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THE TIME DEPENDENT MEAN FIELD METHOD AND THE INTERACTING BOSON MODEL

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The time dependent mean field approximation is applied to calculate excitation energies in the U(5)–O(6) transitional region of the Interacting Boson Model. A classical hamiltonian is constructed using coherent states parametrized in terms of five coordinates and their conjugate momenta. Energies are obtained using a Bohr–Sommerfeld type requantization condition.

Recently renewed interest in the application of time dependent mean field approximations to general many body systems has developed. An important aspect of previous studies [1–4] was the investigation of the accuracy of the various approximations by comparison with exact solutions of certain hamiltonians. In this respect the two-level Lipkin model [5] has served as a valuable solvable schematic model. In this letter we discuss the application of mean field techniques to the interacting boson model (IBA) [6] which has two features not present in the Lipkin model:

- (i) it has been shown to give a realistic description of collective properties of heavy nuclei,
- (ii) it has a much richer group structure than the Lipkin model (however, exact numerical results can still be obtained for not too large boson number N).

The IBA-model exhibits three dynamical symmetries:

$$\begin{array}{ll} \text{I} & \text{SU}(6) \supset \text{U}(5) \supset \text{O}(5) \supset \text{SO}(3), \\ \text{II} & \supset \text{O}(6) \supset \text{O}(5) \supset \text{SO}(3), \\ \text{III} & \supset \text{SU}(3) \supset \text{SO}(3), \end{array} \quad (1)$$

corresponding to the (an)harmonic vibrator, the gamma unstable rotor, and the axially symmetric rotor, respectively. The ground state properties of the IBA-hamiltonian

$$H = \epsilon C_1(\text{U}(5)) + \kappa C_2(\text{O}(6)) + \kappa' C_2(\text{SU}(3)), \quad (2)$$

have been studied as a function of the parameters ϵ , κ and κ' , using a time independent mean field method [7]. It was found that the ground states associated with each of the dynamical symmetries (1) are characterized by certain shape phases, and that the transitional regions exhibit shape phase transitions of different nature (for $N \rightarrow \infty$). In the present letter we wish to describe not only the ground state properties but the whole spectrum of eigenstates of an IBA hamiltonian by applying the full time dependent technique, where for simplicity we restrict ourselves to the limits I and II and the transitional region in between [i.e. $\kappa' = 0$ in (2)]. As a test of the accuracy the results will be compared to exact calculations with the computer code PHINT [8].

In the present study we consider a hamiltonian:

$$H = (1 - \xi) d^\dagger \cdot \tilde{d} + \{\xi/4(N - 1)\} (d^\dagger \cdot d^\dagger - s^\dagger s^\dagger) (\tilde{d} \cdot \tilde{d} - ss), \quad (3)$$

acting within a symmetric representation [N] (N is the number of bosons) of $\text{SU}(n)$ for n even (where $\tilde{b}_\mu = (-)^\mu$

$\times b_{-\mu}$ and $b \cdot c = \sum_{\mu} (-)^{\mu} b_{\mu} c_{-\mu}$. In general s is a scalar and $d_{\mu}^{\dagger}$ transforms as a tensor of rank $n/2 - 1$ under the rotation group. The first term at the rhs of eq. (3) leaves invariant every $U(n-1)$ irreducible subspace of the representation $[N]$ while the second term corresponds to an invariant operator of the $O(n)$ subgroup of $SU(n)$. Both for $\xi = 0$ and $\xi = 1$ the exact energy spectrum can be obtained by group theoretical methods [6]; however, when both terms are present no exact analytic expression can be derived. In the specific case of the IBA-model ($n = 6$) the hamiltonian (3) describes the transition from the vibrational to the γ -unstable limit as ξ runs from 0 to 1.

