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THE INTERACTING BOSON MODEL AND COLLECTIVE HAMILTONIANS

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It is shown that in some cases of physical interest it is not possible to construct regular collective hamiltonians which will exactly reproduce in all their bound eigenstates only the finite set of eigenvalues of the interacting boson hamiltonian with given boson number N .

The interacting boson model in its original form (IBA) with one kind of s- and d-bosons has been successful in calculating spectra of even-even nuclei [1, 2]. A simple hamiltonian, with single boson terms and boson-boson interactions reproduces very well the various collective spectra observed in nuclei. These spectra range between the limits of vibrational spectra with anharmonicities, rotational spectra corresponding to a symmetric rotor and $O(6)$ spectra (γ -unstable). These collective spectra have been obtained many years ago as the (approximate) solutions of a collective hamiltonian in which the variables are β , γ and the three Euler angles [3,4].

The relation between these two approaches has been considered by several authors [5-7]. Starting from the boson hamiltonian, collective coordinates could be defined and a potential energy surface constructed. The question naturally arose whether the apparent equivalence of the two models could be carried even further. Starting with the spectrum of a boson hamiltonian, could a collective hamiltonian be constructed which will reproduce this spectrum not only approximately but exactly. This question seems to have been answered recently by Ginocchio and Kirson [8,9] who obtained for any given IBA hamiltonian a second-order differential operator in β , γ and the Euler angles which has exactly the *same* eigenvalues. The range of the parameter β as well as the boundary conditions on

the solutions of the differential equation should be determined in a way which will define a Sturm-Liouville problem. In that case, discrete eigenvalues are obtained for eigenfunctions which satisfy the boundary conditions. The range of β need not be limited a priori and may well exceed the value imposed by its original definition [3] (it is, in fact, in some cases between $\beta = 0$ and $\beta = \infty$).

Still the collective hamiltonians constructed in refs. [8,9] seem to be plagued in some cases by non-hermiticity. This fact brings forward the question whether it is possible to construct for any given boson hamiltonian and given boson number N a collective hamiltonian which will have in its all bound eigenstates the same finite set of eigenvalues.

The boson hamiltonians commute with the number operator N of the total number of s- and d-bosons. For any given value of N , there is only a finite number of eigenstates, which is simply determined by N . In the case of a collective hamiltonian the number of *bound* states is not simply determined. There will be a finite number of bound states if the potential energy as a function of β goes to zero fast enough for large values of β . In the collective hamiltonians constructed in refs. [8,9] the number N appears as a parameter. To be fully equivalent to the boson hamiltonians they should give rise to a finite number of bound eigenstates with eigenvalues equal to those of the boson hamiltonians.

The aim of this note is to show that at least for

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certain interesting cases of boson hamiltonians it is impossible to construct collective hamiltonians which are self-adjoint and have the same finite set of eigenvalues. This fact, which will be demonstrated below, should not be too surprising. If we consider a collective hamiltonian with a finite number of eigenstates (as should be the case for an actual physical situation), it may be possible to construct an equivalent boson hamiltonian but it will most probably contain terms with higher orders of boson operators. A boson hamiltonian with at most two-boson terms may yield too idealized a spectrum which a collective hamiltonian may not be able to reproduce exactly.

To prove the above statement it is worthwhile to consider first a simpler boson model introduced by Iachello for diatomic molecules [10]. In many cases energy levels of such molecules are given very well by the simple expression

$$a + bv + cv^2 + BL(L+1), \quad (1)$$

where v is the number of vibrational quanta. Such spectra can be obtained by the approximate solutions of the Schrödinger equation with a suitably chosen local potential. Neglecting rotation-vibration interaction, the expression (1) can be obtained by using the Morse potential. Due to the behavior of the Morse potential for large r , there are only a finite number of bound states.

In the algebraic approach of Iachello [10], p-bosons (1^-) and s-bosons (0^+) are introduced and a simple hamiltonian constructed with single-boson terms and boson-boson interactions. This boson hamiltonian is constructed from generators of the U(4) group. Thus, the total number of bosons N , which commutes with the hamiltonian, characterizes the fully symmetric irreducible representations of U(4). If only the generators of O(4) [a subgroup of U(4)] are used, the eigenstates can be further classified by the irreducible representations of O(4). The quantum number $\sigma = N, N-2, N-4, \dots, 1$ or 0 , characterizes these fully symmetric irreducible representations. A simple hamiltonian can be written down with eigenvalues given by

$$A \frac{1}{4}(N-\sigma)(N+\sigma+2) + BL(L+1), \quad (2)$$

where $L = \sigma, \sigma-1, \sigma-2, \dots, 0$. This is equivalent to (1) since v is defined by $\sigma = N - 2v$. In (2) the constants A and B are both positive. Thus for fixed σ , the higher the L the higher the energy and for a given L ,

the higher the σ the lower the energy. The ground-state rotational band is characterized by $\sigma = N$ and the highest L value in it is given by $L = N$.

