - \frac{1}{\varepsilon^3}\right)\tilde{u} + \left(2\varepsilon u + \frac{1}{\varepsilon}\right)\tilde{u}^2, \tag{12b}$$

We claim that covering (12) is equivalent to the covering that was obtained in p. 277 of Ref. 12 for the KdV equation. To prove this, we first put $\tilde{u} = -\tilde{v}/\varepsilon$. We have

$$-\frac{\tilde{v}}{\varepsilon} = -\frac{1}{\varepsilon^2}\tilde{v} - \frac{1}{\varepsilon}u - \frac{1}{\varepsilon}\tilde{v}^2,$$

in other words

$$\tilde{v}_x = u + \left(\tilde{v} + \frac{1}{2\varepsilon}\right)^2 - \frac{1}{4\varepsilon^2}.$$

Next, we put $p = \tilde{v} + 1/(2\varepsilon)$, whence we obtain

$$p_x = u + p^2 - \frac{1}{4\varepsilon^2}, \tag{13a}$$

$$p_t = -u_{xx} - 2u^2 - \frac{1}{2\varepsilon^2}u + \frac{1}{4\varepsilon^4} - 2u_x p - (2u + \frac{1}{\varepsilon^2})p^2. \tag{13b}$$

Dividing (7) by w^2, we conclude that

$$w_x = -1 + (\lambda - u)w^2,$$

$$\frac{w_x}{w^2} = -\frac{1}{w^2} - u + \lambda.$$

On the other hand, we put $p = 1/w$, whence $p_x = -w_x/w^2$, and set $\lambda = 1/(4\varepsilon^2)$. This brings (7) to the same notation as in formulas (13),

$$p_x = u + p^2 - \lambda,$$

$$p_t = -u_{xx} - 2u^2 - 2\lambda u + 4\lambda^2 - 2u_x p - (2u + 4\lambda)p^2.$$

The corresponding one-form of the zero-curvature representation for the KdV equation is equal to

$$\alpha_2^{\mathrm{KdV}} = \begin{pmatrix} 0 & \lambda - u \\ 1 & 0 \end{pmatrix} dx + \begin{pmatrix} -u_x & -4\lambda^2 + 2\lambda u + 2u^2 + u_{xx} \\ -4\lambda - 2u & u_x \end{pmatrix} dt. \tag{14}$$

In Sec. II C we show that this zero-curvature representation is also equivalent to α^{KdV} from Example 1.

C. Gauge transformations

Let G be the Lie group of the Lie algebra $\mathfrak{g}$ (so that $G = SL_2(\mathbb{C})$ in the previous example). Given an equation $\mathcal{E}$, for any zero-curvature representation α there exists the zero-curvature representation α^S such that

$$\alpha^S = \bar{d}S \cdot S^{-1} + S \cdot \alpha \cdot S^{-1}, \quad S \in C^\infty(\mathcal{E}^\infty) \otimes G. \tag{15}$$

The zero-curvature representation α^S is called *gauge-equivalent* to α and S is the *gauge transformation*. Suppose $\alpha = A_i\,dx^i$. The gauge transformation S acts on the components A_i of α as follows

$$A_i^S = \tfrac{\mathrm{d}}{\mathrm{d}x^i}(S)S^{-1} + SA_iS^{-1}. \tag{15$'$}$$

Example 5 (The relation between the coverings which stem from gauge equivalent zero curvature representations): Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $G = SL_2(\mathbb{C})$. Suppose $S \in SL_2(\mathbb{C})$, so that

$$S = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}, \quad \det S = 1.$$

Let $\alpha = \sum_i A_i dx^i$ be a zero-curvature representation for an equation $\mathcal{E}$. Using decomposition (5) for $A_i \in \mathfrak{sl}_2(\mathbb{C})$, we inspect how the gauge transformation S acts on the components of α:

$$A_i^S = \tfrac{\mathrm{d}}{\mathrm{d}x^i}(S)S^{-1} + S(a_e^{(i)} \otimes \rho(e) + a_h^{(i)} \otimes \rho(h) a_e^{(i)} \otimes \rho(f))S^{-1} =$$
$$= \tfrac{\mathrm{d}}{\mathrm{d}x^i}(S)S^{-1} + a_e^{(i)} \otimes (S \cdot \rho(e) \cdot S^{-1}) + a_h^{(i)} \otimes (S \cdot \rho(h) \cdot S^{-1}) + a_e^{(i)} \otimes (S \cdot \rho(f) \cdot S^{-1}).$$

We have that

$$\tfrac{\mathrm{d}}{\mathrm{d}x^i}(S)S^{-1} = \begin{pmatrix} s_{1;i}s_4 - s_{2;i}s_3 & s_{2;i}s_1 - s_{1;i}s_2 \\ s_{3;i}s_4 - s_{4;i}s_3 & s_{4;i}s_1 - s_{3;i}s_2 \end{pmatrix} = \begin{pmatrix} s_{1;i}s_4 - s_{2;i}s_3 & s_{2;i}s_1 - s_{1;i}s_2 \\ s_{3;i}s_4 - s_{4;i}s_3 & -s_{1;i}s_4 + s_{2;i}s_3 \end{pmatrix} =$$
$$= (s_{2;i}s_1 - s_{1;i}s_2)\rho(e) + (s_{1;i}s_4 - s_{2;i}s_3)\rho(h) + (s_{3;i}s_4 - s_{4;i}s_3)\rho(f),$$

$$S \cdot \rho(e) \cdot S^{-1} = \begin{pmatrix} -s_1s_3 & s_1^2 \\ -s_3^2 & s_1s_3 \end{pmatrix} = (s_1^2)\rho(e) + (-s_1s_3)\rho(h) + (-s_3^2)\rho(f),$$

$$S \cdot \rho(h) \cdot S^{-1} = \begin{pmatrix} s_1s_4 + s_2s_3 & -2s_1s_2 \\ 2s_3s_4 & -s_1s_4 - s_2s_3 \end{pmatrix} = (-2s_1s_2)\rho(e) + (s_1s_4 + s_2s_3)\rho(h) + (2s_3s_4)\rho(f),$$

$$S \cdot \rho(f) \cdot S^{-1} = \begin{pmatrix} s_2s_4 & -s_2^2 \\ s_4^2 & -s_2s_4 \end{pmatrix} = (-s_2^2)\rho(e) + (s_2s_4)\rho(h) + (s_4^2)\rho(f).$$

We finally obtain

$$A_i^S = (s_{2;i}s_1 - s_{1;i}s_2 + s_1^2a_e^{(i)} - 2s_1s_2a_h^{(i)} - s_2^2a_f^{(i)}) \otimes \rho(e) +$$
$$+ (s_{1;i}s_4 - s_{2;i}s_3 - s_1s_3a_e^{(i)} + (s_1s_4 + s_2s_3)a_h^{(i)} + s_2s_4a_f^{(i)}) \otimes \rho(h) +$$
$$+ (s_{3;i}s_4 - s_{4;i}s_3 - s_3^2a_e^{(i)} + 2s_3s_4a_h^{(i)} + s_4^2a_f^{(i)}) \otimes \rho(f).$$