Mean field equations can be obtained from a variational principle:

$$\delta S = \delta \int_0^T dt \langle \psi(t) | i \partial / \partial t - H | \psi(t) \rangle = 0. \quad (4)$$

The many-body wave function can be parametrized as a coherent state [9,10]:

$$|N, \alpha_{\mu}\rangle = (N!)^{-1/2} \{ (1 - \alpha \cdot \tilde{\alpha}^*)^{1/2} s^{\dagger} + \tilde{\alpha}^* \cdot d^{\dagger} \}^N |0\rangle, \quad (5)$$

where α represents a set of $n-1$ complex variables.

The expectation value of H and the time derivative appearing in eq. (4) can be easily obtained:

$$\langle N, \alpha_{\mu} | i \partial / \partial t | N, \alpha_{\mu} \rangle = \frac{1}{2} i N (\alpha \cdot \dot{\tilde{\alpha}}^* - \tilde{\alpha}^* \cdot \dot{\alpha}), \quad (6)$$

$$H(\alpha) = \langle N, \alpha_{\mu} | H | N, \alpha_{\mu} \rangle = N(1 - \xi) \alpha \cdot \tilde{\alpha}^* + \frac{1}{4} N \xi (1 - \alpha \cdot \tilde{\alpha}^* - \alpha \cdot \alpha)(1 - \alpha \cdot \tilde{\alpha}^* - \tilde{\alpha}^* \cdot \tilde{\alpha}^*). \quad (7)$$

It turns out that this parametrization is not very convenient for the present application since the mean field equations do not exhibit the symmetries of the hamiltonian in an obvious way. However, it is possible to reexpress the α 's in terms of variables p_i and q_i such that the action S takes the form:

$$S = \int_0^T dt \left\{ \sum_i p_i \dot{q}_i - H(p_i, q_i) \right\}, \quad (8)$$

and the hamiltonian $H(p_i, q_i)$ has a more direct physical interpretation. In this way the mean field equations become equal to Hamilton's equations of motion. We will refer to $H(p_i, q_i)$ as the classical limit of the second quantized hamiltonian (3) and to the solutions of Hamilton's equations of motion as classical trajectories.

To find the approximate eigenvalues of (3) periodic classical trajectories $p_i(t)$, $q_i(t)$ (with certain period T) must be constructed, that satisfy:

$$\int_0^T p_i \dot{q}_i dt = \oint p_i dq_i = 2\pi n_i, \quad (9)$$

where the n_i are positive integers. Then an approximate eigenvalue is given by $E = H(p_i, q_i)$, which is independent of time for a given classical trajectory.

We note that several methods have been proposed to derive eqs. (4, 8, 9). One of them consists of using a path integral representation for the propagator involving coherent states [11]. This approach seems very appealing since coherent states can be considered as intrinsic states convenient for defining a classical (large N) limit for the many particle system [7,12]. Although there are some formal problems concerning these path integrals [11] they allow for a systematic expansion and one can try to obtain corrections to the lowest order results used here.

To obtain the form (8) for the action it turns out to be appropriate in the IBA-model to redefine the five complex parameters α_{μ} as follows:

$$\alpha_{\mu} = \sum_{\nu \rho} D_{\mu \rho}^{(2)}(\phi, \theta, \psi) S_{\nu \rho}(\gamma) (\beta_{\nu} + i \beta'_{\nu}), \quad (10)$$

where $D_{\mu\rho}^{(2)}(\phi, \theta, \psi)$ is a rotation matrix depending on the three Euler angles ϕ, θ, ψ ; $S(\gamma)$ is an $O(5)$ transformation:

$$S(\gamma) = \exp \{ i\gamma \sqrt{2} [(d^\dagger \tilde{d})_2^{(3)} + (d^\dagger \tilde{d})_{-2}^{(3)}] \}, \quad (11)$$

which transforms $\beta_\nu \equiv (0, 0, \beta/\sqrt{2}, 0, 0)$ into

$$\sum_\nu S_{\rho\nu}(\gamma) \beta_\nu = 2^{-1/2} (\beta/\sqrt{2} \sin \gamma, 0, \beta \cos \gamma, 0, \beta/\sqrt{2} \sin \gamma).$$

