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ANOTHER INSTRUCTIVE EXAMPLE IN EIGENVALUE PERTURBATION THEORY

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Additional quantum mechanical counterexamples to the argument that vacuum instability implies divergence of perturbation theory are presented. They differ from previous examples by having the instability present for real values of the coupling.

In a recent letter [1], Herbst and Simon have given simple counterexamples in quantum mechanics to Dyson's heuristic argument [2] that instability implies divergence of perturbation theory. The hamiltonians discussed are

$$H^{(1)}(g) = p^2 - 1 + g^{-2}((gx + 1)^2 - 1)^2 - 2gx,$$

$$H^{(2)}(g) = p^2 - 1 + x^2(g^2x^2 + 1)^2 - 3g^2x^2,$$

where g is real. These have the property of becoming, in their words, formally unstable for suitable g . They have stated that the Rayleigh–Schrödinger series associated with the ground-state energy of $H^{(i)}$ is asymptotic and have then shown that the series converges, contrary to what one might expect on the basis of ref. [2]. In the case of $H^{(1)}$ the Rayleigh–Schrödinger series converges to the wrong answer for all real, non-vanishing values of g .

More recently, Calogero [3] has studied the hamiltonian

$$H^{(3)}(g) = p^2 - 1 + x^2(g^2x^2 - 1)^2 - 3g^2x^2.$$

This system is interesting because the well centered at $x = 0$ is unstable^{*1} for *real* g : there is tunneling to two degenerate, symmetrical, lower-lying wells centered at $\pm [2 + (1 + 9g^2)^{1/2}]^{1/2}/3^{1/2}g$.

He has shown that the exact ground-state energy is -2

instead of 0 for $g^2 \in D$, where $D = \{g^2 | 0 < |g| < \infty, |\arg g^2| < \pi/2\}$. Therefore the associated Rayleigh–Schrödinger series cannot be asymptotic to the ground-state energy of $H^{(3)}$, thereby making any discussion of its convergence properties irrelevant.

In this letter we will discuss the ground-state energy, $E(g)$, and the associated Rayleigh–Schrödinger series, $\sum a_n g^{2n}$, of the tunneling hamiltonian

$$H^{(4)}(g) = p^2 - 1 + x^2(g^2x^2 - (9/2)^{1/2}gx + 1)^2 + 18^{1/2}gx - 3g^2x^2.$$

This system is of interest because there is tunneling, for *real* g , between two degenerate, symmetrical wells centered at

$$x = 1/2^{1/2}g \pm [1/3 + (1 + 36g^2)^{1/2}/6]^{1/2}/g$$

(see fig. 1). We find that $E(g) \equiv 0$ for $g^2 \in D'$, where $D' = \{g^2 | 0 \leq |g| < \infty, |\arg g^2| < \pi/2\}$. As the potential in $H^{(4)}$ has three wells, the proof that $\sum a_n g^{2n}$ is asymptotic to $E(g)$ is not standard. The fact that $E(g) \equiv 0$ and our explicit verification that $a_1 = a_2 = a_3 = 0$ leads us to suspect that it is. If so, then the series $\sum a_n g^{2n}$ is convergent. The important difference between our $H^{(4)}$ and the hamiltonians $H^{(1)}$ and $H^{(2)}$ is that the instability is already present for real g .

The proof that $E(g) \equiv 0$ for $g^2 \in D'$ is very simple and follows the lines of the analysis of Herbst and Simon. We note that

$$H^{(4)} = [A(g)] * A(g),$$

where

^{*1} Instability here refers to the situation when the unperturbed harmonic oscillator well at $x = 0$ is not the absolute minimum of the potential in the associated hamiltonian.

Fig. 1. Sketch of the potential in $H^{(4)}$ for small, positive g .

$$A(g) = d/dx + x - (9/2)^{1/2}gx^2 + g^2x^3.$$

This implies $E(g) \geq 0$. The ground-state wave function, ψ , satisfies

$$A(g)\psi(g, x) = 0,$$

and has the normalizable solution

$$\psi = C \exp\left(-\frac{1}{2}x^2 + 2^{-1/2}gx^3 - \frac{1}{4}g^2x^4\right),$$

for $g^2 \in D'$. Hence $E(g) \equiv 0$.

It might be objected that the unperturbed well at $x = 0$ in $H^{(4)}$ is shifted to the left by $-(9/2)^{1/2}g + O(g^3)$ and downward by $-9g^2/2 + O(g^4)$ for small, positive g . To see that this makes no essential difference, consider the hamiltonian

$$H^{(5)}(g) = p^2 - 1 + x^2(g^2x^2 - (9/2)^{1/2}g|x| + 1)^2 + 18^{1/2}g|x| - 3g^2x^2.$$

Besides the well centered at $x = 0$ the potential in $H^{(5)}$ has two degenerate, symmetrical, lower-lying wells centered at

$$x = \pm [2^{-1/2} + [1/3 + (1 + 36g^2)^{1/2}/6]^{1/2}]/g.$$

Again, one may proceed exactly as above to show that the ground-state energy of $H^{(5)}$ vanishes identically for

$g^2 \in D'$. This result, and our explicit verification that the first three terms in the associated Rayleigh–Schrödinger series $\sum b_n g^n$ vanish, lead us to the presumption that the series is asymptotic. If so, it is convergent.

On the basis of the above examples we can now state the following: for non-relativistic quantum mechanical systems having unstable vacua for suitable values of the coupling, g , the Rayleigh–Schrödinger series associated with the ground-state energy, $E(g)$, may

(a) converge to the correct eigenvalue ($H^{(2)}$, $H^{(4)}$, $H^{(5)}$);

(b) converge to the wrong eigenvalue ($H^{(1)}$);

(c) converge or diverge but not be asymptotic to $E(g)$; the Rayleigh–Schrödinger series has nothing to do with $E(g)$ ($H^{(3)}$);

(d) diverge but be uniquely Borel summable to the correct eigenvalue (the anharmonic oscillators $gx^{2(m+1)}$).

Of course a fifth case where the series may be asymptotic yet not uniquely summable by known theorems is always a possibility.

Finally, we refrain from speculating about the implication of our results for the calculation of the large-order behaviour of the ground-state energies of potentials with degenerate minima based on path-space saddle points [4]. We only note that if the last two terms in $H^{(4)}$ ($H^{(5)}$) are dropped, as is the usual practice with this procedure, the potentials in $H^{(4)}$ ($H^{(5)}$) develop three (five) degenerate minima for real g and that there are real instanton–antiinstanton configurations that interpolate between these classical minima.

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