Passing to the vector field representation of A_i^S by using formula (5$'$), we have

$$V_{A_i^S} = (s_{2;i}s_1 - s_{1;i}s_2 + s_1^2a_e^{(i)} - 2s_1s_2a_h^{(i)} - s_2^2a_f^{(i)}) \otimes \varrho(e) +$$
$$+ (s_{1;i}s_4 - s_{2;i}s_3 - s_1s_3a_e^{(i)} + (s_1s_4 + s_2s_3)a_h^{(i)} + s_2s_4a_f^{(i)}) \otimes \varrho(h) +$$
$$+ (s_{3;i}s_4 - s_{4;i}s_3 - s_3^2a_e^{(i)} + 2s_3s_4a_h^{(i)} + s_4^2a_f^{(i)}) \otimes \varrho(f). \tag{16}$$

In other words, whenever we start from the covering of $\mathcal{E}$ associated with a zero-curvature representation α, such that the differentiation rules for the nonlocality w are

$$\tfrac{\mathrm{d}}{\mathrm{d}x^i}(w) = -a_e^{(i)} + 2a_h^{(i)}w + a_f^{(i)}w^2,$$

we obtain the covering which is associated with α^S:

$$\tfrac{\mathrm{d}}{\mathrm{d}x^i}(w_S) = -(s_{2;i}s_1 - s_{1;i}s_2 + s_1^2 a_e^{(i)} - 2s_1s_2a_h^{(i)} - s_2^2 a_f^{(i)})+ \\ + 2(s_{1;i}s_4 - s_{2;i}s_3 - s_1s_3a_e^{(i)} + (s_1s_4 + s_2s_3)a_h^{(i)} + s_2s_4a_f^{(i)})w_S+ \\ + (s_{3;i}s_4 - s_{4;i}s_3 - s_3^2a_e^{(i)} + 2s_3s_4a_h^{(i)} + s_4^2a_f^{(i)})w_S^2. \quad (17)$$

We shall use this relation between the two coverings in the search of the gauge transformations between known zero-curvature representations for the KdV equation.

Example 6 (Gauge transformations between zero-curvature representations for the KdV equation): Let us find the gauge transformations that bring coverings (7) and (12) to the form (13).

For the transformation (7)→(13) we have

$$p_x = u + p^2 - \lambda = -(s_{2;x}s_1 - s_{1;x}s_2 + s_1^2 - s_2^2(\lambda - u)+ \\ - 2(s_{1;x}s_4 - s_{2;x}s_3 - s_1s_3 + s_2s_4(\lambda - u))p- \\ - (s_{3;x}s_4 - s_{4;x}s_3 - s_3^2 + s_4^2(\lambda - u))p^2).$$

Solving this equation for s_i, we find a unique solution $s_2 = s_3 = i$, $s_1 = s_4 = 0$:

$$S = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \quad (18)$$

The matrices of the zero curvature representations corresponding to the coverings (7) and (13) are related as follows:

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda - u \\ 1 & 0 \end{pmatrix}.$$

On the other hand, for the transformation (12)→(13) we have

$$p_x = u + p^2 - \frac{1}{4\varepsilon^2} = -(s_{2;x}s_1 - s_{1;x}s_2 - s_1^2\frac{u}{\varepsilon} + s_1s_2\frac{1}{\varepsilon} - s_2^2\varepsilon+ \\ - 2(s_{1;x}s_4 - s_{2;x}s_3 + s_1s_3\frac{u}{\varepsilon} - (s_1s_4 + s_2s_3)\frac{1}{2\varepsilon} + s_2s_4\varepsilon)p- \\ - (s_{3;x}s_4 - s_{4;x}s_3 + s_3^2\frac{u}{\varepsilon} - s_3s_4\frac{1}{\varepsilon} + s_4^2\varepsilon)p^2).$$

Solving this equation for s_i, we find a solution $s_1 = i/\sqrt{\varepsilon}$, $s_2 = i/(2\varepsilon\sqrt{\varepsilon})$, $s_3 = 0$, $s_4 = i\sqrt{\varepsilon}$. Therefore,

$$S = \begin{pmatrix} i/\sqrt{\varepsilon} & i/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & -i\sqrt{\varepsilon} \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -i\sqrt{\varepsilon} & -i/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & i/\sqrt{\varepsilon} \end{pmatrix}, \quad (19)$$

The matrices of the zero-curvature representations corresponding to coverings (12) and (13) satisfy the relation

$$\begin{pmatrix} i/\sqrt{\varepsilon} & i/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & -i\sqrt{\varepsilon} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{4\varepsilon^2} - u \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -i\sqrt{\varepsilon} & -i/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & i/\sqrt{\varepsilon} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\varepsilon} & \frac{u}{\varepsilon} \\ -\varepsilon & -\frac{1}{2\varepsilon} \end{pmatrix}.$$

Let us remember that in Example 1 we derived the zero-curvature representation for the KdV equation from its Lax pair. Having done that, we also revised the transition from this zero-curvature representation to the Gardner deformation of the KdV equation. In Sec. III we extend this approach and find the generalizations of Gardner's deformation (11) for graded systems, in particular, for the $N = 1$ and $N = 2$ supersymmetric Korteweg–de Vries equations.

III. GRADED SYSTEMS

A. Lie super-algebras

We recall first the definition of the Lie super-algebra.[14,19,20] Let $\mathcal{A}$ be an algebra over the field $\mathbb{C}$ and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ be the group of residues modulo 2. An algebra $\mathcal{A}$ is called a super-algebra if $\mathcal{A}$ can be decomposed as the direct sum $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ such that

$$\mathcal{A}_{\bar{0}} \cdot \mathcal{A}_{\bar{0}} \subset \mathcal{A}_{\bar{0}}, \quad \mathcal{A}_{\bar{0}} \cdot \mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{1}}, \quad \mathcal{A}_{\bar{1}} \cdot \mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{0}}.$$

A nonzero element of $\mathcal{A}_{\bar{0}}$ or $\mathcal{A}_{\bar{1}}$ is called *homogeneous* (respectively, even or odd). Let $\mathsf{p}(a) = k$ if $a \in \mathcal{A}_k$ for $k \in \mathbb{Z}_2$. The number $\mathsf{p}(a)$ is the parity of a.

The super-algebra $\mathfrak{g}$ is a *Lie super-algebra* if it is endowed with the linear multiplication $\underline{[}\cdot\,, \cdot\underline{]}$ that satisfies the equalities

$$\underline{[}x, y\underline{]} = -(-1)^{\mathsf{p}(x)\mathsf{p}(y)}\underline{[}y, x\underline{]}, \tag{20}$$

$$\underline{[}x, \underline{[}y, z\underline{]}\underline{]} = \underline{[}\underline{[}x, y\underline{]}, z\underline{]} + (-1)^{\mathsf{p}(x)\mathsf{p}(y)}\underline{[}y, \underline{[}x, z\underline{]}\underline{]}, \tag{21}$$

here x, y, and z are arbitrary elements of $\mathcal{A}$ and x, y are presumed homogeneous.

The super-matrix structure of a matrix is achieved whenever the parity is assigned to its rows and columns. We choose the super-matrix structure such that rows (respectively, columns) which are assigned the even parity always precede the rows (columns) of odd parity.[19] The parity of the matrix element is determined by the sum of the parity of its column and the parity of its row. If a matrix has r even and s odd rows and p even and q odd columns, then its dimension is said to be equal to $(r|s) \times (p|q)$. In particular, we shall use the shorthand notation $(p|q)$ for the dimension $(p|q) \times (p|q)$. We denote by $\mathrm{Mat}(p \mid q; \mathcal{A})$ the set of all matrices of dimension $(p|q)$ with elements that belong the super-algebra $\mathcal{A}$.