Let us now see whether a Schrödinger equation can be constructed which will exactly reproduce the eigenvalues (2) in all its bound eigenstates and in particular, in the ground-state band with $\sigma = N$. Using adequate units ($\hbar^2/2m = 1$) this equation can be written in the form

$$[-d^2/dr^2 - 2r^{-1}d/dr + r^{-2}L^2 + V(r)]\phi(r) = E\phi(r). \quad (3)$$

The bound eigenstates of (3) have the form $r^{-1}u_l(r) \times Y_{lm}(\theta, \phi)$ so that

$$[T_r + V(r) + l(l+1)/r^2]u_l(r) = Eu_l(r), \quad (4)$$

where $T_r = -d^2/dr^2$. Let us now consider eigenstates in the ground-state band, with $\sigma = N$. According to (2), it contains states each of which has, for given $l \leq \sigma$, the lowest eigenvalue (yrast states). The eigenfunctions are solutions of (4) with $E = E_0 + Bl(l+1)$ which satisfy the boundary conditions at $r = 0$ and $r = \infty$. Such a normalized solution $u_l(r)$ satisfies the condition

$$\int_0^\infty u_l(r) [T_r + V(r) + l(l+1)/r^2] u_l(r) dr = E_0 + Bl(l+1). \quad (5)$$

The function $r^{-1}u_{l_0}(r)Y_{lm}(\theta, \phi)$ can be used as a variational function for the eigenstate of the hamiltonian in (3) with angular momentum l . We thus obtain the inequality

$$\begin{aligned} & \int_0^\infty u_{l_0} [T_r + V + l(l+1)/r^2] u_{l_0} dr \\ &= \int_0^\infty u_{l_0} [T_r + V + l_0(l_0+1)/r^2] u_{l_0} dr \\ &+ [l(l+1) - l_0(l_0+1)] \int_0^\infty (u_{l_0}^2/r^2) dr \\ &= E_0 + Bl_0(l_0+1) + [l(l+1) - l_0(l_0+1)] \\ &\times \int_0^\infty (u_{l_0}^2/r^2) dr \geq E_0 + Bl(l+1), \end{aligned} \quad (6)$$

which can be simplified into

$$[l(l+1) - l_0(l_0+1)] \left(\int_0^\infty (u_{l_0}^2/r^2) dr - B \right) \geq 0. \quad (7)$$

If there are at least three levels with the eigenvalues (5), we take l_0 in (7) to be the middle one and consider two values of l one satisfying $l > l_0$ and the other $l < l_0$. Provided these levels are the *lowest* levels with these angular momenta, the variational argument can be applied. We then conclude from (7) that

$$\int_0^\infty (u_{l_0}^2/r^2) dr = B, \quad (8)$$

which makes the inequality (6) become an equality. As a result, the functions $r^{-1}u_{l_0}(r)Y_{lm}(\theta, \phi)$ are eigenfunctions of (3) for *any* values of l with eigenvalues $E_0 + Bl(l+1)$. The radial functions of the Schrödinger equation (3) are all equal independent of the centrifugal term $r^{-2}l(l+1)$. This fact is rather strange and cannot be the case for a regular potential. Moreover, the rotational band continues to $l \rightarrow \infty$ in contradiction to the finite spectrum (2) of the boson model. This is also in contradiction to the expected behavior of an equivalent Schrödinger equation. If $V(r) \rightarrow 0$ for $r \rightarrow \infty$, then beyond a certain l value, all members of the band which are bound states, will have positive energy eigenvalues. As already mentioned, there are potentials which give rise to a spectrum whose low lying energy levels are well given by (2). As l increases, however, the centrifugal term becomes more important, modifying the radial functions so that (8), and a consequence (2), cannot any longer hold.