The five parameters $\beta, \gamma, \phi, \theta, \psi$ can be considered to be the coordinates. Substituting (10) into eq. (6) and demanding the following canonical form for the time derivative:

$$(i \partial/\partial t) = N [\frac{1}{2} P_\beta \dot{\beta} - \frac{1}{2} \beta \dot{P}_\beta + P_\gamma \dot{\gamma} + P_\psi \dot{\psi} + P_\theta \dot{\theta} + P_\phi \dot{\phi}], \quad (12)$$

the five parameters β'_ν can be expressed in terms of the momenta:

$$\begin{pmatrix} \beta'_2 \\ \beta'_1 \\ \beta'_0 \\ \beta'_{-1} \\ \beta'_{-2} \end{pmatrix} = \begin{pmatrix} iM_3/4\beta \sin \gamma + P_\gamma/2\beta \\ M_2/4\beta \sin(\gamma - 4\pi/3) - iM_1/4\beta \sin(\gamma - 2\pi/3) \\ P_\beta/\sqrt{2} \\ -M_2/4\beta \sin(\gamma - 4\pi/3) - iM_1/4\beta \sin(\gamma - 2\pi/3) \\ -iM_3/4\beta \sin \gamma + P_\gamma/2\beta \end{pmatrix}. \quad (13)$$

Here M_k are the angular momentum components in the intrinsic frame. After the substitution of eq. (10) with β'_ν given by (13), into eq. (7) the classical hamiltonian takes on the form:

$$H/N = (1 - \xi) (\frac{1}{2} \beta^2 + \frac{1}{2} P_\beta^2 + T_5^2/2\beta^2) + \xi (\frac{1}{4} (1 - \beta^2)^2 + \frac{1}{4} \beta^2 P_\beta^2), \quad (14)$$

where

$$T_5^2 = P_\gamma^2 + \sum_k \frac{M_k^2}{4 \sin^2(\gamma - 2\pi k/3)}. \quad (15)$$

Due to the compactness of the parameter space of $SU(6)$ there is a limitation on the phase space given by:

$$\frac{1}{2} \beta^2 + \frac{1}{2} P_\beta^2 + T_5^2/2\beta^2 \leq 1. \quad (16)$$

We note that T_5^2 in eq. (14) is just the classical limit of the $O(5)$ Casimir operator and is a constant of the motion. Therefore we can first apply the requantization conditions (9) to the coordinates and momenta contained in T_5 [after which one is left with a requantization problem in the two dimensional phase space (β, P_β)]. One finds that T_5^2 has to be replaced by τ^2/N^2 , where τ represents the boson seniority quantum number. This differs from the exact result, $\tau(\tau+3)/N^2$ [6], in a way that is typical for the approximation used.

Since along a trajectory H is conserved, solutions in the (β, P_β) plane are simply constant energy contours of (14) with fixed T_5 . In the cases of $\xi=0$ [$U(5)$ limit] and $\xi=1$ [$O(6)$ limit] one can find analytic expressions for the energies, within the present approximation, by writing P_β as a function of β and H , and quantizing according to: $\oint P_\beta d\beta = 2\pi n_\beta/N$.

For $U(5)$ this results in the exact spectrum:

$$E_{n_\beta, \tau} = 2n_\beta + \tau, \quad (\tau = 0, 1, \dots, N - 2n_\beta). \quad (17)$$

In the $O(6)$ limit one obtains the τ -independent spectrum:

$$E_{n_\beta} = n_\beta(N - n_\beta)/N, \quad (18)$$

to be compared with the exact one [6]:

$$E_{n_\beta}^{\text{exact}} = n_\beta(N + 2 - n_\beta)/(N - 1). \quad (19)$$

For intermediate cases $0 < \xi < 1$, where the loop integral cannot be evaluated analytically, the result can easily be obtained using numerical methods. In figs. 1a and 1b we compare the energies of some levels from the quantized mean field approximation with those obtained from the diagonalization [8] of the IBA hamiltonian (3). The mean field results are shown for two values of T_5^2 , namely $T_5^2 = (\tau/N)^2$ and $T_5^2 = (\tau + 3/2)^2/N^2$. It is seen that the latter choice (which gives a better approximation to the exact value of T_5 , and could be derived from higher order corrections) results in a better agreement with the exact energies.