Let us introduce the super-matrix structure on the space $\mathrm{Mat}(p \mid q; \mathcal{A})$. Consider a matrix $X = \left(\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right) \in \mathrm{Mat}(p \mid q; \mathcal{A})$ and set

$$\mathsf{p}(X) = \bar{0} \quad \text{if } \mathsf{p}(R_{ij}) = \mathsf{p}(U_{ij}) = \bar{0}, \quad \mathsf{p}(T_{ij}) = \mathsf{p}(S_{ij}) = \bar{1};$$

$$\mathsf{p}(X) = \bar{1} \quad \text{if } \mathsf{p}(R_{ij}) = \mathsf{p}(U_{ij}) = \bar{1}, \quad \mathsf{p}(T_{ij}) = \mathsf{p}(S_{ij}) = \bar{0}.$$

Taking into account the graded skew-symmetry (20) of the bracket $\underline{[}\cdot, \cdot\underline{]}$, we define the Lie super-algebra structure on the space $\mathrm{Mat}(p \mid q; \mathcal{A})$ by the formula

$$\underline{[}X, Y\underline{]} = XY - (-1)^{\mathsf{p}(X)\mathsf{p}(Y)}YX, \quad X, Y \in \mathrm{Mat}(p \mid q; \mathcal{A}). \tag{22}$$

The Lie super-algebras $\mathfrak{gl}(m \mid n) \simeq \mathrm{Mat}(m \mid n, \mathbb{C})$ and $\mathfrak{sl}(m \mid n) = \{X \in \mathfrak{gl}(m \mid n) | \operatorname{str} X = 0\}$, where $\operatorname{str}\left(\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right) = \operatorname{tr} R - \operatorname{tr} U$, are called the general linear and special linear Lie super-algebras, respectively.

To calculate the super-commutator $\underline{[}X, Y\underline{]}$ of two nonhomogeneous elements X and Y, we first split $X = X_{\bar{0}} + X_{\bar{1}}$ and $Y = Y_{\bar{0}} + Y_{\bar{1}}$ so that $\mathsf{p}(X_{\bar{0}}) = \mathsf{p}(Y_{\bar{0}}) = \bar{0}$ and $\mathsf{p}(X_{\bar{1}}) = \mathsf{p}(Y_{\bar{1}}) = \bar{1}$. Using (22), we obtain

$$\begin{aligned}\underline{[}X, Y\underline{]} &= \underline{[}X_{\bar{0}} + X_{\bar{1}}, Y_{\bar{0}} + Y_{\bar{1}}\underline{]} = \underline{[}X_{\bar{0}}, X_{\bar{0}}\underline{]} + \underline{[}X_{\bar{0}}, Y_{\bar{1}}\underline{]} + \underline{[}X_{\bar{1}}, Y_{\bar{0}}\underline{]} + \underline{[}X_{\bar{1}}, Y_{\bar{1}}\underline{]} = \\ &= (X_{\bar{0}}Y_{\bar{0}} - Y_{\bar{0}}X_{\bar{0}}) + (X_{\bar{0}}Y_{\bar{1}} - Y_{\bar{1}}X_{\bar{0}}) + (X_{\bar{1}}Y_{\bar{0}} - Y_{\bar{0}}X_{\bar{1}}) + (X_{\bar{1}}Y_{\bar{1}} + Y_{\bar{1}}X_{\bar{1}}).\end{aligned} \tag{23}$$

The super-determinant, or the *Berezinian* of an invertible matrix $X = \left(\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right) \in \mathfrak{gl}(m \mid n)$ is given by the formula[14]

$$\operatorname{sdet}\begin{pmatrix} R & S \\ T & U \end{pmatrix} = \frac{\det(R - SU^{-1}T)}{\det U}.$$

Example 7: In what follows, we shall use the Lie super-algebra $\mathfrak{sl}(1 \mid 2) \simeq \mathfrak{sl}(2 \mid 1)$, see Ref. 21. Its representation in the space $\mathrm{Mat}(2 \mid 1; \mathbb{C})$ is given by the eight basic vectors, four even: E^+, E^-, H, and Z, and four odd: F^+, F^-, $\bar{F}^+$, and $\bar{F}^-$, where

$$E^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad F^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \bar{F}^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \bar{F}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The elements of the basis satisfy the following commutation relations:

$$\begin{array}{lll}
[\![H, E^\pm]\!] = \pm E^\pm & [\![H, F^\pm]\!] = \pm\tfrac{1}{2} F^\pm & [\![H, \bar{F}^\pm]\!] = \pm\tfrac{1}{2}\bar{F}^\pm \\
[\![Z, H]\!] = [\![Z, E^\pm]\!] = 0 & [\![Z, F^\pm]\!] = \tfrac{1}{2} F^\pm & [\![Z, \bar{F}^\pm]\!] = -\tfrac{1}{2}\bar{F}^\pm \\
[\![E^\pm, F^\pm]\!] = [\![E^\pm, \bar{F}^\pm]\!] = 0 & [\![E^\pm, F^\mp]\!] = -F^\pm & [\![E^\pm, \bar{F}^\mp]\!] = \bar{F}^\pm \\
[\![F^\pm, F^\pm]\!] = [\![\bar{F}^\pm, \bar{F}^\pm]\!] = 0 & [\![F^\pm, F^\mp]\!] = [\![\bar{F}^\pm, \bar{F}^\mp]\!] = 0 & [\![F^\pm, \bar{F}^\pm]\!] = E^\pm \\
[\![E^+, E^-]\!] = 2H & [\![F^\pm, \bar{F}^\mp]\!] = Z \mp H. &
\end{array}$$

The Lie super-algebra $\mathfrak{sl}(2 \mid 1)$ contains the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ as a subalgebra. The vectors $E^\pm$ and H form a basis in $\mathfrak{sl}(2, \mathbb{C})$.

The Lie super-group $SL(2|1)$, which corresponds to the Lie super-algebra $\mathfrak{sl}(2 \mid 1)$, consists of the matrices with unit Berezinian: $SL(2 \mid 1) = \{S \in GL(2 \mid 1) \mid \operatorname{sdet} S = 1\}$.

Remark 2: Consider the following three subgroups of the Lie super-group $SL(2|1)$:

$$G_+ = \left\{ \begin{pmatrix} \mathbf{1} & B \\ 0 & 1 \end{pmatrix} \right\}, \quad G_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad G_- = \left\{ \begin{pmatrix} \mathbf{1} & 0 \\ C & 1 \end{pmatrix} \right\}.$$

Each matrix $S \in SL(2|1)$ can be represented[22] as a product $S = S_+ S_0 S_-$, where $S_+ \in G_+$, $S_0 \in G_0$, $S_- \in G_-$. Due to the multiplicativity of the Berezinian, $\operatorname{sdet} S = \operatorname{sdet} S_+ \cdot \operatorname{sdet} S_0 \cdot \operatorname{sdet} S_- = 1$, and in view of the obvious property $\operatorname{sdet} S_+ = \operatorname{sdet} S_- = 1$ for all elements of the groups G_+ and G_-, we conclude that $\operatorname{sdet} S_0 = 1$ for all $S_0 \in G_0$.