The proof given above that it is impossible to construct a regular differential equation to reproduce exactly the eigenvalues (2) is based on the variational principle. Hence, it holds for self-adjoint differential operators not necessarily given by (3). The operator T_r could be a more complicated second-order differential operator. The corresponding eigenfunctions may be orthogonal only when an appropriate weight function is introduced. The coefficient of L^2 in the differential equation need not be r^{-2} but could be any function of r . The argument can be extended also to other cases and in particular to the case of the interaction boson model for nuclei.

The above considerations for the case of O(4) symmetry can be directly applied to nuclear collective hamiltonians with O(6) symmetry. In eq. (5.22) of ref. [8] or eq. (5.10) of ref. [9] the collective hamiltonian is given as

$$H_0(\beta) + (\kappa/\beta^2)C_5, \quad (9)$$

where $H_0(\beta)$ is a second-order differential operator involving only β in which N appears as a parameter. The operator C_5 is the Casimir operator of O(5). It is a differential operator in the angle γ and the Euler angles and its eigenvalues are given by $\tau(\tau+3)$ where $\tau = \sigma, \sigma-1, \dots, 0$. Also for the O(6) case, σ should be limited by $\sigma = N, N-2, N-4, \dots, 1$ or 0. The eigenvalues of the boson hamiltonian in this case are degenerate in L and are given by

$$-\kappa[\sigma(\sigma+4) - \tau(\tau+3)]. \quad (10)$$

The parameter $-\kappa$ is negative and the value of σ which gives the lowest energies is $\sigma = N$. For this value, the higher the value of τ , the higher the energy. The orthogonal eigenfunctions of the differential equation which corresponds to the collective hamiltonian (9) are products of a function $\Phi_{N\sigma\tau}(\beta)$ and a function [11] $\Psi_{\tau LM}(\gamma, \Omega)$ where Ω stands for the three Euler angles.

If a collective hamiltonian has eigenvalues given by (10) with $\sigma \leq N$ we can take, for $\sigma = N$, three states with three different values of τ with angular momenta $L = 2\tau$. Since $L \leq 2\tau$, such states are the lowest among all states with the same L value and hence the conditions for the variational estimates are satisfied. We can then conclude as before, that the wave functions $\Phi_{N\sigma\tau}(\beta)$ must be independent of τ . From this fact follows that all states

$$\Phi_{N\sigma\tau_0}(\beta)\Psi_{\tau LM}(\gamma, \Omega) \quad (11)$$

must be bound eigenstates of the hamiltonian (9) for any given value of τ , as high as it may be. Yet, the wave functions of (9) given in ref. [9] [eq. (5.13)] do not have the form (11) since $\Phi(\beta)$ *does* depend on the value of τ . Hence, we must conclude that the band with $\sigma = N, L = 2\tau$ is not the ground-state band. There must be states below the members of that band even for a given finite value of N . If these eigenvalues of those lower states are also given by (10) there cannot be a lowest band characterized by $\sigma > N$ to which the variational argument can be applied. If that were the case, that band should have continued to arbitrarily large values of τ (and L).

Another case in which it can be shown that no collective hamiltonian can be constructed which has for its bound states the identical spectrum of the boson hamiltonian is the SU(3) limit. In that limit a spectrum like that of a symmetric rotor is obtained, Ginocchio and Kirson have constructed a differential operator

((5.26) of ref. [8]) that acting on a set of functions of β, γ, Ω gives the same functions multiplied by the exact eigenvalues of a boson hamiltonian in the SU(3) limit. Let us examine that operator in the case of the ground state band [characterized by $(\lambda, \mu) = (2N, 0)$] in which the energies are given by ((5.29) of ref. [8]) as

$$-\kappa [N(2N+3) - \frac{3}{8}L(L+1)] . \quad (12)$$

all boson states of this band can be projected from the intrinsic state

$$(s^+ + \sqrt{2}d_0)^N |0\rangle , \quad (13)$$

which has a $K=0$ eigenvalue of the 3-component of the angular momentum. The same set of states can be projected also from the general intrinsic state

$$[s^+ + \beta \cos \gamma d_0^+ + \beta \sin \gamma (d_2^+ + d_{-2}^+)]^N |0\rangle \quad (14)$$

rotated with respect to the original axes by the Euler angles Ω . The eigenvalue $K=0$ will then be the eigenvalue of the angular momentum component along the intrinsic 3-axis, L_3 . For this component L_3 , as well as for L_1 and L_2 , equivalent differential operators in the Euler angles have been constructed (see e.g. ref. [8]). The collective hamiltonian, in which N appears as a parameter, was then expressed as ((5.5), (5.11) and (5.26) of ref. [8])

$$T_N(\beta, \gamma) + V_N(\beta, \gamma) + \frac{\hbar^2}{2} \sum_{k=1}^3 L_k^2 / I_k(\beta, \gamma) . \quad (15)$$