The phase transition occurring in this system for $N \rightarrow \infty$ at $\xi = 1/2$, earlier observed for the ground state [7], manifests itself also in the excited states. One finds for the excitation energies:

$$E_{n_\beta, \tau} = (2n_\beta + \tau) \{(1 - \xi)(1 - 2\xi)\}^{1/2} + O(1/N), \quad \text{for } \xi < \frac{1}{2},$$

$$E_{n_\beta, \tau} = n_\beta(2\xi - 1)^{1/2} + O(1/N), \quad \text{for } \xi > \frac{1}{2}.$$

Thus in leading order in N a $U(5)$ multiplet is not split until ξ reaches $1/2$. For $\xi > 1/2$ any $O(6)$ multiplet remains degenerate. Of course this can only occur when all energies (with $n_\beta \ll N$) vanish at $\xi = 1/2$.

The specific form of the hamiltonian (3) is only essential in the sense that in order to apply requantization in the two dimensional phase space spanned by β and β_β one needs a dynamical $O(5)$ symmetry in order to keep T a constant of the motion. One could easily repeat the procedure for a more general IBA hamiltonian containing also $O(5)$ and $SO(3)$ Casimir invariants. On the other hand inclusion of an $SU(3)$ invariant operator gives rise to less

Fig. 1. Some excitation energies, $E(n, \tau)$ in the $U(5)$ – $O(6)$ transitional region as a function of the strength parameter ξ for $N = 14$. The exact results (left) are obtained with the computer program PHINT [8]; the mean field results (right) are shown for $T_5 = \tau/N$ (broken line) and $T_5 = (\tau + 3/2)/N$ (full line).

Fig. 2. Some classical trajectories ($N = 14$) in the β, P_β plane in the $U(u-1)-O(n)$ transitional region for the case $n = 2$ (left) and $n = 6$ (IBA) (right). The value of the strength parameter $\xi = 0.7$; in the case $SU(6)$ the orbitals are labeled by $(n, \tau = 1)$.

trivial changes. However, also for such a term the classical limit can be obtained using the parametrization (10).

Although so far we have restricted the discussion of the hamiltonian (3) to the case $n = 6$, the present approach can directly be applied for other n values. For example with $n = 4$ the $SU(4)$ symmetric hamiltonian (3) contains two dynamical symmetries (which contain the $SO(3)$ subgroup), namely the $U(3)$ ($\xi = 0$) and $O(4)$ ($\xi = 1$) limits [13].

The parametrization that can be used in this case is very similar to (10). ($S(\gamma)$ is not present and one has rank one rotation matrix and tensors.) The classical limit of (3) is identical to (14) with T_5^2 replaced by L^2 , the total angular momentum in polar coordinates, and the solutions in the β, P_β plane as well as the energy spectrum are similar to those described here for the IBA-model.

If one reduces the group structure even more and takes $n = 2$ as to obtain a two level boson system (similar to the fermion systems studies before [1-4]), one obtains a classical hamiltonian from (3) identical to (14) where the T_5^2 term is completely absent. In this case the singularity at $\beta = 0$ is not present and therefore the classical trajectories can in general connect the regions $\beta > 0$ and $\beta < 0$. This is illustrated in fig. 2 where some trajectories in the β, P_β plane corresponding to excited states for the IBA ($n = 6$) case and the $SU(2)$ case are compared. As a result in $SU(2)$ the phase transition gives rise to a discontinuous behaviour of some excitation energies as a function of ξ in the mean field approximation. This does not occur for $n = 6$ (IBA) if the $\tau = 0$ trajectory is calculated by taking the limit $\tau \rightarrow 0$.

As a more general application of the present technique one could think of the description of time dependent phenomena. Recently a mean field approximation to the S -matrix was discussed in connection with the Lipkin model [14].

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