For the Lie super-group $SL(2|1)$, the dimension of the matrix D is equal to 1×1 and the dimension of the matrix A is equal to 2×2. Let us show that $G_0 \simeq GL(2|0)$. The condition $\operatorname{sdet} S_0 = 1$ for the matrix $S_0 \in SL(2|1)$ implies the equality $\det A = \det D$ of the usual determinants of A and D. Therefore, to each matrix $A \in GL(2|0)$ we can put into correspondence the matrix $S_A \in G_0$ by setting $S_A = \left(\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & \det A \end{smallmatrix}\right)$ and conversely, to each matrix $S = \left(\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & D \end{smallmatrix}\right) \in G_0$ we associate the matrix A from $GL(2|0)$.

B. Zero-curvature representations of graded extension of the KdV equation

The graded extension of the Maurer–Cartan equation (2) has the form

$$\tfrac{\mathrm{d}}{\mathrm{d}x^j} A_i - \tfrac{\mathrm{d}}{\mathrm{d}x^i} A_j + [\![A_i, A_j]\!] = 0, \quad \forall i, j = 1, \ldots, m : i \neq j. \tag{24}$$

Let us study in more detail the geometry of the $N = 1$ and $N = 2$ supersymmetry-invariant generalizations of the Korteweg–de Vries equation.[2,23]

1. N = 1 supersymmetric Korteweg–de Vries equation

The $N = 1$ supersymmetric generalization of the KdV equation (3) is the sKdV equation[23]

$$\phi_t = -\phi_{xxx} - 3(\phi \mathcal{D}\phi)_x, \quad \mathcal{D} = \frac{\partial}{\partial\theta} + \theta\frac{d}{dx}, \tag{25}$$

where $\phi(x, t, \theta) = \xi + \theta u$ is a complex fermionic super-field, θ is the Grassmann (or anti-commuting) variable such that $\theta^2 = 0$, the unknown u is the bosonic field, and ξ is the fermionic field. By using the expansion $\phi(x, t, \theta) = \xi + \theta u$ in (25), we obtain

$$\underline{u_t} = \underline{-u_{xxx} - 6uu_x} + 3\xi\xi_{xx}, \tag{26a}$$

$$\xi_t = -\xi_{xxx} - 3(u\xi)_x. \tag{26b}$$

The KdV equation (3) is underlined in (26a).

Example 8 (Zero-curvature representation and Gardner's deformation of the sKdV equation): We claim that the sKdV equation (26) admits the $\mathfrak{sl}(2 \mid 1)$-valued zero-curvature representation

$$\alpha^{N=1} = A_1^{N=1} dx + B_1^{N=1} dt,$$

where

$$A_1^{N=1} = \begin{pmatrix} -\frac{1}{2\varepsilon} & -u + \frac{1}{4\varepsilon^2} & \xi \\ 1 & -\frac{1}{2\varepsilon} & 0 \\ 0 & -\xi & -\frac{1}{\varepsilon} \end{pmatrix},$$

$$B_1^{N=1} = \begin{pmatrix} \frac{1}{2}\varepsilon^{-3} - u_x & 2u^2 + u_{xx} - \xi\xi_x + \frac{1}{2}\varepsilon^{-2}u - \frac{1}{4}\varepsilon^{-4} & -\xi_{xx} - 2\xi u - \frac{1}{2}\varepsilon^{-1}\xi_x - \frac{1}{2}\varepsilon^{-2}\xi \\ -2u - \varepsilon^{-2} & \frac{1}{2}\varepsilon^{-3} + u_x & -\xi_x - \xi\varepsilon^{-1} \\ -\xi_x + \xi\varepsilon^{-1} & \xi_{xx} + 2\xi u - \frac{1}{2}\varepsilon^{-1}\xi_x + \frac{1}{2}\varepsilon^{-2}\xi & \varepsilon^{-3} \end{pmatrix}.$$

Let us construct the generalization $S^{N=1} \in SL(2|1)$ of gauge transformation (19) where we had $S \in SL_2(\mathbb{C}) \simeq SL(2 \mid 0)$. Taking into account Remark 2, we consider the ansatz $S^{N=1} = S_+^{N=1} S_0^{N=1} S_-^{N=1}$, where $S_\nu \in G_\nu$, $\nu \in \{+, 0, -\}$. Bearing in mind that $SL_2(\mathbb{C}) \simeq SL(2 \mid 0) \subset GL(2 \mid 0)$, we construct S by using the following scheme:

(1) we obtain an element $S_0^{N=1}$ by the multiplication of S from right and left by some matrices from $GL(2|0)$;
(2) we specify the matrices $S_+^{N=1}$ and $S_-^{N=1}$.

We construct the matrix $S^{N=1}$ as follows

$$S^{N=1} = \begin{pmatrix} -1 & -\frac{1}{2}\varepsilon^{-1} & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S_+^{N=1}} \underbrace{\begin{pmatrix} i\sqrt{\varepsilon} & i\sqrt{\varepsilon}/\varepsilon^2 & 0 \\ 0 & i\sqrt{\varepsilon} & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix} \overbrace{\begin{pmatrix} i/\sqrt{\varepsilon} & i/(2\varepsilon\sqrt{\varepsilon}) & 0 \\ 0 & -i\sqrt{\varepsilon} & 0 \\ 1 & 0 & 1 \end{pmatrix}}^{S} \begin{pmatrix} 1 & \varepsilon^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S_0^{N=1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S_-^{N=1}}. \tag{27}$$

By applying the gauge transformation $S^{N=1}$ to the zero-curvature representation $\alpha^{N=1}$, we obtain the gauge-equivalent zero-curvature representation β for the sKdV equation (26):

$$\beta^{N=1} = (\alpha^{N=1})^{S^{N=1}} = A_2^{N=1} dx + B_2^{N=1} dt, \tag{28}$$

where

$$A_2^{N=1} = \begin{pmatrix} 0 & \varepsilon^{-1}u & \varepsilon^{-1}\xi \\ -\varepsilon & -\varepsilon^{-1} & 0 \\ 0 & \xi & -\varepsilon^{-1} \end{pmatrix},$$

$$B_2^{N=1} = \begin{pmatrix} u_x - u\varepsilon^{-1} & \frac{1}{\varepsilon}(-2u^2 - u_{xx} + \xi\xi_x) - \frac{1}{\varepsilon^2}u_x - \frac{1}{\varepsilon^3}u & \frac{1}{\varepsilon}(-\xi_{xx} - 2\xi u) - \frac{1}{\varepsilon^2}\xi_x - \frac{1}{\varepsilon^3}\xi \\ 2u\varepsilon + \varepsilon^{-1} & u_x + u\varepsilon^{-1} + \varepsilon^{-3} & \xi_x + \xi\varepsilon^{-1} \\ -\xi_{;x}\varepsilon + \xi & -\xi_{xx} - 2\xi u & \varepsilon^{-3} \end{pmatrix}.$$