In the SU(3) limit the coefficients of the L_k^2 are given by ((5.26) ref. [8])

$$\frac{\hbar^2}{2I_k} = \frac{\kappa}{2\beta^2} \left(\frac{1 - \beta \cos(\gamma - 2\pi k/3)/\sqrt{2}}{\sin(\gamma - 2\pi k/3)} \right)^2 . \quad (16)$$

The eigenstates of the operator (15) in the ground-state band have all $K=0$. The only terms in (15) which do not commute with L_3 are L_1^2 and L_2^2 . Expressed in terms of the raising and lowering components they are given by

$$\begin{aligned} 4L_1^2 &= L_+^2 + L_-^2 + L_+L_- + L_-L_+ , \\ 4L_2^2 &= -L_+^2 - L_-^2 + L_+L_- + L_-L_+ , \end{aligned} \quad (17)$$

we can now rewrite the hamiltonian (15) as

$$\begin{aligned} T_N(\beta, \gamma) + V_N(\beta, \gamma) + (\hbar^2/2I_3)L_3^2 \\ + \frac{1}{8}\hbar^2(1/I_1 + 1/I_2)(L_+L_- + L_-L_+) \\ + \frac{1}{8}\hbar^2(1/I_1 - 1/I_2)(L_+^2 + L_-^2) , \end{aligned} \quad (18)$$

where the last term connects states with given eigenvalue K of L_3 with states having $K+2$ and $K-2$. States of the ground-state band of SU(3) are eigenstates of L_3 with $K=0$ and thus for such states, the last term in (18) does not contribute and can be omitted. Due to

$$L_+L_- + L_-L_+ = 2L_1^2 + 2L_2^2 = 2L^2 - 2L_3^2 ,$$

the terms in (18) can be simplified for the $K=0$ eigenvalues of L_3 , and written as

$$T_N(\beta, \gamma) + V_N(\beta, \gamma) + \frac{1}{4}\hbar^2[1/I_1(\beta, \gamma) + 1/I_2(\beta, \gamma)]L^2 . \quad (19)$$

The hamiltonian (19) is very similar to the one in (3) which was considered above. The radial coordinate r is replaced here by β and γ . It has in the ground-state band the eigenvalue (12) and the variational argument can be directly applied. The eigenfunctions of (19) have the well known general form $\Phi_{NL}(\beta, \gamma)D_{MO}^{(L)}(\Omega)$. If the band consists of at least three states, one with L_0 , one above and one below it, we conclude that $\Phi_{NL}(\beta, \gamma)$ must be independent of L and that

$$\Phi_{NL_0}(\beta, \gamma)D_{MO}^{(L)}(\Omega) \quad (20)$$

must be eigenstates for any value of L (not limited by $L \leq 2N$ as in the boson model). Also in this case we must conclude that even for a given finite value of N , the hamiltonian (15) has bound states with eigenvalues not equal to those of the boson model. Either the values of L in (12) are unlimited and exceed the limit $L \leq 2N$ of the boson model, or the $K=0, \lambda=2N$ "ground-state band" considered here does not contain the yrast states. There must then be states with eigenvalues lower than those of the boson model. If these eigenvalues belong to $K=0$ states and have the form $-\kappa[\frac{1}{4}\lambda(\lambda+3) - \frac{3}{8}L(L+1)]$ with $\lambda > 2N$ then it may happen that there is no lowest band of this kind. Otherwise, the variational argument would lead to a band with arbitrarily high values of L .

It seems that also for the interacting boson model for nuclei there are two possibilities of obtaining collective hamiltonians. One way is to write collective

hamiltonians with regular potential energy terms in β which will have a finite number of bound eigenstates. The eigenvalues of such collective hamiltonians may be very good approximations to those of the boson hamiltonians for low lying levels but could not be exactly equal to them. The other possibility is to obtain collective hamiltonians which will exactly reproduce the eigenvalues of the boson hamiltonians but will have in addition an infinite number of bound states also for finite values of N .

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