Let us recall that formula (9) yields the representation of the matrices $A_2^{N=1}$ and $B_2^{N=1}$ in terms of vector fields. By this argument, from the zero-curvature representation $\beta^{N=1}$ we obtain the two-dimensional covering over the sKdV equation (26); one of the two new nonlocal variables is bosonic (let us denote it by $\tilde{u}$) and the other, $\tilde{\xi}$ is fermionic:

$$\begin{aligned}
\tilde{u}_x &= -\,\tilde{u}^2\varepsilon + (\tilde{u} - u)\varepsilon^{-1} - \tilde{\xi}\xi,\\
\tilde{\xi}_x &= -\,\tilde{\xi}\tilde{u}\varepsilon + (\tilde{\xi} - \xi)\varepsilon^{-1},\\
\tilde{u}_t &= \frac{1}{\varepsilon^3}(2\tilde{u}^2u\varepsilon^4 + \tilde{u}^2\varepsilon^2 - 2\tilde{u}u\varepsilon^2 - 2\tilde{u}u_x\varepsilon^3 - \tilde{u} + 2u^2\varepsilon^2 + u + u_{xx}\varepsilon^2 + u_x\varepsilon - \tilde{\xi}\tilde{u}\xi_x\varepsilon^4 +\\
&\quad + \tilde{\xi}\xi_{xx}\varepsilon^3 + \tilde{\xi}\xi\tilde{u}\varepsilon^3 + 2\tilde{\xi}\xi u\varepsilon^3 - \xi\xi_x\varepsilon^2),\\
\tilde{\xi}_t &= \frac{1}{\varepsilon^3}(-\tilde{u}\xi_x\varepsilon^3 + \xi_{xx}\varepsilon^2 + \xi_x\varepsilon + 2\tilde{\xi}\tilde{u}u\varepsilon^4 + \tilde{\xi}\tilde{u}\varepsilon^2 - \tilde{\xi}u\varepsilon^2 - \tilde{\xi}u_x\varepsilon^3 - \tilde{\xi} - \xi\tilde{u}\varepsilon^2 + 2\xi u\varepsilon^2 + \xi).
\end{aligned}$$

We now express the local variables u and ξ from $\tilde{u}_x$ and $\tilde{\xi}_x$ and substitute them in $\tilde{u}_t$ and $\tilde{\xi}_t$. We thus obtain the Gardner deformation[3] of sKdV equation (26):

$$\mathcal{E}_\varepsilon = \Big\{\underline{\tilde{u}_t} = \underline{6\tilde{u}^2\tilde{u}_x\varepsilon^2 - 6\tilde{u}\tilde{u}_x - \tilde{u}_{xxx}} - 3\tilde{\xi}\tilde{u}\tilde{\xi}_{xx}\varepsilon^2 + 3\tilde{\xi}\tilde{\xi}xx - 3\tilde{\xi}\tilde{\xi}_x\tilde{u}_x\varepsilon^2,$$

$$\tilde{\xi}_t = 3\tilde{u}^2\tilde{\xi}_x\varepsilon^2 - 3\tilde{u}\tilde{\xi}_x - \tilde{\xi}_{xxx} + 3\tilde{\xi}\tilde{u}\tilde{u}_x\varepsilon^2 - 3\tilde{\xi}\tilde{u}_x\Big\},$$

$$\mathfrak{m}_\varepsilon = \Big\{\underline{u} = \underline{\tilde{u} - \varepsilon\tilde{u}_x} + \varepsilon^2(\tilde{\xi}\tilde{\xi}_x\varepsilon^2 \underline{- \tilde{u}^2}), \quad \xi = \tilde{\xi} - \varepsilon\tilde{\xi}_x - \varepsilon^2\tilde{\xi}\tilde{u}\Big\} : \mathcal{E}_\varepsilon \to \mathcal{E}_{\text{sKdV}}.$$

This deformation can also be obtained by using super-field formalism.[3] The original Gardner deformation (11) of the KdV equation (3) is underlined in the above formulas.

2. N = 2 supersymmetric Korteweg–de Vries equation

Let us consider the four-component generalization of the KdV equation (3), namely, the $N = 2$ supersymmetric Korteweg–de Vries equation (SKdV):[2]

$$\boldsymbol{u}_t = -\boldsymbol{u}_{xxx} + 3\big(\boldsymbol{u}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}\big)_x + \frac{a-1}{2}\big(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}^2\big)_x + 3a\boldsymbol{u}^2\boldsymbol{u}_x, \qquad \mathcal{D}_i = \frac{\partial}{\partial\theta_i} + \theta_i \cdot \frac{\mathrm{d}}{\mathrm{d}x}, \tag{29}$$

where

$$\boldsymbol{u}(x, t; \theta_1, \theta_2) = u_0(x, t) + \theta_1 \cdot u_1(x, t) + \theta_2 \cdot u_2(x, t) + \theta_1\theta_2 \cdot u_{12}(x, t) \tag{30}$$

is the complex bosonic super-field, θ_1, θ_2 are Grassmann variables such that $\theta_1^2 = \theta_2^2 = \theta_1\theta_2 + \theta_2\theta_1 = 0$, u_0, u_{12} are bosonic fields, and u_1, u_2 are fermionic fields. Expansion (30) converts (29) to the four-component system

$$u_{0;t} = -u_{0;xxx} + \big(au_0^3 - (a+2)u_0u_{12} + (a-1)u_1u_2\big)_x, \tag{31a}$$

$$u_{1;t} = -u_{1;xxx} + \big(\ (a+2)u_0u_{2;x} + (a-1)u_{0;x}u_2 - 3u_1u_{12} + 3au_0^2u_1\big)_x, \tag{31b}$$

$$u_{2;t} = -u_{2;xxx} + \big(-(a+2)u_0u_{1;x} - (a-1)u_{0;x}u_1 - 3u_2u_{12} + 3au_0^2u_2\big)_x, \tag{31c}$$

$$\underline{u_{12;t}} = \underline{-u_{12;xxx} - 6u_{12}u_{12;x}} + 3au_{0;x}u_{0;xx} + (a+2)u_0u_{0;xxx} + 3u_1u_{1;xx} + 3u_2u_{2;xx} + 3a\big(u_0^2u_{12} - 2u_0u_1u_2\big)_x. \tag{31d}$$

The KdV equation is underlined in (31d). The SKdV equation is most interesting (in particular, bi-Hamiltonian, whence completely integrable) if $a \in \{-2, 1, 4\}$, see Refs. 2, 4, and 24. Let us consider the bosonic limit $u_1 = u_2 = 0$ of system (31): by setting $a = -2$ we obtain the triangular system which consists of the modified KdV equation upon u_0 and the equation of KdV-type; in the case $a = 1$ we obtain the Krasil'shchik–Kersten system; for $a = 4$, we obtain the third equation in the Kaup–Boussinesq hierarchy. A Gardner deformation of the Kaup–Boussinesq system was constructed in Ref. 4.

The Gardner deformation problem for the $N = 2$ supersymmetric $a = 4$ KdV equation was formulated in Ref. 2. In the paper[4] it was shown that one cannot construct such a deformation under the assumptions that, first, the deformation is polynomial in $\mathcal{E}$, second, it involves only the super-fields but not their components, and third, it contains the known deformation (11) under the reduction $u_0 = 0$, $u_1 = u_2 = 0$. Therefore, we shall find a graded generalization of Gardner's deformation (11) for the system of four Eqs. (31) treating it in components but not as the single Eq. (29) upon the super-field.

The SKdV equation (31) admits[25] the $\mathfrak{sl}(2\mid 1)$-valued zero-curvature representation $\alpha^{N=2} = A\,dx + B\,dt$ such that

$$A = \begin{pmatrix} \eta - iu_0 & \eta^2 - 2i\eta u_0 - u_0^2 - u_{12} & -u_2 - iu_1 \\ 1 & \eta - iu_0 & 0 \\ 0 & u_2 - iu_1 & 2\eta - 2iu_0 \end{pmatrix}, \tag{32a}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \tag{32b}$$

where the elements of B are as follows:

$$b_{11} = -4\eta^3 - 2\eta u_{0;x}i - 4u_0^3 i + 6u_0u_{12}i + 4u_0u_{0;x} + u_{0;xx}i - u_{12;x} + 4u_2u_1 i,$$

$$b_{12} = -4\eta^4 + 4\eta^3u_0 i + 2\eta^2u_{12} - 4\eta u_0^3 i + 8\eta u_0u_{12}i + 2\eta u_{0;xx}i - 4u_0^4 - 2u_0^2u_{12} - {}$$
$$- 4u_0u_{0;xx} + 2u_{12}^2 - 4u_{0;x}^2 + u_{12;xx} - u_2u_{2;x} + 4u_2u_1\eta i - 8u_2u_1u_0 - u_1u_{1;x},$$

$$b_{13} = -\eta u_{2;x} - \eta u_{1;x}i - 5u_0u_{2;x}i + 5u_0u_{1;x} + u_{2;xx} + u_{1;xx}i + 2u_2\eta^2 + 2u_2\eta u_0 i - {}$$
$$- 8u_2u_0^2 + 2u_2u_{12} - 4u_2u_{0;x}i + 2u_1\eta^2 i - 2u_1\eta u_0 - 8u_1u_0^2 i + 2u_1u_{12}i + 4u_1u_{0;x},$$

$$b_{21} = 2(-2\eta^2 - 2\eta u_0 i + 2u_0^2 - u_{12}),$$

$$b_{22} = -4\eta^3 + 2\eta u_{0;x}i - 4u_0^3 i + 6u_0u_{12}i - 4u_0u_{0;x} + u_{0;xx}i + u_{12;x} + 4u_2u_1 i,$$

$$b_{23} = u_{2;x} + u_{1;x}i - 2u_2\eta - 4u_2u_0 i - 2u_1\eta i + 4u_1u_0,$$

$$b_{31} = u_{2;x} - u_{1;x}i + 2u_2\eta + 4u_2u_0 i - 2u_1\eta i + 4u_1u_0,$$

$$b_{32} = -\eta u_{2;x} + \eta u_{1;x} i - 5u_0 u_{2;x} i - 5u_0 u_{1;x} - u_{2;xx} + u_{1;xx} i - 2u_2\eta^2 - 2u_2\eta u_0 i + \\ + 8u_2 u_0^2 - 2u_2 u_{12} - 4u_2 u_{0;x} i + 2u_1\eta^2 i - 2u_1\eta u_0 - 8u_1 u_0^2 i + 2u_1 u_{12} i - 4u_1 u_{0;x},$$

$$b_{33} = 2(-4\eta^3 - 4u_0^3 i + 6u_0 u_{12} i + u_{0;xx} i + 4u_2 u_1 i).$$

We claim that the parameter $\eta \in \mathbb{C}$ is non-removable from A and B under gauge transformation.

Remark 3: Let us recall that the vectors Z, H, and $E^{\pm}$ that belong to $\mathfrak{sl}(2 \mid 1)$ generate a basis in $\mathfrak{gl}(2, \mathbb{C})$ (see the respective formulas on p. 22 in Example 7). We notice that the vector Z commutes with any other vector from $\mathfrak{gl}(2, \mathbb{C})$.

The reduction $u_0 = u_1 = u_2 = 0$ converts zero-curvature representation (32) to the $\mathfrak{gl}(2, \mathbb{C})$-valued zero-curvature representation of the KdV equation (3),

$$A_{\mathrm{KdV}} = \begin{pmatrix} \eta & \eta^2 - u_{12} & 0 \\ 1 & \eta & 0 \\ 0 & 0 & 2\eta \end{pmatrix},$$

$$B_{\mathrm{KdV}} = \begin{pmatrix} -4\eta^3 - u_{12;x} & -4\eta^4 + 2\eta^2 u_{12} + 2u_{12}^2 + u_{12;xx} & 0 \\ 2(-2\eta^2 - u_{12}) & -4\eta^3 + u_{12;x} & 0 \\ 0 & 0 & -8\eta^3 \end{pmatrix}.$$

Taking into account Remark 3, we obtain the $\mathfrak{sl}(2, \mathbb{C})$-valued zero-curvature representation (14) for the KdV equation (3) by omitting the summands $\eta \otimes Z dx$ and $-4\eta^3 \otimes Z dt$ in A_{KdV} and B_{KdV} and by denoting $\eta^2 = \lambda$.

Proposition 1: The $N = 2$ supersymmetric $a = 4$ Korteweg–de Vries equation (31) admits the $(1|1)$-dimensional $\mathbb{Z}_2$-graded covering, which is given in formulas (35) and (36) and which is such that, under the reduction $u_0 = u_1 = u_2 = 0$ of (31) to the KdV equation (3) and the consistent trivialization $f := 0$ in (35a) and (36a), see also (37), it reduces to the known Gardner deformation of (3) in the form of (12).

Proof: Let us extend the gauge transformation (19), which was determined by the element S of the Lie group $SL(2, \mathbb{C})$. We let

$$S^{N=2} = \begin{pmatrix} -1 & -\frac{1}{2}\varepsilon^{-1} & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix}. \tag{33}$$

Acting by gauge transformation (33) on zero-curvature representation (32), we obtain the graded zero-curvature representation that contains the "small" zero-curvature representation which, in turn, originates from (12) and is gauge-equivalent to (14) for the KdV equation (3). Specifically, we have that

$$A = \begin{pmatrix} iu_0 & \varepsilon^{-1}(u_0^2 + u_{12}) - i\varepsilon^{-2}u_0 & \varepsilon^{-1}(u_2 - iu_1) \\ -\varepsilon & iu_0 - \varepsilon^{-1} & 0 \\ 0 & u_2 + iu_1 & 2iu_0 - \varepsilon^{-1} \end{pmatrix}, \tag{34a}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \tag{34b}$$

where the elements of the matrix B are as follows,

$$b_{11} =4iu_0^3 - 6iu_0u_{12} + 4u_0u_{0;x} - iu_{0;xx} - u_{12;x} - 4iu_2u_1 + \varepsilon^{-1}(2u_0^2 - u_{12} - iu_{0;x}) - i\varepsilon^{-2}u_0,$$

$$b_{12} =\varepsilon^{-1}(4u_0^4 + 2u_0^2u_{12} + 4u_0u_{0;xx} - 2u_{12}^2 + 4u_{0;x}^2 - u_{12;xx} + u_2u_{2;x} + 8u_2u_1u_0 + u_1u_{1;x})+$$
$$+\varepsilon^{-2}(2iu_0^3 - 4iu_0u_{12} + 4u_0u_{0;x} - iu_{0;xx} - u_{12;x} - 2iu_2u_1) + \varepsilon^{-3}(u_0^2 - u_{12} - iu_{0;x})-$$
$$-i\varepsilon^{-4}u_0,$$

$$b_{13} =\varepsilon^{-1}(-5iu_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} + iu_{1;xx} + 8u_2u_0^2 - 2u_2u_{12} - 4iu_2u_{0;x} - 8iu_1u_0^2+$$
$$+2iu_1u_{12} - 4u_1u_{0;x}) + \varepsilon^{-2}(-u_{2;x} + iu_{1;x} - 3iu_2u_0 - 3u_1u_0) + \varepsilon^{-3}(-u_2 + iu_1),$$

$$b_{21} =2\varepsilon(-2u_0^2 + u_{12}) + 2iu_0 + \varepsilon^{-1},$$

$$b_{22} =4iu_0^3 - 6iu_0u_{12} - 4u_0u_{0;x} - iu_{0;xx} + u_{12;x} - 4iu_2u_1 + \varepsilon^{-1}(-2u_0^2 + u_{12} + iu_{0;x})+$$
$$+i\varepsilon^{-1}u_0 + \varepsilon^{-3},$$

$$b_{23} =u_{2;x} - iu_{1;x} + 4iu_2u_0 + 4u_1u_0 + \varepsilon^{-1}(u_2 - iu_1),$$

$$b_{31} =\varepsilon(-u_{2;x} - iu_{1;x} + 4iu_2u_0 - 4u_1u_0) + u_2 + iu_1,$$

$$b_{32} =5iu_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} - iu_{1;xx} + 8u_2u_0^2 - 2u_2u_{12} + 4iu_2u_{0;x} + 8iu_1u_0^2 - 2iu_1u_{12}-$$
$$-4u_1u_{0;x} + \varepsilon^{-1}u_0(iu_2 - u_1),$$

$$b_{33} =2(4iu_0^3 - 6iu_0u_{12} - iu_{0;xx} - 4iu_2u_1) + \varepsilon^{-3}.$$

The projective substitution (8) yields the two-dimensional covering over the $N = 2$, $a = 4$ SKdV equation. Under the reduction $u_0 = u_1 = u_2 = 0$ the covering contains (12), which is equivalent to Gardner's deformation (11) of the KdV equation (3). The x-components of the derivation rules for the nonlocalities w and f are

$$\underline{w_x} =\underline{-\varepsilon w^2 + \varepsilon^{-1}(w - u_{12})} - fu_2 - ifu_1 - \varepsilon^{-1}u_0^2 - \varepsilon^{-2}iu_0, \tag{35a}$$

$$f_x = -\varepsilon wf - iu_0f + \varepsilon^{-1}(f - u_2 + iu_1); \tag{35b}$$

here and in what follows we underline the covering (12) that encodes the "small" Gardner deformation for the KdV equation. The t-components of the "large" covering over the $N = 2$, $a = 4$ SKdV are

$$\underline{w_t} =\varepsilon(-4w^2u_0^2\underline{+2w^2u_{12}} - fwu_{2;x} - ifwu_{1;x} + 4ifu_2wu_0 - 4fu_1wu_0) + 2iw^2u_0+$$
$$+8wu_0u_{0;x}\underline{-2wu_{12;x}} - 5ifu_0u_{2;x} + 5fu_0u_{1;x} + fu_{2;xx} + ifu_{1;xx} + fu_2w - 8fu_2u_0^2+$$
$$+2fu_2u_{12} - 4ifu_2u_{0;x} + ifu_1w - 8ifu_1u_0^2 + 2ifu_1u_{12} + 4fu_1u_{0;x} + \varepsilon^{-1}(\underline{w^2} + 4wu_0^2-$$
$$\underline{-2wu_{12}} - 2iwu_{0;x} - 4u_0^4 - 2u_0^2u_{12} - 4u_0u_{0;xx}\underline{+2u_{12}^2} - 4u_{0;x}^2\underline{+u_{12;xx}} - ifu_2u_0+$$
$$+fu_1u_0 - u_2u_{2;x} - 8u_2u_1u_0 - u_1u_{1;x}) + \varepsilon^{-2}(-2iwu_0 - 2iu_0^3 + 4iu_0u_{12} - 4u_0u_{0;x}+$$
$$+iu_{0;xx}\underline{+u_{12;x}} + 2iu_2u_1) + \varepsilon^{-3}(\underline{-w} - u_0^2\underline{+u_{12}} + iu_{0;x}) + \varepsilon^{-4}iu_0, \tag{36a}$$

$$
\begin{aligned}
f_t =&2\varepsilon w(-2fu_0^2 + fu_{12}) + (-wu_{2;x} + iwu_{1;x} + 2ifwu_0 - 4ifu_0^3 + 6ifu_0u_{12} + 4fu_0u_{0;x} +\\
&+ ifu_{0;xx} - fu_{12;x} + 4ifu_2u_1 - 4iu_2wu_0 - 4u_1wu_0) + \varepsilon^{-1}(5iu_0u_{2;x} + 5u_0u_{1;x} + u_{2;xx} -\\
&- iu_{1;xx} + fw + 2fu_0^2 - fu_{12} - ifu_{0;x} - u_2w - 8u_2u_0^2 + 2u_2u_{12} + 4iu_2u_{0;x} + iu_1w +\\
&+ 8iu_1u_0^2 - 2iu_1u_{12} + 4u_1u_{0;x}) + \varepsilon^{-2}(u_{2;x} - iu_{1;x} - ifu_0 + 3iu_2u_0 + 3u_1u_0) +\\
&+ \varepsilon^{-3}(-f + u_2 - iu_1). \qquad (36b)
\end{aligned}
$$

It is noteworthy that the reduction $u_0 = u_1 = u_2 = 0$ in (31) eliminates the presence of the fermionic variables f in (35a) and (36a) so that there remains only (12) in the bosonic sector:

$$w_x = -\varepsilon w^2 + \varepsilon^{-1}(w - u_{12}), \qquad (37a)$$

$$w_t = 2\varepsilon w^2 u_{12} - 2wu_{12;x} + \varepsilon^{-1}(w^2 - 2wu_{12} + 2u_{12}^2 + u_{12;xx}), \qquad (37b)$$

$$f_x = -\varepsilon wf + \varepsilon^{-1} f, \qquad (37c)$$

$$f_t = 2\varepsilon wfu_{12} - fu_{12;x} + \varepsilon^{-1} f(u_{12} - w) - \varepsilon^{-3} f. \qquad (37d)$$

This proves our claim. □

In contrast with Gardner's deformation of $N = 1$ sKdV equation (see Example 8), covering (35) and (36), which we obtain for $N = 2$ supersymmetric $a = 2$ KdV equation, cannot be expressed in terms of the super-field.

We finally remark that the reduction $u_0 = 0$, $u_1 = 0$ (and the change of notation $u_2 \to \xi$, $u_{12} \to u$) maps this covering over the $N = 2$, $a = 4$ SKdV equation to the covering which was constructed in Example 8 for the $N = 1$ supersymmetric Korteweg–de Vries equation (25).

IV. CONCLUSION

By now the Gardner deformation problem for the $N = 2$ supersymmetric $a = 4$ Korteweg–de Vries equation (see Ref. 3) is solved. In this paper we have found the solution which is an alternative to our previous result in Ref. 4. Namely, we introduced the nonlocal bosonic and fermionic variables in such a way that the rules to differentiate them are consistent by virtue of the super-equation at hand and second, the entire system retracts to the standard KdV equation and the classical Gardner deformation for it[1] under setting to zero the fermionic nonlocal variable and the first three components of the $N = 2$ superfield in (29). At the same time, the structure under study is equivalent to the $\mathfrak{sl}(2 \mid 1)$-valued zero-curvature representation for this super-equation; the zero-curvature representation contains the non-removable spectral parameter, which manifests the integrability.

Our new solution of Mathieu's open problem 2 (see Ref. 3) relies on the interpretation of both Gardner's deformations and zero-curvature representations in similar terms, as a specific type of nonlocal structures over the equation of motion.[13] However, we emphasize that generally there is no one-to-one correspondence between the two constructions, so that the interpretation of deformations in the Lie-algebraic language is not always possible. Because this correlation between the two approaches to the integrability was not revealed in the canonical formulation of the deformation problem,[3] there appeared some attempts to solve it within the classical scheme but the progress was partial.[2,26] Still, the use of zero-curvature representations in this context could have given the sought deformation long ago.

Let us also notice that projective substitution (8) correlates the super-dimension of the Lie algebra in a zero-curvature representation for a differential equation with the numbers of bosonic and fermionic nonlocalities over the same system: a subalgebra of $\mathfrak{gl}(p \mid q)$ yields at most $p - 1$

bosonic and q fermionic variables. This implies that, for a covering over the $N = 2$ supersymmetric KdV equation (29) to extend the Gardner deformation (11) in its classical sense $\mathfrak{m}_\varepsilon : \mathcal{E}_\varepsilon \to \mathcal{E}$ (see Refs. 1, 2, and 9), the extension $\mathcal{E}_\varepsilon$ must be the system of evolution equations upon two bosonic and two fermionic fields. Therefore, one may have to use the $\mathfrak{sl}(3 \mid 2)$-valued zero-curvature representations. This outlines the working approach to a yet another method of solving the Gardner deformation problem for the $N = 2$ supersymmetric Korteweg–de Vries systems (29), which we leave as a new open problem.

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[1] C. S. Gardner, "Korteweg–de Vries equation and generalizations. I. A remarkable nonlinear substitution," J. Math. Phys. **9**, 1202–1204 (1968).
[2] P. Labelle and P. Mathieu, "A new supersymmetric Korteweg–de Vries equation," J. Math. Phys. **32**(4), 923–927 (1991).
[3] P. Mathieu, "Open problems for the super KdV equations," in *Bäcklund and Darboux Transformations. The Geometry of Solitons*, CRM Proceedings and Lecture Notes Vol. 29, edited by A. Coley, D. Levi, R. Milson, C. Rogers, and P. Winternitz (AMS, 2001), pp. 325–334; e-print arXiv:math-ph/0005007.
[4] V. Hussin, A. V. Kiselev, A. O. Krutov, and T. Wolf, "$N = 2$ supersymmetric $a = 4$-Korteweg–de Vries hierarchy derived via Gardner's deformation of Kaup–Boussinesq equation," J. Math. Phys. **51**(8), 083507 (2010); e-print arXiv:0911.2681 [nlin.SI].
[5] I. M. Gelfand and L. A. Dikiĭ, "Asymptotic properties of the resolvent of Sturm-Liouville equations, and the algebra of Korteweg-de Vries equations," Russ. Math. Surveys **30**(5), 77–113 (1975).
[6] G. Wilson, "On two constructions of conservation laws for Lax equations," Q. J. Math. Oxford Ser. **32**(128), 491–512 (1981).
[7] M. Roelofs, "Prolongation structures of supersymmetric systems," Ph.D. dissertation (University of Twente, 1993).
[8] I. S. Krasil'shchik and P. H. M. Kersten, *Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations* (Kluwer, Dordrecht, 2000).
[9] A. V. Kiselev, "Algebraic properties of Gardner's deformations for integrable systems," Theor. Math. Phys. **152**(1), 963–976 (2007); e-print arXiv:nlin.SI/0610072.
[10] M. Marvan, "On the horizontal gauge cohomology and non-removability of the spectral parameter," Acta Appl. Math. **72**, 51–65 (2002).
[11] T. Miva, M. Jimbo, and E. Date, *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras* (Cambridge University Press, Cambridge, 2000).
[12] A. V. Bocharov, A. M. Verbovetsky, A. M. Vinogradov *et al.*, in *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, edited by I. S. Krasil'shchik and A. M. Vinogradov (AMS, Providence, RI, 1999).
[13] A. V. Kiselev, "The twelve lectures in the (non)commutative geometry of differential equations," preprint IHÉS/M/12/13, (2012), p. 140.
[14] F. A. Berezin, "Introduction to superanalysis," *Mathematical Physics and Applied Mathematics* (Reidel, Dordrecht/Boston, MA, 1987), Vol. 9.
[15] L. D. Faddeev, and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer-Verlag, Berlin, 1987).
[16] Bäcklund (auto)transformations between PDE appear in the same context. In Ref. 9 we argued that the former, when regarded as the diagrams, are dual to the diagram description of Gardner's deformations.
[17] V. E. Zakharov, and A. B. Shabat, "Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II," Funct. Anal. Appl. **13**(3), 166–174 (1979).
[18] M. Marvan, "A direct procedure to compute zero-curvature representations. The case $\mathfrak{sl}_2$," in *Proceedings of a Conference on Secondary Calculus and Cohomological Physics*, *Moscow, 24–31 August, 1997* (Diffiety Institute of the Russian Academy of Natural Sciences, Moscow, 1997), p. 9; see http://www.emis.de/proceedings/SCCP97 for ELibEMS.
[19] D. A. Leites, "Lie superalgebras," J. Math. Sci. **30**(6), 2481–2512 (1985).
[20] Yu. I. Manin, "Holomorphic supergeometry and Yang–Mills superfields," J. Math. Sci. **30**(2), 1927–1975 (1985).
[21] L. Frappat, P. Sorba, and A. Sciarrino, "Dictionary on Lie superalgebras," preprint arXiv:hep-th/9607161 (1996), p. 146.
[22] D. A. Leites, "Introduction to the theory of supermanifolds," Russ. Math. Surveys **35**(1), 1–64 (1980).
[23] P. Mathieu, "Supersymmetric extension of the Korteweg–de Vries equation," J. Math. Phys. **29**(11), 2499–2506 (1988).
[24] A. V. Kiselev and V. Hussin, "Hirota's virtual multisoliton solutions of $N = 2$ supersymmetric Korteweg–de Vries equations," Theor. Math. Phys. **159**(3), 832–840 (2009); e-print arXiv:0810.0930 [nlin.SI].
[25] A. Das, W.-J. Huang, and S. Roy, "Zero curvature condition of $\mathfrak{osp}(2/2)$ and the associated supergravity theory," Int. J. Mod. Phys. **18**, 4293–4311 (1992).
[26] S. Andrea, A. Restuccia, and A. Sotomayor, "The Gardner category and nonlocal conservation laws for $N = 1$ super KdV," J. Math. Phys. **46**, 103517 (2005).